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Analysis of the Preference Shift of Customer Brand Selection and Its Matrix Structure -Expansion to the second order lag

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Abstract

It is often observed that consumers select the upper class brand when they buy the next time. Suppose that the former buying data and the current buying data are gathered. Also suppose that the upper brand is located upper in the variable array. Then the transition matrix becomes an upper triangle matrix under the supposition that the former buying variables are set input and the current buying variables are set output. The goods of the same brand group would compose the Block Matrix in the transition matrix. Condensing the variables of the same brand group into one, analysis becomes easier to handle and the transition of Brand Selection can

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be easily grasped. In this paper, equation using transition matrix stated by the Block Matrix is expanded to the second order lag and the method of condensing the variable stated above is also applied to this new model. Planners for products need to know its brand position whether their brand is upper or lower than other products. Matrix structure makes it possible to ascertain this by calculating consumers' activities for brand selection. Thus, this proposed approach makes it possible to execute an effective marketing plan and/or establishing new brand.

Mathematics Subject Classification: 03C85, 15A23, 15A24

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1 Introduction

It is often observed that consumers select the upper class brand when they buy the next time. Focusing the transition matrix structure of brand selection, their activities may be analyzed. In the past, there are many researches about brand selection [1-5]. But there are few papers concerning the analysis of the transition matrix structure of brand selection. In this paper, we make analysis of the preference shift of customer brand selection. The goods of the same brand group would compose the Block Matrix in the transition matrix. Condensing the variables of the same brand group into one, analysis becomes easier to handle and the transition of Brand Selection can be easily grasped. Then we confirm them by the questionnaire investigation for automobile purchasing case. If we can identify the feature of the matrix structure of brand selection, it can be utilized for the marketing strategy.

Suppose that the former buying data and the current buying data are gathered. Also suppose that the upper brand is located upper in the variable array. Then the transition matrix becomes an upper triangular matrix under the supposition that the former buying variables are set input and the current buying variables are set output. If the top brand were selected from the lower brand in jumping way, corresponding part in the upper triangular matrix would be 0. These are verified by the numerical examples with simple models.

Planners for products need to know its brand position whether their brand is upper or lower than other products. Matrix structure makes it possible to ascertain this by calculating consumers' activities for brand selection. Thus, this proposed approach makes it possible to execute an effective marketing plan and/or establishing new brand.

A quantitative analysis concerning brand selection has been executed by [4, 5]. [5] examined purchasing process by Markov Transition Probability with the input of advertising expense. [4] made analysis by the Brand Selection Probability model using logistics distribution.

The goods of the same brand group would compose the Block Matrix in the transition matrix. Condensing the variables of the same brand group into one, analysis becomes easier to handle and the transition of Brand Selection can be easily grasped. In this paper, equation using transition matrix stated by the Block Matrix is expanded to the second order lag and the method of condensing the variable stated above is also applied to this new model. Such research cannot be found as long as searched.

The rest of the paper is organized as follows. Matrix structure is clarified for the selection of brand in section 2. A block matrix structure is analyzed when brands are handled in a group in section 3. Expansion of the block matrix structure to the second order lag is executed in section 4. A block matrix structure when condensing the variables of the same brand group is analyzed in section 5. Its expansion to the second order lag in stated in section 6. Numerical calculation is executed in section 7. Section 8 is a summary.

2 Brand Selection and Its Matrix Structure

2.1 Upper Shift of Brand Selection

It is often observed that consumers select the upper class brand when they buy the next time. Now, suppose that x is the most upper class brand, y is the second upper brand, and z is the lowest brand. Consumer's behavior of selecting brand would be $z \to y$, $y \to x$, $z \to x$ etc. $x \to z$ might be few. Suppose that x is the current buying variable, and x_b is the previous buying variable. Shift to x is executed from x_b, y_b, or z_b.

Therefore, x is stated in the following equation.

$$x = a_{11}x_b + a_{12}y_b + a_{13}z_b$$

Similarly,

$$y = a_{22}y_b + a_{23}z_b$$

And

$$z = a_{33}z_b$$

These are re-written as follows.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \tag{1}$$

Set :

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad \mathbf{X_b} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

then, $\mathbf{X}$ is represented as follows.

$$\mathbf{X} = \mathbf{A}\mathbf{X_b} \tag{2}$$

Here,

$$\mathbf{X} \in \mathbf{R}^3, \mathbf{A} \in \mathbf{R}^{3\times 3}, \mathbf{X_b} \in \mathbf{R}^3,$$

$\mathbf{A}$ is an upper triangular matrix.

To examine this, generating the following data, which are all consisted by the upper brand shift data,

$$\mathbf{X}^{\mathbf{i}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \cdots \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{3}$$

$$\mathbf{X}_{\mathbf{b}}^{\mathbf{i}} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \cdots \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{4}$$

$$i = 1, \qquad 2, \qquad \cdots \qquad N$$

parameter can be estimated using least square method.

Suppose

$$\mathbf{X}^{i} = \mathbf{A}\mathbf{X}_{\mathbf{b}}^{i} + \boldsymbol{\varepsilon}^{i} \tag{5}$$

and

$$J = \sum_{i=1}^{N} \boldsymbol{\varepsilon}^{iT} \boldsymbol{\varepsilon}^{i} \rightarrow Min \tag{6}$$

$\hat{\mathbf{A}}$ which is an estimated value of $\mathbf{A}$ is obtained as follows.

$$\hat{\mathbf{A}} = \left(\sum_{i=1}^{N} \mathbf{X}^{i} \mathbf{X}_{\mathbf{b}}^{iT} \right) \left(\sum_{i=1}^{N} \mathbf{X}_{\mathbf{b}}^{i} \mathbf{X}_{\mathbf{b}}^{iT} \right)^{-1} \tag{7}$$

In the data group of the upper shift brand, estimated value $\hat{\mathbf{A}}$ should be an upper triangular matrix. If the following data, that have the lower shift brand, are added only a few in equation (3) and (4),

$$\mathbf{X}^{i} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{X}_{\mathbf{b}}^{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$\hat{\mathbf{A}}$ would contain minute items in the lower part triangle.

2.2 Sorting Brand Ranking by Re-arranging Row

In a general data, variables may not be in order as x, y, z. In that case, large and small value lie scattered in $\hat{\mathbf{A}}$. But re-arranging this, we can set in order by shifting row. The large value parts are gathered in an upper triangular matrix, and the small value parts are gathered in a lower triangular matrix.

$$\begin{array}{cc} & \hat{\mathbf{A}} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \begin{pmatrix} \bigcirc & \bigcirc & \bigcirc \\ \varepsilon & \bigcirc & \bigcirc \\ \varepsilon & \varepsilon & \bigcirc \end{pmatrix} \end{array} \xleftarrow{\text{Shifting row}} \begin{array}{cc} & \hat{\mathbf{A}} \\ \begin{pmatrix} z \\ x \\ y \end{pmatrix} & \begin{pmatrix} \varepsilon & \varepsilon & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \\ \varepsilon & \bigcirc & \bigcirc \end{pmatrix} \end{array} \qquad (8)$$

2.3 In the case that brand selection shifts in jump

It is often observed that some consumers select the most upper class brand from the most lower class brand and skip selecting the middle class brand. We suppose v, w, x, y, z brands (suppose they are laid from the upper position to the lower position as $v > w > x > y > z$).

In the above case, the selection shifts would be

$$v \leftarrow z$$

$$v \leftarrow y$$

Suppose there is no shift from z to y, corresponding part of the transition matrix is 0 (i.e. a_{45}=0). Similarly, if there is no shift from z to y, from z to w, from y to x, form y to w, from x to w, then the matrix structure would be as follows.

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \begin{pmatrix} v_b \\ w_b \\ x_b \\ y_b \\ z_b \end{pmatrix} \tag{9}$$

3 Block Matrix structure in Brand Groups

Next, we examine the case in brand groups. Matrices are composed by Block Matrix.

(1) Brand shift group — in the case of two groups

Suppose brand selection shifts from Corolla class to Mark II class in car. In this case, it does not matter which company's car they choose. Thus, selection of cars are executed in a group and brand shift is considered to be done from group to group. Suppose brand groups at time n are as follows.

$\mathbf{X}$ consists of p varieties of goods, and $\mathbf{Y}$ consists of q varieties of goods.

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12} \\ \mathbf{0}, & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \end{pmatrix} \tag{10}$$

Here,

$\mathbf{X_n} \in \mathbf{R}^p \ (n=1,2,\cdots)$, $\mathbf{Y_n} \in \mathbf{R}^q \ (n=1,2,\cdots)$, $\mathbf{A_{11}} \in \mathbf{R}^{p\times p}$, $\mathbf{A_{12}} \in \mathbf{R}^{p\times q}$, $\mathbf{A_{22}} \in \mathbf{R}^{q\times q}$

Make one more step of shift, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^2, & \mathbf{A_{11}}\mathbf{A_{12}} + \mathbf{A_{12}}\mathbf{A_{22}} \\ \mathbf{0}, & \mathbf{A_{22}}^2 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-2}} \\ \mathbf{Y_{n-2}} \end{pmatrix} \tag{11}$$

Make one more step of shift again, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^3, & \mathbf{A_{11}}^2\mathbf{A_{12}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{12}}\mathbf{A_{22}}^2 \\ \mathbf{0}, & \mathbf{A_{22}}^3 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-3}} \\ \mathbf{Y_{n-3}} \end{pmatrix} \tag{12}$$

Similarly,

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^4, & \mathbf{A_{11}}^3\mathbf{A_{12}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{12}}\mathbf{A_{22}}^3 \\ \mathbf{0}, & \mathbf{A_{22}}^4 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-4}} \\ \mathbf{Y_{n-4}} \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^5, & \mathbf{A_{11}}^4\mathbf{A_{12}} + \mathbf{A_{11}}^3\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^3 + \mathbf{A_{12}}\mathbf{A_{22}}^4 \\ \mathbf{0}, & \mathbf{A_{22}}^5 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-5}} \\ \mathbf{Y_{n-5}} \end{pmatrix} \tag{14}$$

Finally, we get generalized equation for s-step shift as follows.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^s, & \mathbf{A_{11}}^{s-1}\mathbf{A_{12}} + \sum_{k=2}^{s-1}\mathbf{A_{11}}^{s-k}\mathbf{A_{12}}\mathbf{A_{22}}^{k-1} + \mathbf{A_{12}}\mathbf{A_{22}}^{s-1} \\ \mathbf{0}, & \mathbf{A_{22}}^s \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-s}} \\ \mathbf{Y_{n-s}} \end{pmatrix} \tag{15}$$

If we replace $n-s \to n, n \to n+s$ in equation (15), we can make s-step forecast.

(2) Brand shift group — in the case of three groups

Suppose brand selection is executed in the same group or to the upper group, and also suppose that brand position is $x > y > z$ (x is upper position). Then brand selection transition matrix would be expressed as

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12}, & \mathbf{A}_{13} \\ \mathbf{0}, & \mathbf{A}_{22}, & \mathbf{A}_{23} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix} \tag{16}$$

Where

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}, \qquad \mathbf{Z_n} = \begin{pmatrix} z_1^n \\ z_2^n \\ \vdots \\ z_r^n \end{pmatrix}$$

Here,

$\mathbf{X_n} \in \mathbf{R}^p \quad (n = 1,2,\cdots)$, $\mathbf{Y_n} \in \mathbf{R}^q \quad (n = 1,2,\cdots)$, $\mathbf{Z_n} \in \mathbf{R}^r \quad (n = 1,2,\cdots)$,

$\mathbf{A_{11}} \in R^{p\times p}$, $\mathbf{A_{12}} \in R^{p\times q}$, $\mathbf{A_{13}} \in R^{p\times r}$, $\mathbf{A_{22}} \in R^{q\times q}$, $\mathbf{A_{23}} \in R^{q\times r}$,

$\mathbf{A_{33}} \in R^{r\times r}$

These are re-stated as

$$\mathbf{W_n} = \mathbf{A}\mathbf{W_{n-1}} \tag{17}$$

where, $\mathbf{W_n} = \begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12}, & \mathbf{A}_{13} \\ \mathbf{0}, & \mathbf{A}_{22}, & \mathbf{A}_{23} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33} \end{pmatrix}$, $\mathbf{W_{n-1}} = \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix}$

Hereinafter, we shift steps as is done in previous section.

In the general description, we state as

$$\mathbf{W_n} = \mathbf{A^{(s)}}\mathbf{W_{n-s}} \tag{18}$$

Here,

$$\mathbf{A}^{(\mathbf{s})}=\begin{pmatrix}\mathbf{A}_{11}{}^{(\mathbf{s})}, & \mathbf{A}_{12}{}^{(\mathbf{s})}, & \mathbf{A}_{13}{}^{(\mathbf{s})}\\ \mathbf{0}, & \mathbf{A}_{22}{}^{(\mathbf{s})}, & \mathbf{A}_{23}{}^{(\mathbf{s})}\\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}{}^{(\mathbf{s})}\end{pmatrix}, \qquad \mathbf{W}_{\mathbf{n-s}}=\begin{pmatrix}\mathbf{X}_{\mathbf{n-s}}\\ \mathbf{Y}_{\mathbf{n-s}}\\ \mathbf{Z}_{\mathbf{n-s}}\end{pmatrix}$$

From definition,

$$\mathbf{A}^{(1)}=\mathbf{A} \tag{19}$$

In the case $s=2$, we obtain

$$\begin{aligned}\mathbf{A}^{(2)}&=\begin{pmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}, & \mathbf{A}_{13}\\ \mathbf{0}, & \mathbf{A}_{22}, & \mathbf{A}_{23}\\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}\end{pmatrix}\begin{pmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}, & \mathbf{A}_{13}\\ \mathbf{0}, & \mathbf{A}_{22}, & \mathbf{A}_{23}\\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}\end{pmatrix}\\ &=\begin{pmatrix}\mathbf{A}_{11}^{2}, & \mathbf{A}_{11}\mathbf{A}_{12}+\mathbf{A}_{12}\mathbf{A}_{22}, & \mathbf{A}_{11}\mathbf{A}_{13}+\mathbf{A}_{12}\mathbf{A}_{23}+\mathbf{A}_{13}\mathbf{A}_{33}\\ \mathbf{0}, & \mathbf{A}_{22}^{2}, & \mathbf{A}_{22}\mathbf{A}_{23}+\mathbf{A}_{23}\mathbf{A}_{33}\\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^{2}\end{pmatrix}\end{aligned} \tag{20}$$

Next, in the case $s=3$, we obtain

$$\mathbf{A}^{(3)}=\begin{pmatrix}\mathbf{A}_{11}^{3}, & \mathbf{A}_{11}^{2}\mathbf{A}_{12}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}+\mathbf{A}_{12}\mathbf{A}_{22}^{2}, & \mathbf{A}_{11}^{2}\mathbf{A}_{13}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}+\mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}+\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}+\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}+\mathbf{A}_{13}\mathbf{A}_{33}^{2}\\ \mathbf{0}, & \mathbf{A}_{22}^{3}, & \mathbf{A}_{22}^{2}\mathbf{A}_{23}+\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}+\mathbf{A}_{23}\mathbf{A}_{33}^{2}\\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^{3}\end{pmatrix} \tag{21}$$

In the case $s=4$, equations become wide-spread, so we express each Block Matrix as follows.

$$\left.\begin{aligned}&\mathbf{A}_{11}^{(4)}=\mathbf{A}_{11}^{4}\\ &\mathbf{A}_{12}^{(4)}=\mathbf{A}_{11}^{3}\mathbf{A}_{12}+\mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2}+\mathbf{A}_{12}\mathbf{A}_{22}^{3}\\ &\mathbf{A}_{13}^{(4)}=\mathbf{A}_{11}^{3}\mathbf{A}_{13}+\mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23}+\mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}+\mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}+\mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{2}\\ &\qquad+\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}+\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}+\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2}+\mathbf{A}_{13}\mathbf{A}_{33}^{3}\\ &\mathbf{A}_{22}^{(4)}=\mathbf{A}_{22}^{4}\\ &\mathbf{A}_{23}^{(4)}=\mathbf{A}_{22}^{3}\mathbf{A}_{23}+\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33}+\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2}+\mathbf{A}_{23}\mathbf{A}_{33}^{3}\\ &\mathbf{A}_{33}^{(4)}=\mathbf{A}_{33}^{4}\end{aligned}\right\} \tag{22}$$

In the case $s=5$, we obtain the following equations similarly.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(5)} = \mathbf{A}_{11}^{5} \\
&\mathbf{A}_{12}^{(5)} = \mathbf{A}_{11}^{4}\mathbf{A}_{12} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}^{2} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{3} + \mathbf{A}_{12}\mathbf{A}_{22}^{4} \\
&\mathbf{A}_{13}^{(5)} = \mathbf{A}_{11}^{4}\mathbf{A}_{13} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^{3}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33}^{2} \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{3} \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{13}\mathbf{A}_{33}^{4} \\
&\mathbf{A}_{23}^{(5)} = \mathbf{A}_{22}^{4}\mathbf{A}_{23} + \mathbf{A}_{22}^{3}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{23}\mathbf{A}_{33}^{4} \\
&\mathbf{A}_{33}^{(5)} = \mathbf{A}_{33}^{5}
\end{aligned}\right\} \tag{23}$$

In the case $s = 6$, we obtain

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(6)} = \mathbf{A}_{11}^{6} \\
&\mathbf{A}_{13}^{(6)} = \mathbf{A}_{11}^{5}\mathbf{A}_{13} + \mathbf{A}_{11}^{4}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^{4}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{3}\mathbf{A}_{13}\mathbf{A}_{33}^{2} \\
&\qquad + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33}^{3} \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{4} \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^{4}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{4} + \mathbf{A}_{13}\mathbf{A}_{33}^{5}
\end{aligned}\right\} \tag{24}$$

We get generalized equations for s-step shift as follows.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(s)} = \mathbf{A}_{11}^{s} \\
&\mathbf{A}_{12}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{12} + \sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1} + \mathbf{A}_{12}\mathbf{A}_{22}^{s-1} \\
&\mathbf{A}_{13}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{13} + \mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{\mathbf{1(k+1)}}\mathbf{A}_{\mathbf{(k+1)3}}\right) + \sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{S-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right) + \mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right] \\
&\mathbf{A}_{22}^{(s)} = \mathbf{A}_{22}^{s} \\
&\mathbf{A}_{23}^{(s)} = \sum_{K-1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1} \\
&\qquad \mathbf{A}_{33}^{(s)} = \mathbf{A}_{33}^{s}
\end{aligned}\right\} \tag{25}$$

Expressing them in matrix, it follows.

$$\mathbf{A}^{(S)} = \begin{pmatrix} \mathbf{A}_{11}^{s}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{12} + \sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1} + \mathbf{A}_{12}\mathbf{A}_{22}^{s-1}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{13} + \mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{1(\mathbf{k}+1)}\mathbf{A}_{(\mathbf{K}+1)3}\right) + \sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{s-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right) + \mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right] \\ \mathbf{0}, & \mathbf{A}_{22}^{s}, & \sum_{k=1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^{s} \end{pmatrix} \quad (26)$$

Generalizing them to m groups, they are expressed as

$$\begin{pmatrix} \mathbf{X}_{\mathbf{n}}^{(1)} \\ \mathbf{X}_{\mathbf{n}}^{(2)} \\ \vdots \\ \mathbf{X}_{\mathbf{n}}^{(m)} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{\mathbf{11}} & \mathbf{A}_{\mathbf{12}} & \cdots & \mathbf{A}_{\mathbf{1m}} \\ \mathbf{A}_{\mathbf{21}} & \mathbf{A}_{\mathbf{22}} & \cdots & \mathbf{A}_{\mathbf{2m}} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{\mathbf{m1}} & \mathbf{A}_{\mathbf{m2}} & \cdots & \mathbf{A}_{\mathbf{mm}} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{\mathbf{n-1}}^{(1)} \\ \mathbf{X}_{\mathbf{n-1}}^{(2)} \\ \vdots \\ \mathbf{X}_{\mathbf{n-1}}^{(m)} \end{pmatrix} \quad (27)$$

$$\mathbf{X}_{\mathbf{n}}^{(1)} \in R^{k_1}, \quad \mathbf{X}_{\mathbf{n}}^{(2)} \in R^{k_2}, \quad \cdots, \quad \mathbf{X}_{\mathbf{n}}^{(m)} \in R^{k_m}, \quad \mathbf{A}_{\mathbf{ij}} \in R^{k_i \times k_j} \; (i = 1, \cdots, m)(j = 1, \cdots, m)$$

4 Expansion of the Block Matrix structure to the second order lag

Expansion of the above stated Block Matrix model to the second order lag is executed in the following method. Here we take three groups case.

Generating Eq.(16) and Eq.(18),we state the model as follows. Here we set P=3.

$$\begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix} = \begin{pmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{D}, & \mathbf{E}, & \mathbf{F} \\ \mathbf{G}, & \mathbf{H}, & \mathbf{J} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{n-1} \\ \mathbf{Y}_{n-1} \\ \mathbf{Z}_{n-1} \end{pmatrix} \quad (28)$$

Where

$$\mathbf{X}_n = \begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix}, \mathbf{Y}_n = \begin{pmatrix} y_1^n \\ y_2^n \\ y_3^n \end{pmatrix}, \mathbf{Z}_\mathbf{n} = \begin{pmatrix} z_1^n \\ z_2^n \\ z_3^n \end{pmatrix} \tag{29}$$

Here,

$\mathbf{X}_n \in \mathbf{R}^3 \left(n = 1, 2, \cdots\right)$, $\mathbf{Y}_n \in \mathbf{R}^3 \left(n = 1, 2, \cdots\right)$, $\mathbf{Z}_n \in \mathbf{R}^\mathbf{3} \left(n = 1, 2, \cdots\right)$,

$\left\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\right\} \in \mathbf{R}^{3\times3}$. These are re-stated as

$$\mathbf{W}_n = \mathbf{P}\mathbf{W}_{n-1} \tag{30}$$

$$\mathbf{W}_n = \begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix} \tag{31}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{D}, & \mathbf{E}, & \mathbf{F} \\ \mathbf{G}, & \mathbf{H}, & \mathbf{J} \end{pmatrix} \tag{32}$$

$$\mathbf{W}_{n-1} = \begin{pmatrix} \mathbf{X}_{n-1} \\ \mathbf{Y}_{n-1} \\ \mathbf{Z}_{n-1} \end{pmatrix} \tag{33}$$

if N amount of data exist, we can derive the following the equation similarly as Eq.(5),

$$\mathbf{W}_n^i = \mathbf{P}\mathbf{W}_{n-1}^i + \boldsymbol{\varepsilon}_n^i \left(i = 1, 2, \cdots, N\right) \tag{34}$$

And

$$J_n = \sum_{i=1}^{N} \boldsymbol{\varepsilon}_n^{iT} \boldsymbol{\varepsilon}_n^i \rightarrow Min \tag{35}$$

$\hat{\mathbf{P}}$ which is an estimated value of $\mathbf{P}$ is obtained as follows.

$$\hat{\mathbf{P}} = \left(\sum_{i=1}^{N} \mathbf{W}_n^i \mathbf{W}_{n-1}^{iT}\right)\left(\sum_{i=1}^{N} \mathbf{W}_{n-1}^i \mathbf{W}_{n-1}^{iT}\right)^{-1} \tag{36}$$

Now, we expand Eq.(34) to the second order lag model as follows.

$$\mathbf{W}_n^i = \mathbf{P}_1 \mathbf{W}_{n-1}^i + \mathbf{P}_2 \mathbf{W}_{n-2}^i + \boldsymbol{\varepsilon}_n^i \tag{37}$$

Here

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1 \\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1 \\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2 \\ \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2 \\ \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2 \end{pmatrix} \tag{38}$$

It we set

$$\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2) \tag{39}$$

then $\hat{\mathbf{P}}$ can be estimated as follows.

$$\mathbf{P} = \left(\sum_{i=1}^{N} \mathbf{W}_t^i \begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \end{pmatrix}^T\right)\left(\sum_{i=1}^{N} \begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \end{pmatrix}\begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \end{pmatrix}^T\right)^{-1} \tag{40}$$

We further develop this equation as follows.

$$
\begin{aligned}
\mathbf{P} &= \left(\mathbf{P}_1, \mathbf{P}_2\right) \\
&= \begin{pmatrix} \mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1, & \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2 \\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1, & \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2 \\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1, & \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2 \end{pmatrix} \\
&= \left(\sum_{i=1}^{N} \mathbf{W}_t^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_t^i \mathbf{W}_{t-2}^{iT} \right) \begin{pmatrix} \sum_{i=1}^{N} \mathbf{W}_{t-1}^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-1}^i \mathbf{W}_{t-2}^{iT} \\ \sum_{i=1}^{N} \mathbf{W}_{t-2}^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-2}^i \mathbf{W}_{t-2}^{iT} \end{pmatrix}^{-1} \\
&= \left(\sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_t^i \\ \mathbf{y}_t^i \\ \mathbf{z}_t^i \end{pmatrix} \left(\mathbf{x}_{t-1}^{iT}, \mathbf{y}_{t-1}^{iT}, \mathbf{z}_{t-1}^{iT} \right), \quad \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_t^i \\ \mathbf{y}_t^i \\ \mathbf{z}_t^i \end{pmatrix} \left(\mathbf{x}_{t-2}^{iT}, \mathbf{y}_{t-2}^{iT}, \mathbf{z}_{t-2}^{iT} \right) \right) \\
&\times \begin{pmatrix} \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_{t-1}^i \\ \mathbf{y}_{t-1}^i \\ \mathbf{z}_{t-1}^i \end{pmatrix} \left(\mathbf{x}_{t-1}^{iT}, \mathbf{y}_{t-1}^{iT}, \mathbf{z}_{t-1}^{iT} \right), & \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_{t-1}^i \\ \mathbf{y}_{t-1}^i \\ \mathbf{z}_{t-1}^i \end{pmatrix} \left(\mathbf{x}_{t-2}^{iT}, \mathbf{y}_{t-2}^{iT}, \mathbf{z}_{t-2}^{iT} \right) \\ \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_{t-2}^i \\ \mathbf{y}_{t-2}^i \\ \mathbf{z}_{t-2}^i \end{pmatrix} \left(\mathbf{x}_{t-1}^{iT}, \mathbf{y}_{t-1}^{iT}, \mathbf{z}_{t-1}^{iT} \right), & \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_{t-2}^i \\ \mathbf{y}_{t-2}^i \\ \mathbf{z}_{t-2}^i \end{pmatrix} \left(\mathbf{x}_{t-2}^{iT}, \mathbf{y}_{t-2}^{iT}, \mathbf{z}_{t-2}^{iT} \right) \end{pmatrix}^{-1} \\
&= \left(\begin{array}{ccc:ccc} \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{x}_t^i \mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{y}_t^i \mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N} \mathbf{z}_t^i \mathbf{z}_{t-2}^{iT} \end{array} \right),
\end{aligned}
$$

$$\times\left(\begin{array}{ccc|ccc} \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{z}_{t-2}^{iT} \\ \hline \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} \\ \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} \end{array}\right) \tag{41}$$

We set this as:

$$\mathbf{P}=\begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 & \mathbf{K}_3 & \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 \\ \mathbf{K}_4 & \mathbf{K}_5 & \mathbf{K}_6 & \mathbf{L}_4 & \mathbf{L}_5 & \mathbf{L}_6 \\ \mathbf{K}_7 & \mathbf{K}_8 & \mathbf{K}_9 & \mathbf{L}_7 & \mathbf{L}_8 & \mathbf{L}_9 \end{pmatrix}\begin{pmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \\ \mathbf{M}_4 & \mathbf{M}_5 & \mathbf{M}_6 & \mathbf{N}_4 & \mathbf{N}_5 & \mathbf{N}_6 \\ \mathbf{M}_7 & \mathbf{M}_8 & \mathbf{M}_9 & \mathbf{N}_7 & \mathbf{N}_8 & \mathbf{N}_9 \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \\ \mathbf{Q}_4 & \mathbf{Q}_5 & \mathbf{Q}_6 & \mathbf{R}_4 & \mathbf{R}_5 & \mathbf{R}_6 \\ \mathbf{Q}_7 & \mathbf{Q}_8 & \mathbf{Q}_9 & \mathbf{R}_7 & \mathbf{R}_8 & \mathbf{R}_9 \end{pmatrix}^{-1} \tag{42}$$

Then when all consist of the same level shifts or the upper level shifts (suppose $\mathbf{X} > \mathbf{Y} > \mathbf{Z}$),

$\boldsymbol{K}_4, \boldsymbol{K}_7, \boldsymbol{K}_8, \boldsymbol{L}_4, \boldsymbol{L}_7, \boldsymbol{L}_8, \boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{M}_6, \boldsymbol{N}_4, \boldsymbol{N}_7, \boldsymbol{N}_8, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4$

are all 0.

As

$$\boldsymbol{M}_4 = \boldsymbol{M}_2^T,\ \boldsymbol{M}_7 = \boldsymbol{M}_3^T,\ \boldsymbol{M}_8 = \boldsymbol{M}_6^T,\ \boldsymbol{R}_6 = \boldsymbol{R}_2^T,\ \boldsymbol{R}_7 = \boldsymbol{R}_3^T,\ \boldsymbol{R}_8 = \boldsymbol{R}_6^T,\ \boldsymbol{Q}_2 = \boldsymbol{N}_4^T,\ \boldsymbol{Q}_3 = \boldsymbol{N}_7^T,\ \boldsymbol{Q}_6 = \boldsymbol{N}_8^T,$$

therefore they are all 0.

$\boldsymbol{M}_1, \boldsymbol{M}_5, \boldsymbol{M}_9, \boldsymbol{R}_1, \boldsymbol{R}_5, \boldsymbol{R}_9$ become diagonal Matrices.

Using a symbol "$*$" as a diagonal matrix, **P** becomes as follows by using the relation stated above.

$$\mathbf{P}=\left(\begin{array}{ccc:ccc}\mathbf{K}_1, & \mathbf{K}_2, & \mathbf{K}_3, & \mathbf{L}_1, & \mathbf{L}_2, & \mathbf{L}_3 \\ \mathbf{0}, & \mathbf{K}_5, & \mathbf{K}_6, & \mathbf{0}, & \mathbf{L}_5, & \mathbf{L}_6 \\ \mathbf{0}, & \mathbf{0}, & \mathbf{K}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{L}_9\end{array}\right)\left(\begin{array}{ccc:ccc} *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_1, & \mathbf{N}_2, & \mathbf{N}_3 \\ \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_5, & \mathbf{N}_6 \\ \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_9 \\ \hdashline \mathbf{Q}_1, & \mathbf{0} & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0} \\ \mathbf{Q}_4, & \mathbf{Q}_5, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0} \\ \mathbf{Q}_7, & \mathbf{Q}_8, & \mathbf{Q}_9, & \mathbf{0}, & \mathbf{0}, & * \end{array}\right)^{-1} \tag{43}$$

5 Matrix Structure When Condensing the Variables of the Same Class

Suppose the customer select Bister from Corolla (Bister is an upper class brand automobile than Corolla.) when he/she buy next time. In that case, there are such brand automobiles as bluebird, Gallant sigma and 117 coupes for the corresponding same brand class group with Bister in other companies.

Someone may select another automobile from the same brand class group. There is also the case that the consumer select another company's automobile of the same brand class group when he/she buy next time.

Matrix structure would be, then, as follows.

Suppose w, x, y in the same brand class group in the example of 2.3. If there exist following shifts:

$$y, x, w, v \leftarrow z_b$$
$$x, w \leftarrow y_b$$
$$y, w \leftarrow x_b$$
$$x, y \leftarrow w_b$$
$$v \leftarrow y_b$$
$$v \leftarrow x_b$$
$$v \leftarrow w_b$$
$$y \leftarrow y_b$$
$$y \leftarrow x_b$$
$$y \leftarrow w_b$$

then, transition equation is expressed as follows.

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \begin{pmatrix} v_b \\ w_b \\ x_b \\ y_b \\ z_b \end{pmatrix} \tag{44}$$

Expressing these in Block Matrix form, it becomes as follows.

$$\mathbf{X} = \begin{pmatrix} a_{11} & \mathbf{A}_{12} & a_{15} \\ \mathbf{0} & \mathbf{A}_{22} & \mathrm{A}_{23} \\ \mathbf{0} & \mathbf{0} & a_{55} \end{pmatrix} \mathbf{X}_b \tag{45}$$

Where,

$$\mathbf{X} \in \mathbf{R}^5, \mathbf{A}_{12} \in \mathbf{R}^{1\times 3}, \mathbf{A}_{22} \in \mathbf{R}^{3\times 3}, \mathbf{A}_{23} \in \mathbf{R}^{3\times 1}, \mathbf{X}_b \in \mathbf{R}^5$$

As w, x, y are in the same class brand group, condensing these variables into one, and expressing it as $\overline{w}$, then Eq.(10) becomes as follows.

$$\begin{pmatrix} v \\ \overline{w} \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & \overline{a}_{12} & a_{15} \\ 0 & \overline{a}_{22} & \overline{a}_{23} \\ 0 & 0 & a_{55} \end{pmatrix} \begin{pmatrix} v_b \\ \overline{w}_b \\ z \end{pmatrix} \tag{46}$$

As a_{ij} satisfies the following equation :

$$\sum_{i=1}^{5} a_{ij} = 1 \qquad (\forall\ j) \tag{47}$$

Condensed version of Block Matrix $\mathbf{A}_{22}$ are as follows, where sum of each item of Block matrix is taken and they are divided by the number of variables.

$$\overline{a}_{22} = \frac{1}{3} \sum_{i=2}^{4} \sum_{j=2}^{4} a_{ij} \tag{48}$$

$$\overline{a}_{12} = \frac{1}{3} \sum_{j=2}^{4} a_{1j} \tag{49}$$

$$\overline{a}_{23} = \sum_{i=2}^{4} a_{i5} \tag{50}$$

Generalizing this, it becomes as follows.

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_r \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1r} \\ \mathbf{0} & \mathbf{A}_{21} & \cdots & \mathbf{A}_{2r} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{1,b} \\ \mathbf{X}_{2,b} \\ \vdots \\ \mathbf{X}_{r,b} \end{pmatrix} \tag{51}$$

where

$$\mathbf{X}_1 \in \mathbf{R}^{p_1}, \cdots, \mathbf{X}_r \in \mathbf{R}^{p_r}, \quad \mathbf{A}_{ij} \in \mathbf{R}^{p_i \times p_j} (i = 1, \cdots, r), (j = 1, \cdots, r)$$

When the variables of each Block Matrix are condensed into one, transition matrix is expressed as follows.

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_r \end{pmatrix} = \begin{pmatrix} \bar{a}_{11}, & \bar{a}_{12}, & \cdots & \bar{a}_{1r} \\ 0, & \bar{a}_{22}, & \cdots & \bar{a}_{2r} \\ \vdots & \vdots & & \vdots \\ 0, & 0, & \cdots & \bar{a}_{rr} \end{pmatrix} \begin{pmatrix} \bar{x}_{1,b} \\ \bar{x}_{2,b} \\ \vdots \\ \bar{x}_{r,b} \end{pmatrix} \tag{52}$$

Where

$$\bar{a}_{ij} = \frac{1}{p_j} \sum_{k=1}^{p_i} \sum_{l=1}^{p_j} a_{kl}{}^{ij} \qquad (i = 1, \cdots, r), (j = i, \cdots, r) \tag{53}$$

Here, $a_{kl}^{ij} (k = 1, \cdots, P_i), (l = 1, \cdots, P_i)$ means the item of $\mathbf{A}_{ij}$.

Taking these operations, the variables of the same brand class group are condensed into one and the transition condition among brand class can be grasped easily and clearly. Judgment when and where to put the new brand becomes easy and may be executed properly.

6 Expansion to the Second Order Lag

Here we take up Eq. (37), (38), (39) and expand the method stated in section 5.

When the variables of each matrix are condensed into one, transition matrix is expressed as follows in the same way stated in section 5.

$$\begin{pmatrix} \bar{x}_n \\ \bar{y}_n \\ \bar{z}_n \end{pmatrix} = \begin{pmatrix} \bar{a}_1, & \bar{b}_1, & \bar{c}_1 \\ \bar{d}_1, & \bar{e}_1, & \bar{f}_1 \\ \bar{g}_1, & \bar{h}_1, & \bar{j}_1 \end{pmatrix} \begin{pmatrix} \bar{x}_{n-1} \\ \bar{y}_{n-1} \\ \bar{z}_{n-1} \end{pmatrix} + \begin{pmatrix} \bar{a}_2, & \bar{b}_2, & \bar{c}_2 \\ \bar{d}_2, & \bar{e}_2, & \bar{f}_2 \\ \bar{g}_2, & \bar{h}_2, & \bar{j}_2 \end{pmatrix} \begin{pmatrix} \bar{x}_{n-2} \\ \bar{y}_{n-2} \\ \bar{z}_{n-2} \end{pmatrix} \tag{54}$$

Where

$$\bar{a}_1 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^1,\ \ \bar{b}_1 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}^1$$

$$\vdots$$

$$\bar{a}_2 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2,\ \ \bar{b}_2 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}^2$$

$$\vdots$$

Here,

$a_{ij}^1, b_{ij}^1, \cdots, a_{ij}^2, b_{ij}^2, \cdots$ are items of each Block Matrix $\mathbf{A}_1, \mathbf{B}_1, \cdots, \mathbf{A}_2, \mathbf{B}_2, \cdots,$ respectively and $\bar{a}_1, \bar{b}_1, \cdots \bar{a}_2, \bar{b}_2, \cdots$ are the condensed variables of each Block Matrix $\mathbf{A}_1, \mathbf{B}_1, \cdots, \mathbf{A}_2, \mathbf{B}_2 \cdots$.

7 Numerical Example

We consider the case that brand selection shifts to the same class or upper classes. As above-referenced, transition matrix must be an upper triangular matrix. Suppose following events occur.

	X_{t-2} to	X_{t-1}		X_{t-1} to	X_t	
(1)	L_1	L_1	2 *events*	L_1	L_1	2 *events*
(2)	L_1	L_2	1 *event*	L_2	L_2	1 *event*
(3)	L_2	L_2	3 *events*	L_2	L_3	3 *events*
(4)	L_3	L_3	1 *event*	L_3	L_3	1 *event*
(5)	L_2	L_2	2 *events*	L_2	L_2	2 *events*
(6)	L_1	L_1	1 *event*	L_1	M_1	1 *event*
(7)	L_1	L_1	2 *events*	L_1	M_2	2 *events*
(8)	L_1	L_1	3 *events*	L_1	M_3	3 *events*
(9)	L_2	L_2	1 *event*	L_2	M_1	1 *event*

	X_{t-2} to X_{t-1}			X_{t-1} to X_t		
(10)	L_2	L_2	1 *event*	L_2	M_2	1 *event*
(11)	L_2	L_2	2 *events*	L_2	M_3	2 *events*
(12)	L_3	L_3	1 *event*	L_3	M_1	1 *event*
(13)	L_3	L_3	3 *events*	L_3	M_2	3 *events*
(14)	L_3	L_3	2 *events*	L_3	M_3	2 *events*
(15)	L_1	L_2	1 *event*	L_2	M_1	1 *event*
(16)	L_1	L_2	1 *event*	L_2	M_2	1 *event*
(17)	L_1	L_3	1 *event*	L_3	M_3	1 *event*
(18)	L_2	L_3	2 *events*	L_3	M_1	2 *events*
(19)	L_2	L_3	2 *events*	L_3	M_2	2 *events*
(20)	L_2	L_3	1 *event*	L_3	M_3	1 *event*
(21)	M_1	M_1	3 *events*	M_1	M_1	3 *events*
(22)	M_1	M_2	3 *events*	M_2	M_2	3 *events*
(23)	M_1	M_2	2 *events*	M_2	M_3	2 *events*
(24)	M_2	M_2	1 *event*	M_2	M_2	1 *event*
(25)	M_2	M_3	1 *event*	M_3	M_3	1 *event*
(26)	M_3	M_3	2 *events*	M_3	M_3	2 *events*
(27)	M_1	M_1	2 *events*	M_1	U_1	2 *events*
(28)	M_1	M_1	1 *event*	M_1	U_2	1 *event*
(29)	M_1	M_1	3 *events*	M_1	U_3	3 *events*
(30)	M_1	M_2	2 *events*	M_2	U_1	2 *events*
	X_{t-2} to X_{t-1}			X_{t-1} to X_t		
(31)	M_1	M_2	1 *event*	M_2	U_2	1 *event*
(32)	M_1	M_2	2 *events*	M_2	U_3	2 *events*
(33)	M_1	M_3	1 *event*	M_3	U_1	1 *event*
(34)	M_1	M_3	1 *event*	M_3	U_2	1 *event*
(35)	M_2	M_2	2 *events*	M_2	U_1	2 *events*
(36)	M_2	M_2	2 *events*	M_2	U_2	2 *events*
(37)	M_2	M_2	3 *events*	M_2	U_3	3 *events*
(38)	M_2	M_3	2 *events*	M_3	U_1	2 *events*
(39)	M_2	M_3	1 *event*	M_3	U_2	1 *event*
(40)	M_2	M_3	2 *events*	M_3	U_3	2 *events*

(41)	M_3	M_3	$1\ event$	M_3	U_1	$1\ event$
(42)	M_3	M_3	$2\ events$	M_3	U_2	$2\ events$
(43)	M_3	M_3	$2\ events$	M_3	U_3	$2\ events$
(44)	U_1	U_1	$1\ event$	U_1	U_1	$1\ event$
(45)	U_1	U_1	$1\ event$	U_1	U_2	$1\ event$
(46)	U_1	U_1	$2\ events$	U_1	U_3	$2\ events$
(47)	U_2	U_2	$1\ event$	U_2	U_2	$1\ event$
(48)	U_2	U_2	$2\ events$	U_2	U_3	$2\ events$
(49)	U_2	U_2	$2\ events$	U_2	U_1	$2\ events$
(50)	U_3	U_3	$3\ events$	U_3	U_3	$3\ events$
(51)	U_3	U_3	$2\ events$	U_3	U_2	$2\ events$
(52)	U_3	U_3	$1\ event$	U_3	U_1	$1\ event$
(53)	M_3	M_3	$1\ event$	M_3	M_2	$1\ event$
(54)	M_3	M_3	$2\ events$	M_3	M_1	$2\ events$
(55)	M_3	M_2	$1\ event$	M_2	M_2	$1\ event$
(56)	M_3	M_2	$1\ event$	M_2	M_1	$1\ event$
(57)	M_2	M_2	$2\ events$	M_2	M_1	$2\ events$
(58)	L_3	L_3	$3\ events$	L_3	L_2	$3\ events$
(59)	L_3	L_2	$2\ events$	L_2	L_1	$2\ events$
(60)	L_3	L_1	$1\ event$	L_1	L_1	$1\ event$
(61)				L_1	L_1	$2\ events$
(62)				L_1	L_2	$2\ events$
(63)				L_1	L_3	$1\ event$
(64)				L_1	M_1	$1\ event$
(65)				L_1	M_2	$3\ events$
(66)				L_2	L_2	$2\ events$
(67)				L_2	L_3	$1\ event$
(68)				L_2	L_1	$2\ events$
(69)				L_2	M_1	$1\ event$
(70)				L_2	U_1	$1\ event$
(71)				L_1	U_2	$3\ events$
(72)				M_1	M_1	$2\ events$

	X_{t-1}	to X_t	
(73)	L_3	L_3	$2\ events$
(74)	M_2	M_2	$1\ event$
(75)	M_3	M_3	$1\ event$
(76)	M_1	M_2	$3\ events$
(77)	M_1	U_1	$3\ events$
(78)	M_2	U_3	$2\ events$
(79)	M_3	U_2	$1\ event$
(80)	U_1	U_1	$1\ event$
(81)	U_2	U_2	$2\ events$
(82)	U_3	U_3	$2\ events$
(83)	U_1	U_2	$1\ event$
(84)	U_1	U_3	$2\ events$
(85)	U_3	U_2	$2\ events$
(86)	U_3	U_1	$1\ event$
(87)	U_2	U_1	$1\ event$
(88)	U_2	U_3	$2\ events$
(89)	L_3	L_1	$3\ events$
(90)	M_2	L_1	$1\ event$

Vector $\begin{pmatrix}\boldsymbol{X}_{t-2}\\ \boldsymbol{Y}_{t-2}\\ \boldsymbol{Z}_{t-2}\end{pmatrix}, \begin{pmatrix}\boldsymbol{X}_{t-1}\\ \boldsymbol{Y}_{t-1}\\ \boldsymbol{Z}_{t-1}\end{pmatrix}, \begin{pmatrix}\boldsymbol{X}_{t}\\ \boldsymbol{Y}_{t}\\ \boldsymbol{Z}_{t}\end{pmatrix}$ in these cases are expressed as follows.

We show (1) and (2) cases for example.

$$
\begin{array}{cccccc}
X_{t-2} & & X_{t-1} & & X_t & \\
\begin{pmatrix} \\ \mathbf{X}_{t-2} \\ \\ \\ \mathbf{Y}_{t-2} \\ \\ \\ \mathbf{Z}_{t-2} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & &
\begin{pmatrix} \\ \mathbf{X}_{t-1} \\ \\ \\ \mathbf{Y}_{t-1} \\ \\ \\ \mathbf{Z}_{t-1} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & &
\begin{pmatrix} \\ \mathbf{X}_{t} \\ \\ \\ \mathbf{Y}_{t} \\ \\ \\ \mathbf{Z}_{t} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} &
\end{array} \tag{1}
$$

$$
\begin{pmatrix} \\ \mathbf{X}_{t-2} \\ \\ \\ \mathbf{Y}_{t-2} \\ \\ \\ \mathbf{Z}_{t-2} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad
\begin{pmatrix} \\ \mathbf{X}_{t-1} \\ \\ \\ \mathbf{Y}_{t-1} \\ \\ \\ \mathbf{Z}_{t-1} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad
\begin{pmatrix} \\ \mathbf{X}_{t} \\ \\ \\ \mathbf{Y}_{t} \\ \\ \\ \mathbf{Z}_{t} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2}
$$

Substituting these to equation (40), we obtain the following estimated Matrix.

$$
\mathbf{P} = \begin{pmatrix}
2 & 3 & 2 & 5 & 8 & 6 & 0 & 1 & 0 & 1 & 2 & 1 & 5 & 4 & 1 & 0 & 0 & 0 \\
2 & 3 & 4 & 1 & 4 & 6 & 0 & 3 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 0 & 0 & 0 \\
4 & 4 & 5 & 3 & 7 & 4 & 0 & 0 & 0 & 2 & 2 & 3 & 5 & 5 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 & 2 & 3 & 3 & 0 & 0 & 0 & 3 & 0 & 3 & 2 & 3 & 1 \\
0 & 0 & 0 & 3 & 5 & 0 & 5 & 2 & 5 & 0 & 0 & 0 & 3 & 1 & 2 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 & 2 & 4 & 3 & 2 & 4 & 0 & 0 & 0 & 2 & 1 & 2 & 4 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1
\end{pmatrix}
$$

$$
\times\begin{pmatrix}
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 10 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 9 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 10\\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 9 & 10 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 10 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13
\end{pmatrix}^{-1}
$$

$$
=\begin{pmatrix}
0.250 & 0.200 & 0.200 & 0.522 & 0.751 & 0.912 & 0.016 & 0.087 & 0.040 & 0 & 0.200 & -0.033 & -0.430 & -0.561 & -0.802 & -0.036 & -0.070 & -0.045\\
0.250 & 0.400 & 0.400 & 0.104 & 0.282 & 0.521 & 0.047 & 0.262 & 0.120 & 0 & -0.200 & -0.067 & -0.085 & -0.184 & -0.314 & -0.107 & -0.211 & -0.136\\
0.500 & 0.400 & 0.400 & 0.130 & 0.174 & 0.130 & 0 & 0 & 0 & 0 & 0 & 0.100 & 0.087 & 0.155 & 0.029 & 0 & 0 & 0\\
0 & 0 & 0 & 0.151 & -0.304 & -0.497 & 0.062 & 0.086 & 0.132 & 0 & 0 & 0 & 0.270 & 0.376 & 0.715 & 0.093 & 0.112 & -0.042\\
0 & 0 & 0 & 0.184 & 0.186 & -0.107 & 0.191 & -0.001 & 0.143 & 0 & 0 & 0 & -0.014 & -0.013 & 0.225 & 0.111 & 0.164 & 0.106\\
0 & 0 & 0 & -0.007 & 0.125 & 0.277 & 0.009 & -0.063 & 0.066 & 0 & 0 & 0 & 0.012 & -0.119 & -0.085 & 0.337 & 0.231 & 0.112\\
0 & 0 & 0 & 0 & 0 & 0 & 0.289 & 0.340 & 0.231 & 0 & 0 & 0 & 0 & 0 & 0 & -0.130 & -0.301 & -0.021\\
0 & 0 & 0 & 0 & 0 & 0 & 0.120 & 0.275 & 0.126 & 0 & 0 & 0 & 0 & 0 & 0 & -0.076 & -0.079 & 0.082\\
0 & 0 & 0 & 0 & 0 & 0 & 0.111 & 0.228 & 0.215 & 0 & 0 & 0 & 0 & 0 & 0 & -0.149 & -0.009 & -0.132
\end{pmatrix}
$$

The Block Matrices make upper triangular matrix as is supposed. We can confirm that $\boldsymbol{K}_4, \boldsymbol{K}_7, \boldsymbol{K}_8, \boldsymbol{L}_4, \boldsymbol{L}_7, \boldsymbol{L}_8, \boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{M}_4, \boldsymbol{M}_6, \boldsymbol{M}_7, \boldsymbol{M}_8, \boldsymbol{N}_4,$ $\boldsymbol{N}_7, \boldsymbol{N}_8, \boldsymbol{Q}_2, \boldsymbol{Q}_3,$ $\boldsymbol{Q}_6, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4, \boldsymbol{R}_6, \boldsymbol{R}_7, \boldsymbol{R}_8$ are all 0. $\boldsymbol{M}_1, \boldsymbol{M}_5, \boldsymbol{M}_9, \boldsymbol{R}_1, \boldsymbol{R}_5, \boldsymbol{R}_9$ become diagonal Matrices as we have assumed.

Condensing the variables into one as stated in section 6, we obtain the following equation.

$$\begin{pmatrix} \bar{x}_n \\ \bar{y}_n \\ \bar{z}_n \end{pmatrix} = \begin{pmatrix} 1 & 1.175 & 0.190 \\ 0 & 0.003 & 0.208 \\ 0 & 0 & 0.645 \end{pmatrix} \begin{pmatrix} \bar{x}_{n-1} \\ \bar{y}_{n-1} \\ \bar{z}_{n-1} \end{pmatrix} + \begin{pmatrix} 0 & -0.702 & -0.202 \\ 0 & 0.456 & 0.408 \\ 0 & 0 & -0.272 \end{pmatrix} \begin{pmatrix} \bar{x}_{n-1} \\ \bar{y}_{n-1} \\ \bar{z}_{n-1} \end{pmatrix}$$

This is by far a simple one compared with the result obtained so far.

The variables of the same brand class are condensed into one.

The matrix of $\mathbf{P}_1$ and $\mathbf{P}_2$ are 9×9 respectively, but the condensed versions become 3×3 Matrix. Therefore the brand transition condition among brand class can be grasped easily and clearly.

8 Conclusion

It is often observed that consumers select the upper class brand when they buy the next time. Suppose that the former buying data and the current buying data are gathered. Also suppose that the upper brand is located upper in the variable array. Then the transition matrix become an upper triangle matrix under the supposition that former buying variables are set input and current buying variables are set output. If the top brands are selected from the lower brand in jumping way, corresponding part in an upper triangle matrix would be 0. The goods of the same brand group would compose the Block Matrix in the transition matrix. Condensing the variables of the same brand group into one, analysis becomes easier to handle and the transition of Brand Selection can be easily grasped.

In this paper, equation using transition matrix stated by the Block Matrix is expanded to the second order lag and the method of condensing the variable stated above is also applied to this new model. Unless planner for products does not notice its brand position whether it is upper or lower than other products, matrix structure make it possible to identify those by calculating consumers' activities for brand selection. Thus, this proposed approach enables to make effective marketing plan and/or establishing new brand. Various fields should be examined hereafter.

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