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Degree of Approximation of Fourier Series by Hausdörff and Nörlund Product Means

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Abstract

In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha, r)$ by product summability $(E,q)(N,p_n)$ of Fourier series associated with f has been established.

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1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$.

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Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty, as\ n \longrightarrow \infty, (P_{-i} = p_{-i} = 0, i \geq 0).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v, \tag{1}$$

defines the sequence $\{t_n\}$ of the (N, p_n)-mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \longrightarrow s,\ as\ n \longrightarrow \infty, \tag{2}$$

then the series $\sum a_n$ is said to be (N, p_n) summable to s.

The conditions for regularity of Nörlund summability (N, p_n) are easily seen to be [1]

$$(i)\ \ \frac{p_n}{P_n} \rightarrow 0\ \ as\ \ n \rightarrow \infty, \tag{3}$$

$$(ii)\ \sum_{k=0}^{n} p_k = 0(P_n) as\ \ n \rightarrow \infty. \tag{4}$$

The sequence-to-sequence transformation [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v, \tag{5}$$

defines the sequence $\{T_n\}$ of the (E, q) mean of the sequence $\{s_n\}$. If

$$T_n \rightarrow s,\ \ as\ \ n \rightarrow \infty, \tag{6}$$

then the series Σa_n is said to be (E, q) summable to s. Clearly (E, q) method is regular [1]. Further, the (E, q) transformation of the (N, p_n) transform of $\{s_n\}$ is defined by

$$\begin{aligned} \tau_n &= \frac{1}{(1+q)^n} \Sigma_{k=0}^{n} \binom{n}{k} q^{n-k} T_k \\ &= \frac{1}{(1+q)^n} \Sigma_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \Sigma_{v=0}^{k} p_{k-v} s_v \right\} \end{aligned} \tag{7}$$

If

$$\tau_n \rightarrow s,\ as\ n \rightarrow \infty, \tag{8}$$

then $\sum a_n$ is said to be $(E,q)(N,p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π, L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\ nx + b_n sin\ nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{9}$$

Let $s_n(f:x)$ be the n-th partial sum of (9). The L_∞-norm of a function $f:R\to R$ is defined by

$$\parallel f \parallel_\infty = sup\{|f(x)| : x \in R\} \tag{10}$$

and the L_υ-norm is defined by

$$\parallel f \parallel_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \upsilon \geq 1. \tag{11}$$

The degree of approximation of a function $f:R\to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\cdot\|$ is defined by

$$\parallel P_n - f \parallel_\infty = sup\{|p_n(x) - f(x)| : x\epsilon R\} \tag{12}$$

and the degree of approximation $E_n(f)$ a function $f\epsilon L_\upsilon$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon. \tag{13}$$

This method of approximation is called Trigonometric Fourier approximation.

A function $f\epsilon Lip\ \ \alpha$ if

$$|f(x+1) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1. \tag{14}$$

and $f\epsilon Lip(\alpha,r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = 0(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1, t > 0. \tag{15}$$

We use the following notations throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x), \tag{16}$$

and

$$K_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

Further, the method $(E,q)(N,P_n)$ is assumed to be regular and this case is supposed through out the paper.

2 Known Theorems

Dealing with the degree of approximation by the product $(E, q)(C, 1)$-mean of Fourier series, Nigam et. al [3] proved the following theorem.

Theorem 2.1. *If a function f is 2π-periodic and of class $Lip\alpha$, then its degree of approximation by $(E, q)(C, 1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by*

$$\|E_n^q C_n^1 - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1,$$

where $E_n^q C_n^1$ represents the (E, q) transform of $(C, 1)$ transform of $s_n(f : x)$.

Subsequently Misra et. al. [2] have proved the following theorem on degree of approximation by the product mean $(E, q)(N, p_n)$ of Fourier series:

Theorem 2.2. *If f is a 2π-Periodic function of class $Lip\alpha$, then degree of approximation by the product $(E, q)(N, p_n)$ summability means on its Fourier series (defined above) is given by*

$$\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1,$$

where τ_n as defined in (7).

3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean $(E, q)(N, p_n)$ of the Fourier series of a function of class $Lip(\alpha, r)$. We prove:

Theorem 3.1. *If f is a 2π- periodic function of the class $Lip(\alpha, r)$, then degree of approximation by the product $(E, q)(N, p_n)$ summability means on its Fourier series (9) is given by*

$$\| \tau_n - f \|_\infty = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right), 0 < \alpha < 1, r \le 1,$$

where τ_n is as defined in (7).

4 Required Lemmas

We require the following Lemmas for the proof the theorem.

Lemma 4.1.

$$|K_n(t)| = O(n), 0 \leq t \leq \frac{1}{n+1}.$$

Proof For $0 \leq t \leq \frac{1}{n+1}$, we have $sin\ nt \leq n\ sin\ t$ then

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(2k+1)\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\right\}\right| \\
&\leq \frac{(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O(n).
\end{aligned}$$

This proves the lemma. □

Lemma 4.2.

$$|K_n(t)| = O\left(\frac{1}{t}\right), for \frac{1}{n+1} \leq t \leq \pi.$$

Proof For $\frac{1}{n+1} \leq t \leq \pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$, $\sin nt \leq 1$. Then

$$
\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi p_{k-\upsilon}}{t}\right\}\right| \\
&= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\right\}\right| \\
&= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}
$$

This proves the lemma. □

5 Proof of theorem 3.1

Using Riemann-Lebesgue theorem, for the n-th partial sum $s_n(f:x)$ of the Fourier series (9) of $f(x)$ and following Titchmarch [4], we have

$$
s_n(f:x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt.
$$

Using (1), the (N,p_n) transform of $s_n(f:x)$ is given by

$$
t_n - f(x) = \frac{1}{2\pi P_n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}p_{n-k}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt.
$$

Denoting the $(E,q)(N,p_n)$ transform of $s_n(f:x)$ by τ_n, we have

$$\begin{aligned}
\|\tau_n - f\| &= \frac{1}{2\pi(1+q)^n}\int_0^\pi \phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt \\
&= \int_0^\pi \phi(t)K_n(t)dt = \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\phi(t)K_n(t)dt \\
&= I_1 + I_2. \qquad (17)
\end{aligned}$$

Now

$$\begin{aligned}
|I_1| &= \frac{1}{2\pi(1+q)^n}\left|\int_0^{\frac{1}{n+1}} \phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right| \\
&= \left|\int_0^{\frac{1}{n+1}}\phi(t)K(t)dt\right| \\
&= \left(\int_0^{\frac{1}{n+1}}(\phi(t))^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{n+1}}(K_n(t))^s dt\right)^{\frac{1}{s}}, \text{using Holder's inequality} \\
&= O\left(\frac{1}{(n+1)^\alpha}\right)\left(\frac{n^s}{n+1}\right)^{\frac{1}{s}} \\
&= O\left(\frac{1}{(n+1)^{\frac{1}{s}-1+\alpha}}\right) = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)
\end{aligned}$$

Next

$$\begin{aligned}
|I_2| &\le \left(\int_{\frac{1}{n+1}}^{\pi}\left(\phi(t)\right)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{n+1}}^{\pi}(K_n(t))^s dt\right)^{\frac{1}{s}}, \quad \text{using Holder's inequality} \\
&= O\left(\frac{1}{(n+1)^\alpha}\right)\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)^s dt\right)^{\frac{1}{s}}, \quad \text{using Lemma 4.2} \\
&= O\left(\frac{1}{(n+1)^\alpha}\right)\left(\left[t^{-s+1}\right]_{\frac{1}{n+1}}^{\pi}\right)^{\frac{1}{s}}, \\
&= O\left(\frac{1}{(n+1)^\alpha}\right)\left(\frac{1}{n+1}\right)^{\frac{1-s}{s}} = O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right) = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right).
\end{aligned}$$

Then from (17) and the above inequalities, we have

$$\begin{aligned}
|\tau_n - f(x)| &= O\left(\frac{1}{n+1^{\alpha-\frac{1}{r}}}\right), \quad \text{for} \quad 0<\alpha<1, \; r\ge 1, \\
\|\, \tau_n - f(x)\,\|_\infty &= \sup_{-\pi<x<\pi}|\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), 0<\alpha<1, r\ge 1.
\end{aligned}$$

This completes the proof of the theorem. □

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