

Stress Distributions Due to a Concentrated Force on Viscoelastic Half-Space

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Abstract

A model of a viscoelastic infinite half-space with a concentrated tangential force applied on the boundary, namely, the viscoelastic Cerruti's problem, is presented in this paper, with the derivation of the stress distributions by applying the elastic-viscoelastic correspondence principle to the displacements from the classic Cerruti's problem. In the background viscoelastic materials, based on elastic-viscoelastic correspondence principle, the displacements of the classic Cerruti's problem should have the similar expressions to elastic solutions but vary with time. Two auxiliary functions were used to replace the time components in displacements, which reduces the complexity. By satisfying the boundary conditions and balance conditions, the two auxiliary functions can be determined

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after solving two Volterra integral equations. With displacements known, the strains and stresses can be obtained. The results were also show that the two solutions match.

Keywords: stress distribution, linear viscoelastic, concentrated force, half-space, Cerruti, correspondence principle

1 Introduction

Cerruti's problem is of great importance in many researches since it gives the stress distribution inside a body. For some cutting problems, the tangential force plays an important role in the initiation of the cutting crack rather than the normal force. These problems can be modeled as the Cerruti's problem where a tangential force applied on the boundary of the half body. The solution to Cerruti's problem offers a prediction of the internal stress distributions and therefore researches can know where fracture initiates according to certain criterion. In many applications, the cutting operations are involved in the processing of many viscoelastic materials. Cutting yield and quality can be ensured via the control of the cutting force in the automation process by using robotic devices. The internal stress distribution in a viscoelastic body is not only directly related to the external cutting force; it is also the fundamental cause of cutting fractures. Thus, modeling the relationship between the external cutting force and internal stress distribution can help predict cutting fractures and identify different materials along the cutting path. Finally, a control algorithm can be applied to avoid cutting into the hard materials to ensure the cutting yield and quality.

In actual cases, the contacting force on the cutting edge cannot be viewed as concentrated force, but a distributive force. Therefore the cutting problem can be modeled as a belt-shaped force acting on the surface of a half-space. The study of

the stress distribution due to a concentrated point force will be the first step in modeling the cutting stress distribution problem. A cutting force can be considered as the resultant of a normal force and a tangential force acting on the contact surface between the tool and the material. The stress distribution in an elastic half-space due to a normal point-force has been modeled by Boussinesq; the stress distribution due to a tangential point-force by Cerruti. Correspondingly, they are called Boussinesq's problem and Cerruti's problem. For viscoelastic materials, Talybly (2010) developed a method to solve viscoelastic problems by substituting the multiplication of external force and viscoelastic material functions with time-dependent functions. He formulated the solutions to Boussinesq's viscoelastic problem in cylindrical coordinates, and two time-dependent functions have been used to replace the relaxation functions and Poisson's ratio. During the derivation, the equations to solve the determining functions were obtained from equilibrium equations and boundary conditions and were uniquely solved via the Volterra integral equation of second kind (Zhang 1994).

For viscoelastic materials, the formulation of the stress distribution due to the tangential point force on a half-space is called Cerruti's viscoelastic problem. This paper will concentrate on the formulation of stress distribution in the material in Cerruti's viscoelastic problem. The method used by Talybly (2010) will be adopted in our formulation. The symmetrical property in Boussinesq's problem allowed for the usage of cylindrical coordinates, which greatly reduced the formulation complicity. In Cerruti's problem, due to the asymmetry resulting from the tangential force, only Cartesian coordinates could be used. This makes our formulation much more complicated.

Furthermore, using the solution to Boussinesq's viscoelastic problem by Talybly (2010) and that to Cerruti's viscoelastic problem formulated in this paper, a solution to the problem in viscoelastic materials with both normal and tangential point-forces can be obtained. This generalized solution can be used in many further problems; for example, the food slicing cut problems formulated by Zhou

and McMurray (2011). The remainder of this paper is as follows: Cerruti's viscoelastic problem is explained in Section 2 and the solution to this problem is derived in Section 3. The solution to Boussinesq-Cerruti's viscoelastic problem is shown in Section 4. The verification of elastic-viscoelastic corresponding principle is shown in Section 5. The conclusions are drawn in Section 6.

2 Statement of Viscoelastic Cerruti's Problem

2.1 Elastic Cerruti's Problem

In Cerruti's elastic problem, a reference frame O-xyz is defined on the half-space as shown in Figure 1, where the boundary of the half-space is at $z = 0$, a tangential concentrated force P_x , which could be a function of time, is applied at the origin along the x-axis, and the positive z-axis points towards the interior of the half-space. The solution to this problem for elastic materials was given by Cerruti in 1882 by the use of singularities from potential theory and the results were also presented by Love (1927). The displacement distributions at point (x, y, z) inside the half-space body is shown in (1) and the stress distribution can then be obtained by using the kinematic equations and constitutive equations for elastic materials:

$$\begin{bmatrix} u^e \\ v^e \\ w^e \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{1}{R} + \frac{1}{R+z} + \frac{x^2}{R^3} - \frac{x^2}{R(R+z)^2} & -\frac{2}{R+z} + \frac{2x^2}{R(R+z)^2} \\ \frac{xy}{R^3} - \frac{xy}{R(R+z)^2} & \frac{2xy}{R(R+z)^2} \\ \frac{xz}{R^3} + \frac{x}{R(R+z)} & \frac{-2x}{R(R+z)} \end{bmatrix} \begin{bmatrix} \frac{P_x}{G} \\ \frac{P_x y}{G} \end{bmatrix} \quad (1)$$

where u^e , v^e and w^e denote the displacements in the positive x-, y-, and z-axis directions in the elastic case, $R = \sqrt{x^2 + y^2 + z^2}$, P_x is the external tangential force,

G is the elastic shear modulus and, ν is the Poisson's ratio.

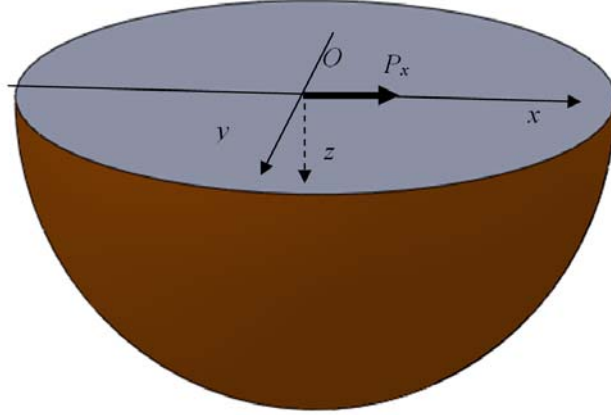


Figure 1: Model of Cerruti's problem

The stresses at point (x, y, z) satisfy the boundary conditions shown in (2) for Cerruti's problem:

$$\left\{ \begin{array}{ll} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_{zx} dx dy + P_x = 0; & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y\sigma_z - z\sigma_{zy}) dx dy = 0; \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_{zy} dx dy = 0; & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x\sigma_z - z\sigma_{zx}) dx dy = 0; \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_z dx dy = 0; & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y\sigma_{zx} - x\sigma_{zy}) dx dy = 0, \end{array} \right. \quad (2)$$

where the three equations in left column represent $\sum F_x = 0$, $\sum F_y = 0$, and $\sum F_z = 0$, respectively, and the three equations in right column represent $\sum M_x = 0$, $\sum M_y = 0$, and $\sum M_z = 0$, respectively.

2.2 Cerruti's Viscoelastic Problem

In our derivation, the viscoelastic effect is taken into consideration based on the original Cerruti's elastic problem. The difference between elastic and viscoelastic problems lies in the constitutive equations. In an elastic material, the relationship between stress and strain can be described by Hooke's Law. However, in a viscoelastic problem, the material will show elastic behaviors, like solids, and also show viscous behaviors, like fluids. This changes the relationship between stress and strain. During the calculation of the stress, instead of multiplying strain with material properties, such as Young's Modulus and Poisson's ratio, the convolutions of strain with relaxation functions are used. These relationships are shown in (2):

$$\left\{ \begin{array}{l} \sigma_x = \int_0^t G_2(t-\tau) \frac{\partial \varepsilon}{\partial \tau} d\tau + \int_0^t G_1(t-\tau) \frac{\partial(\varepsilon_x - \varepsilon)}{\partial \tau} d\tau, \quad \sigma_{yz} = \sigma_{zy} = \int_0^t G_1(t-\tau) \frac{\partial \varepsilon_{yz}}{\partial \tau} d\tau; \\ \sigma_y = \int_0^t G_2(t-\tau) \frac{\partial \varepsilon}{\partial \tau} d\tau + \int_0^t G_1(t-\tau) \frac{\partial(\varepsilon_y - \varepsilon)}{\partial \tau} d\tau, \quad \sigma_{xz} = \sigma_{zx} = \int_0^t G_1(t-\tau) \frac{\partial \varepsilon_{xz}}{\partial \tau} d\tau; \\ \sigma_z = \int_0^t G_2(t-\tau) \frac{\partial \varepsilon}{\partial \tau} d\tau + \int_0^t G_1(t-\tau) \frac{\partial(\varepsilon_z - \varepsilon)}{\partial \tau} d\tau, \quad \sigma_{xy} = \sigma_{yx} = \int_0^t G_1(t-\tau) \frac{\partial \varepsilon_{xy}}{\partial \tau} d\tau, \end{array} \right. \quad (3)$$

where G_1 and G_2 are the deviatoric and volumetric relaxation functions, respectively, and $\varepsilon = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$ is the mean strain.

Therefore, we state Cerruti's viscoelastic problem as follows: finding out the solutions to the stress distribution of a half-space body under a concentrated tangential force applied to the surface, with boundary conditions (2) and stress-strain relationships (3) satisfied.

3 Solutions to Cerruti's Viscoelastic Problem

3.1 Displacements

For the linear viscoelastic case, by applying the elastic-viscoelastic correspondence principle and replacing $\frac{P_x}{G}$ and $\frac{P_x \nu}{G}$ with $\varphi(t)$ and $\psi(t)$, (4) can be obtained from (1):

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} u_\varphi & 2u_\psi \\ v_\varphi & 2v_\psi \\ w_\varphi & 2w_\psi \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}, \quad (4)$$

where u , v and w denote the displacements in the positive direction of the x -, y -, and z -axis in the viscoelastic case and:

$$\begin{cases} u_\varphi = \frac{1}{R} + \frac{1}{R+z} + \frac{x^2}{R^3} - \frac{x^2}{R(R+z)^2}; & u_\psi = -\frac{1}{R+z} + \frac{x^2}{R(R+z)^2}; \\ v_\varphi = \frac{xy}{R^3} - \frac{xy}{R(R+z)^2}; & v_\psi = \frac{xy}{R(R+z)^2}; \\ w_\varphi = \frac{xz}{R^3} + \frac{x}{R(R+z)}; & w_\psi = -\frac{x}{R(R+z)}. \end{cases}$$

In the following, all the derivations will be conducted with the terms of φ and ψ carried out individually.

3.2 Normal Strains

The normal strains are obtained via the derivative of displacements with coordinates. The obtained normal strains are shown in (5):

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{\partial u_\varphi}{\partial x} & 2 \frac{\partial u_\psi}{\partial x} \\ \frac{\partial v_\varphi}{\partial x} & 2 \frac{\partial v_\psi}{\partial x} \\ \frac{\partial w_\varphi}{\partial x} & 2 \frac{\partial w_\psi}{\partial x} \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} \quad (5)$$

where

$$\begin{cases} \frac{\partial u_\varphi}{\partial x} = -\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} - \frac{3x^3}{R^5}; \\ \frac{\partial u_\psi}{\partial x} = \frac{3x}{R(R+z)^2} - \frac{x^3}{R^3(R+z)^2} - \frac{2x^3}{R^2(R+z)^3}, \\ \frac{\partial v_\varphi}{\partial y} = -\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} + \frac{x}{R^3} - \frac{3xy^2}{R^5}; \\ \frac{\partial v_\psi}{\partial y} = \frac{x}{R(R+z)^2} - \frac{xy^2}{R^3(R+z)^2} - \frac{2xy^2}{R^2(R+z)^3}, \\ \frac{\partial w_\varphi}{\partial z} = -\frac{xz}{R^3(R+z)} - \frac{x}{R^2(R+z)} + \frac{x}{R^3} - \frac{3xz^2}{R^5} = -\frac{3xz^2}{R^5}; \\ \frac{\partial w_\psi}{\partial z} = \frac{xz}{R^3(R+z)} + \frac{x}{R^2(R+z)} = \frac{x(R+z)}{R^3(R+z)} = \frac{x}{R^3}. \end{cases}$$

Thus the mean strain is:

$$\varepsilon = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z) = \frac{1}{4\pi} \left(-\frac{2x}{3R^3} \right) \varphi(t) + \frac{1}{2\pi} \left(\frac{2x}{3R^3} \right) \psi(t)$$

The mean stress is:

$$\sigma = G_2 * d\varepsilon = \frac{1}{4\pi} \left(-\frac{2x}{3R^3} \right) G_2 * d\varphi(t) + \frac{1}{2\pi} \left(\frac{2x}{3R^3} \right) G_2 * d\psi(t)$$

3.3 Shear Strains

The shear strains ε_{yz} , ε_{xz} , and ε_{xy} can be obtained as:

$$\begin{bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \right) \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{1}{2} \left(\frac{\partial v_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial y} \right) & 2 \frac{1}{2} \left(\frac{\partial v_\psi}{\partial z} + \frac{\partial w_\psi}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial y} \right) & 2 \frac{1}{2} \left(\frac{\partial u_\psi}{\partial z} + \frac{\partial w_\psi}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) & 2 \frac{1}{2} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} \quad (6)$$

where:

$$\begin{cases} \frac{1}{2} \left(\frac{\partial v_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial y} \right) = -\frac{3xyz}{R^5}; \\ \frac{1}{2} \left(\frac{\partial v_\psi}{\partial z} + \frac{\partial w_\psi}{\partial y} \right) = 0. \end{cases}$$

$$\begin{cases} \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial x} \right) = -\frac{3x^2z}{R^5}; \\ \frac{1}{2} \left(\frac{\partial u_\psi}{\partial z} + \frac{\partial w_\psi}{\partial x} \right) = 0. \end{cases}$$

$$\begin{cases} \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) = -\frac{y}{R(R+z)^2} + \frac{x^2y}{R^3(R+z)^2} + \frac{2x^2y}{R^2(R+z)^3} - \frac{3x^2y}{R^5}; \\ \frac{1}{2} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) = \frac{y}{R(R+z)^2} - \frac{x^2y}{R^3(R+z)^2} - \frac{2x^2y}{R^2(R+z)^3}. \end{cases}$$

Noted is that the coefficients of $\psi(t)$ in ε_{yz} and ε_{xz} are zero, then (6) can be rewritten as:

$$\begin{bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} -\frac{3x^2z}{R^5} & 0 \\ -\frac{3xyz}{R^5} & 0 \\ \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} \quad (7)$$

3.4 Stresses

Based on the strains shown in (5) and (7), the stresses can be obtained from the viscoelastic constitutive equations shown in (3) (Zhang 1994, p. 63) as:

$$\begin{aligned}
 \sigma_x &= \frac{1}{4\pi} \left(-\frac{2x}{3R^3} \right) G_2 * d\varphi(t) + \frac{1}{2\pi} \left(\frac{2x}{3R^3} \right) G_2 * d\psi(t) \\
 &\quad + \frac{1}{4\pi} \left(-\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} - \frac{3x^3}{R^5} + \frac{2x}{3R^3} \right) G_1 * d\varphi(t) \\
 &\quad + \frac{1}{2\pi} \left(\frac{3x}{R(R+z)^2} - \frac{x^3}{R^3(R+z)^2} - \frac{2x^3}{R^2(R+z)^3} - \frac{2x}{3R^3} \right) G_1 * d\psi(t); \\
 \sigma_y &= \frac{1}{4\pi} \left(-\frac{2x}{3R^3} \right) G_2 * d\varphi(t) + \frac{1}{2\pi} \left(\frac{2x}{3R^3} \right) G_2 * d\psi(t) \\
 &\quad + \frac{1}{4\pi} \left(-\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} + \frac{x}{R^3} - \frac{3xy^2}{R^5} + \frac{2x}{3R^3} \right) G_1 * d\varphi(t) \\
 &\quad + \frac{1}{2\pi} \left(\frac{x}{R(R+z)^2} - \frac{xy^2}{R^3(R+z)^2} - \frac{2xy^2}{R^2(R+z)^3} - \frac{2x}{3R^3} \right) G_1 * d\psi(t); \\
 \sigma_z &= \frac{1}{4\pi} \left(-\frac{2x}{3R^3} \right) G_2 * d\varphi(t) + \frac{1}{2\pi} \left(\frac{2x}{3R^3} \right) G_2 * d\psi(t) \\
 &\quad + \frac{1}{4\pi} \left(-\frac{3xz^2}{R^5} + \frac{2x}{3R^3} \right) G_1 * d\varphi(t) \\
 &\quad + \frac{1}{2\pi} \left(\frac{x}{R^3} - \frac{2x}{3R^3} \right) G_1 * d\psi(t); \\
 \sigma_{yz} &= G_1 * d\varepsilon_{yz} = \frac{1}{4\pi} \left(-\frac{3xyz}{R^5} \right) G_1 * d\varphi(t); \\
 \sigma_{xz} &= G_1 * d\varepsilon_{xz} = \frac{1}{4\pi} \left(-\frac{3x^2z}{R^5} \right) G_1 * d\varphi(t); \\
 \sigma_{xy} &= G_1 * d\varepsilon_{xy} = \frac{1}{4\pi} \left(-\frac{y}{R(R+z)^2} + \frac{x^2y}{R^3(R+z)^2} + \frac{2x^2y}{R^2(R+z)^3} - \frac{3x^2y}{R^5} \right) G_1 * d\varphi(t) \\
 &\quad + \frac{1}{2\pi} \left(\frac{y}{R(R+z)^2} - \frac{x^2y}{R^3(R+z)^2} - \frac{2x^2y}{R^2(R+z)^3} \right) G_1 * d\psi(t).
 \end{aligned} \tag{8}$$

3.5 Boundary Conditions

Now let's consider the boundary conditions (2) for Cerruti's viscoelastic problem. First, we evaluate all left terms in (2). Since the convolution will not take part in the integral of coordinates, we get:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_{zx} dx dy + P_x(t) = \frac{1}{4\pi} G_1 * d\varphi(t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(-\frac{3x^2 z}{R^5} \right) dx dy + P_x(t) = -\frac{1}{2} G_1 * d\varphi(t) + P_x(t) \quad (9)$$

Equation (9) is obtained via the direct integration using the left-hand side of the first equation in (3). It represents the resultant force in the x direction on the O - x - y plane. It is worthy to mention that the integrals of the left-hand side of the other five equations in (3) give the value of zero since they are all odd functions of x and the integrations are over $x \in (-\infty, \infty)$. By assigning (9) to be zero based on (3), we get:

$$G_1 * d\varphi(t) = 2P_x(t) \quad (10)$$

3.6 Equilibrium Equation

Substituting (8) into the first equilibrium equation

$$E = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0,$$

yields:

$$\begin{aligned} E = & \frac{1}{4\pi} \frac{\partial}{\partial x} \left(\frac{2x}{3R^3} \right) G_2 * d(-\varphi(t) + 2\psi(t)) \\ & + \frac{1}{4\pi} \left[\frac{\partial}{\partial x} \left(\frac{\partial u_\varphi}{\partial x} + \frac{2x}{3R^3} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial x} \right) \right] G_1 * d\varphi(t) \\ & + \frac{1}{2\pi} \left[\frac{\partial}{\partial x} \left(\frac{\partial u_\psi}{\partial x} - \frac{2x}{3R^3} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) \right] G_1 * \psi(t) = 0, \end{aligned}$$

where the coefficients to $G_1 * d\varphi$ and $G_1 * \psi(t)$ are:

$$\left\{ \begin{aligned} & \frac{\partial}{\partial x} \left(\frac{\partial u_\varphi}{\partial x} + \frac{2x}{3R^3} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial x} \right) \\ & = \frac{\partial}{\partial x} \left(\frac{\partial u_\varphi}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\varphi}{\partial y} + \frac{\partial v_\varphi}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_\varphi}{\partial x} \right) + \frac{\partial}{\partial x} \frac{2x}{3R^3} \\ & = \frac{\partial}{\partial x} \frac{2x}{3R^3}; \\ & \frac{\partial}{\partial x} \left(\frac{\partial u_\psi}{\partial x} - \frac{2x}{3R^3} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) \\ & = \frac{\partial}{\partial x} \left(\frac{\partial u_\psi}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u_\psi}{\partial y} + \frac{\partial v_\psi}{\partial x} \right) - \frac{\partial}{\partial x} \frac{2x}{3R^3} = \frac{\partial}{\partial x} \frac{x}{R^3} - \frac{\partial}{\partial x} \frac{2x}{3R^3} \\ & = \frac{\partial}{\partial x} \frac{x}{3R^3}. \end{aligned} \right.$$

Thus there is:

$$E = \frac{1}{4\pi} \frac{\partial}{\partial x} \left(\frac{2x}{3R^3} \right) G_2 * d(-\varphi(t) + 2\psi(t)) + \frac{1}{4\pi} \frac{\partial}{\partial x} \left(\frac{2x}{3R^3} \right) G_1 * d(\varphi(t) + \psi(t)) = 0.$$

By eliminating the same coefficients containing the coordinates, there is:

$$G_2 * d(\varphi(t) - 2\psi(t)) - G_1 * d(\varphi(t) + \psi(t)) = 0. \quad (11)$$

The second equilibrium equation $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$ will yield the same equation relationship as shown in (10). The third equilibrium equation $\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$ has already been satisfied.

3.7 Solution to Cerruti's Viscoelastic Problem

$\varphi(t)$ and $\psi(t)$ are linearly independent in Equations (10) and (11). Thus,

$\varphi(t)$ and $\psi(t)$ could be uniquely solved using the Volterra integral equations of the second kind (Zhang 1994, p.150-161). It is also interesting to mention that although the ways to apply external force are different, the determining equations (10) and (11) in Cerruti's viscoelastic problem are the same as those given by Boussinesq's viscoelastic problem.

Given the initial condition $P_x(0) = 0$, all stress and strain components are zero at $t = 0$ and $\varphi(0) = 0$, $\psi(0) = 0$, the solution to $\varphi(t)$ can be obtained as follow:

$$\varphi(t) = \frac{2}{G_1(0)} \left(P_x(t) + \int_0^t \Gamma(t-\tau) P_x(\tau) d\tau \right)$$

where $\Gamma(t)$ is a resolvent of the kernel $L(t) = -\frac{1}{G_1(0)} \frac{dG_1(t)}{dt}$.

Noted is that the detailed derivations are not included in this part for the purpose of concision since the detailed derivations and the proof of the uniqueness can be found in the paper by Talybly (2010).

Manipulating equations (10) and (11), there is:

$$(G_1 + 2G_2) * d(\varphi(t) - 2\psi(t)) = 6P_x(t) \quad (12)$$

Thus, similar to the solution to $\varphi(t)$, the solution to $\psi(t)$ can be obtained as:

$$\psi(t) = \frac{1}{2} \varphi(t) - \frac{3}{G_1(0) + 2G_2(0)} \left(P_x(t) + \int_0^t U(t-\tau) P_x(\tau) d\tau \right)$$

where $U(t)$ is a resolvent of the function:

$$M(t) = -\frac{1}{G_1(0) + 2G_2(0)} \frac{d}{dt} (G_1(t) + 2G_2(t)).$$

$U(t)$ could be written as a series composed of iterated kernels:

$$U(t) = \sum_{n=0}^{\infty} U_{n+1}(t)$$

where:

$$U_{n+1}(t) = \int_0^t U_1(t-\tau)U_n(\tau)d\tau, n=1,2,3\dots; U_1(t) = M(t)$$

Taking advantage of (10) and (11), the term of G_2 can be eliminated and the stress distributions (11) can be rewritten as follows:

$$\left\{ \begin{array}{l} \sigma_x = \frac{1}{2\pi} \left[\left(\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x(t) - G_1 * d\psi(t)) - \frac{3x^3}{R^5} P_x(t) \right]; \\ \sigma_y = \frac{1}{2\pi} \left[\left(-\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x(t) - G_1 * d\psi(t)) - \frac{3xy^2}{R^5} P_x(t) \right]; \\ \sigma_z = -\frac{3xz^2}{2\pi R^5} P_x(t); \\ \sigma_{yz} = -\frac{3xyz}{2\pi R^5} P_x(t); \\ \sigma_{xz} = -\frac{3x^2z}{2\pi R^5} P_x(t); \\ \sigma_{xy} = \frac{1}{2\pi} \left[\left(-\frac{y}{R(R+z)^2} + \frac{x^2y}{R^3(R+z)^2} + \frac{2x^2y}{R^2(R+z)^3} \right) (P_x(t) - G_1 * d\psi(t)) - \frac{3x^2y}{R^5} P_x(t) \right]. \end{array} \right. \quad (13)$$

Comparing equations (13) with those of elastic Cerruti's solution (Johnson, 1985 p.69-70), we found that the term $P_x(t) - G_1 * d\psi$ in the viscoelastic problem plays the same role as the term $(I - 2\mu)P$ in the elastic problem.

4 Solution to Generalized Viscoelastic Problem

We now consider Boussinesq-Cerruti viscoelastic problem, in which two tangential forces, $P_1(t)$ and $P_2(t)$, along x - and y - axis respectively and one normal force, $P_3(t)$, along z -axis, are applied at the origin. The displacement and stress distributions are obtained using the superposition of the displacement and stress distributions by $P_1(t)$, $P_2(t)$, and $P_3(t)$. The stress and displacement distributions for the problem when only one tangential force $P_1(t) = P_x(t)$ along x -axis has been

discussed in previous discussions. For disambiguation, we rewrite the solution with the proper superscript (i), ($i = 1, 2$ or 3), to denote the applied forces. Therefore we have:

$$\sigma_x^{(1)} = \sigma_x, \sigma_y^{(1)} = \sigma_y, \sigma_z^{(1)} = \sigma_z, \sigma_{yz}^{(1)} = \sigma_{yz}, \sigma_{xz}^{(1)} = \sigma_{xz}, \sigma_{xy}^{(1)} = \sigma_{xy}. \quad (14)$$

where the expressions of the right hand side are the same as in (13), only with $P_x(t)$ replaced by $P_i(t)$.

When there is a single tangential force $P_2(t)$ applied on the O - x - y plane at point O along y -axis, the stress distributions are obtained through the calculation of the frame rotation. In this method, as shown in Figure 2, we first rotate the frames around z -axis by 90° and denote the new coordinate system as O - x' - y' - z' .

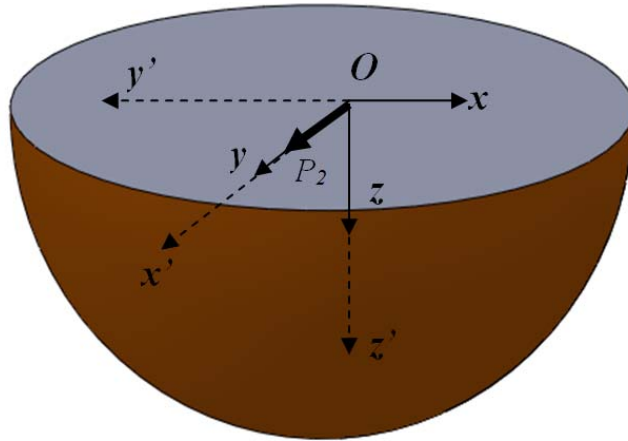


Figure 2: Model for coordinate's transformation

The corresponding Jacob matrix is shown as follow:

$$R = \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) & 0 \\ \sin(90^\circ) & \cos(90^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The coordinates are transformed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -y' \\ x' \\ z' \end{bmatrix} \text{ and } \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} y \\ -x \\ z \end{bmatrix}, \quad (15)$$

The corresponding stress tensors are transformed as:

$$\begin{pmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z \end{pmatrix} = R \begin{pmatrix} \sigma_{x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{y'x'} & \sigma_{y'} & \sigma_{y'z'} \\ \sigma_{z'x'} & \sigma_{z'y'} & \sigma_{z'} \end{pmatrix} R^T = \begin{pmatrix} \sigma_{y'} & -\sigma_{y'x'} & -\sigma_{y'z'} \\ -\sigma_{x'y'} & \sigma_{x'} & \sigma_{x'z'} \\ -\sigma_{z'y'} & \sigma_{z'x'} & \sigma_{z'} \end{pmatrix}. \quad (16)$$

Correspondingly, with the same resolvent of kernels as in Section 3.7, the determining functions can be rewritten as:

$$\varphi^{(i)}(t) = \frac{2}{G_1(0)} \left(P_i(t) + \int_0^t \Gamma(t-\tau) P_i(\tau) d\tau \right), i=1,2,3,$$

$$\psi^{(i)}(t) = \frac{1}{2} \varphi^{(i)}(t) - \frac{3}{G_1(0) + 2G_2(0)} \left(P_i(t) + \int_0^t U(t-\tau) P_i(\tau) d\tau \right), i=1,2,3.$$

As shown in Figure 2, in coordinates $O-x'y'z'$, $P_2(t)$ is applied at the origin along x' -axis. Based on the solution we obtained in (13), the stresses in the $O-x'y'z'$ system can be obtained as:

$$\left\{ \begin{aligned} \sigma_x^{(2)} &= \frac{1}{2\pi} \left[\left(\frac{3x'}{R(R+z')^2} + \frac{x'^3}{R^3(R+z')^2} + \frac{2x'^3}{R^2(R+z')^3} + \frac{x'}{R^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3x'^3}{R^5} P_2(t) \right]; \\ \sigma_{y'}^{(2)} &= \frac{1}{2\pi} \left[\left(\frac{x'}{R(R+z')^2} + \frac{x'y'^2}{R^3(R+z')^2} + \frac{2x'y'^2}{R^2(R+z')^3} + \frac{x'}{R^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3x'y'^2}{R^5} P_2(t) \right]; \\ \sigma_{z'}^{(2)} &= \frac{3x'z'^2}{2\pi R^5} P_2(t); \\ \sigma_{y'z'}^{(2)} &= \frac{3x'y'z'}{2\pi R^5} P_2(t); \\ \sigma_{x'z'}^{(2)} &= \frac{3x'^2 z'}{2\pi R^5} P_2(t); \\ \sigma_{x'y'}^{(2)} &= \frac{1}{2\pi} \left[\left(\frac{y'}{R(R+z')^2} + \frac{x'^2 y'}{R^3(R+z')^2} + \frac{2x'^2 y'}{R^2(R+z')^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3x'^2 y'}{R^5} P_2(t) \right]. \end{aligned} \right. \quad (17)$$

where: $R' = \sqrt{x'^2 + y'^2 + z'^2}$.

Using (15) and (16) in O -xyz system, the stress tensors (17) can be obtained as:

$$\left\{ \begin{array}{l} \sigma_x^{(2)} = \frac{1}{2\pi} \left[\left(-\frac{y}{R(R+z)^2} + \frac{x^2 y}{R^3(R+z)^2} + \frac{2x^2 y}{R^2(R+z)^3} + \frac{y}{R^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3x^2 y}{R^5} P_2(t) \right]; \\ \sigma_y^{(2)} = \frac{1}{2\pi} \left[\left(-\frac{3y}{R(R+z)^2} + \frac{y^3}{R^3(R+z)^2} + \frac{2y^3}{R^2(R+z)^3} + \frac{y}{R^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3y^3}{R^5} P_2(t) \right]; \\ \sigma_z^{(2)} = -\frac{3yz^2}{2\pi R^5} P_2(t); \\ \sigma_{yz}^{(2)} = -\frac{3y^2 z}{2\pi R^5} P_2(t); \\ \sigma_{xz}^{(2)} = -\frac{3xyz}{2\pi R^5} P_2(t); \\ \sigma_{xy}^{(2)} = \frac{1}{2\pi} \left[\left(-\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} \right) (P_2(t) - G_1 * d\psi^{(2)}(t)) - \frac{3xy^2}{R^5} P_2(t) \right]. \end{array} \right. \quad (18)$$

When a single normal force, $P_3(t)$, is applied at point O and along the positive z -axis, the stress distributions are shown in (19). Noted is that this expression has been given by Talybly (2010).

$$\left\{ \begin{array}{l} \sigma_x^{(3)} = \frac{1}{2\pi} \left\{ \frac{1}{x^2 + y^2} \left[\left(1 - \frac{z}{R} \right) \frac{x^2 - y^2}{x^2 + y^2} + \frac{zy^2}{R^3} \right] (P_3(t) - G_1 * d\psi^{(3)}(t)) - \frac{3zx^2}{R^5} P_3(t) \right\}; \\ \sigma_y^{(3)} = \frac{1}{2\pi} \left\{ \frac{1}{x^2 + y^2} \left[\left(1 - \frac{z}{R} \right) \frac{y^2 - x^2}{x^2 + y^2} + \frac{zx^2}{R^3} \right] (P_3(t) - G_1 * d\psi^{(3)}(t)) - \frac{3zy^2}{R^5} P_3(t) \right\}; \\ \sigma_z^{(3)} = -\frac{3z^3}{2\pi R^5} P_3(t); \\ \sigma_{yz}^{(3)} = -\frac{3yz^2}{2\pi R^5} P_3(t); \\ \sigma_{xz}^{(3)} = -\frac{3xz^2}{2\pi R^5} P_3(t); \\ \sigma_{xy}^{(3)} = \frac{1}{2\pi} \left\{ \frac{1}{x^2 + y^2} \left[\left(1 - \frac{z}{R} \right) \frac{xy}{x^2 + y^2} - \frac{xyz}{R^3} \right] (P_3(t) - G_1 * d\psi^{(3)}(t)) - \frac{3xyz}{R^5} P_3(t) \right\}; \end{array} \right. \quad (19)$$

Based on (14), (18) and (19), we can provide the solution of the stress

distributions to Boussinesq-Cerruti viscoelastic problem as follows:

$$\left\{ \begin{array}{l} \bar{\sigma}_x^{(3)}(t) = \sum_{i=1}^3 \sigma_x^{(i)}(t), \\ \bar{\sigma}_y^{(3)}(t) = \sum_{i=1}^3 \sigma_y^{(i)}(t), \\ \bar{\sigma}_z^{(3)}(t) = \sum_{i=1}^3 \sigma_z^{(i)}(t), \\ \bar{\sigma}_{yz}^{(3)}(t) = \sum_{i=1}^3 \sigma_{yz}^{(i)}(t), \\ \bar{\sigma}_{xz}^{(3)}(t) = \sum_{i=1}^3 \sigma_{xz}^{(i)}(t), \\ \bar{\sigma}_{yx}^{(3)}(t) = \sum_{i=1}^3 \sigma_{yx}^{(i)}(t). \end{array} \right.$$

Similarly, the displacement distributions to Boussinesq-Cerruti viscoelastic problem (\bar{u} , \bar{v} and \bar{w}) as follows:

$$\left\{ \begin{array}{l} \bar{u} = \sum_{i=1}^3 u^{(i)}, \\ \bar{v} = \sum_{i=1}^3 v^{(i)}, \\ \bar{w} = \sum_{i=1}^3 w^{(i)}, \end{array} \right.$$

where the components are:

$$\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{1}{R} + \frac{1}{R+z} + \frac{x^2}{R^3} - \frac{x^2}{R(R+z)^2} & -\frac{2}{R+z} + \frac{2x^2}{R(R+z)^2} \\ \frac{xy}{R^3} - \frac{xy}{R(R+z)^2} & \frac{2xy}{R(R+z)^2} \\ \frac{xz}{R^3} + \frac{x}{R(R+z)} & \frac{-2x}{R(R+z)} \end{bmatrix} \begin{bmatrix} \varphi^{(1)}(t) \\ \psi^{(1)}(t) \end{bmatrix},$$

$$\begin{bmatrix} u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{xy}{R^3} - \frac{xy}{R(R+z)^2} & \frac{2xy}{R(R+z)^2} \\ \frac{1}{R} + \frac{1}{R+z} + \frac{y^2}{R^3} - \frac{y^2}{R(R+z)^2} & \frac{-2}{R+z} + \frac{2y^2}{R(R+z)^2} \\ \frac{yz}{R^3} + \frac{y^2}{R(R+z)} & \frac{-2y}{R(R+z)} \end{bmatrix} \begin{bmatrix} \varphi^{(2)}(t) \\ \psi^{(2)}(t) \end{bmatrix},$$

$$\begin{bmatrix} u^{(3)} \\ v^{(3)} \\ w^{(3)} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{xz}{R^3} - \frac{x}{R(R+z)} & \frac{2x}{R(R+z)} \\ \frac{yz}{R^3} - \frac{y}{R(R+z)} & \frac{2y}{R(R+z)} \\ \frac{z}{R^3} + \frac{2}{R} & -\frac{2}{R} \end{bmatrix} \begin{bmatrix} \varphi^{(3)}(t) \\ \psi^{(3)}(t) \end{bmatrix}.$$

The results will be used in our future research about the formulation of the cuttings for linear viscoelastic materials, in which a distributive force is used instead of concentrated force. We could obtain the stress responses $\sigma_{ij}(\vec{r}, t)$ to a certain distributive force by considering the point-force \vec{P} as a function of ξ and ζ , replacing x with $x - \xi$, y with $y - \zeta$, and R with $\bar{R} = \sqrt{(x - \xi)^2 + (y - \zeta)^2 + z^2}$, and then integrating the corresponding stress or displacement components for ξ and ζ in x - y plane.

5 Discussion on Correspondence Principle

The correspondence principle (Lakes, 1998) states that if a solution to a linear elasticity problem is known, the solution to the corresponding problem for a linearly viscoelastic material can be obtained by replacing each quantity which can depend on time by its Laplace transform multiplied by the transform variable (p or s), and then by transforming back to the time domain.

A restriction is that the boundaries under prescribed displacements or forces

may not vary with time, though the stresses and displacements are functions of time. An exceptional case exists when the boundary condition can be separated into spatial and time components. For example, in the viscoelastic Cerruti's problem, the tangential force can be expressed as:

$$P(x, y, t) = P_0(x, y) f(t)$$

where

$$P_0(x, y) = \begin{cases} P_0 & \text{if } x=0 \text{ and } y=0 \\ 0 & \text{else} \end{cases}$$

In those cases, Laplace transform simply transform the time component into frequency domain while keeping the spatial profile of the prescribed loads and displacements. Therefore, the solution to viscoelastic problem still has the same profile as the solution to elastic problem, but it is multiplied by a time-dependent component that describes how the profile changes with time.

In this paper, the boundary conditions for viscoelastic Cerruti's problem satisfy the above requirement, and therefore theoretically, correspondence principle can be used to directly obtain the viscoelastic solution from the elastic solution. The solution to elastic Cerruti's problem is written in (20).

$$\left\{ \begin{array}{l} \sigma_x(\vec{r}) = \frac{1}{2\pi} \left[\left(-\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x - 2\mu P_x) - \frac{3x^3}{R^5} P_x \right]; \\ \sigma_y(\vec{r}) = \frac{1}{2\pi} \left[\left(-\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x - 2\mu P_x) - \frac{3xy^2}{R^5} P_x \right]; \\ \sigma_z(\vec{r}) = -\frac{3P_x xz^2}{2\pi R^5}; \\ \sigma_{yz}(\vec{r}) = -\frac{3P_x xyz}{2\pi R^5}; \\ \sigma_{xz}(\vec{r}) = -\frac{3P_x x^2 z}{2\pi R^5}; \\ \sigma_{xy}(\vec{r}) = \frac{1}{2\pi} \left[\left(-\frac{y}{R(R+z)^2} + \frac{x^2 y}{R^3(R+z)^2} + \frac{2x^2 y}{R^2(R+z)^3} \right) (P_x - 2\mu P_x) - \frac{3x^2 y}{R^5} P_x \right]. \end{array} \right. \quad (20)$$

Applying correspondence principle, equations (20) in Laplace domain are:

$$\left\{ \begin{array}{l} \sigma_x^*(\vec{r},s) = \frac{1}{2\pi} \left[\left(\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x^*(s) - 2\mu^*(s)P_x^*(s)) - \frac{3x^3}{R^5} P_x^*(s) \right]; \\ \sigma_y^*(\vec{r},s) = \frac{1}{2\pi} \left[\left(\frac{x}{R(R+z)^2} + \frac{xy^2}{R^3(R+z)^2} + \frac{2xy^2}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x^*(s) - 2\mu^*(s)P_x^*(s)) - \frac{3xy^2}{R^5} P_x^*(s) \right]; \\ \sigma_z^*(\vec{r},s) = -\frac{3xz^2}{2\pi R^5} P_x^*(s); \\ \sigma_{yz}^*(\vec{r},s) = -\frac{3xyz}{2\pi R^5} P_x^*(s); \\ \sigma_{xz}^*(\vec{r},s) = -\frac{3x^2z}{2\pi R^5} P_x^*(s); \\ \sigma_{xy}^*(\vec{r},s) = \frac{1}{2\pi} \left[\left(\frac{y}{R(R+z)^2} + \frac{x^2y}{R^3(R+z)^2} + \frac{2x^2y}{R^2(R+z)^3} \right) (P_x^*(s) - 2\mu^*(s)P_x^*(s)) - \frac{3x^2y}{R^5} P_x^*(s) \right]. \end{array} \right. \quad (21)$$

where the superscript ‘*’ represents the function in frequency domain.

From (21), we notice that only the first two and the last equations have the multiplication between two time-dependent functions, the external force and the time-dependent Poison’s ratio. Considering a simple case where the Poison’s ratio is held constant, equation (21) can be easily transformed back into time-domain. In this case, the only difference between the viscoelastic problem and elastic problem is whether the external force is a function of time.

In a general case, the Poison’s ratio cannot be treated as a constant for a viscoelastic material (J. Kim, H.S. Lee and N.P. Kim, 2010), for there are two independent time-dependent material properties that characterize the viscoelastic material. In these cases, the direct multiplication of the external force and the time-dependent Poison’s ratio in frequency domain becomes a convolution in time-domain. Taking the first equation in (21) as an example, after the inverse Laplace transform, we have:

$$\sigma_x(\vec{r},t) = \frac{1}{2\pi} \left[\left(\frac{3x}{R(R+z)^2} + \frac{x^3}{R^3(R+z)^2} + \frac{2x^3}{R^2(R+z)^3} + \frac{x}{R^3} \right) (P_x(t) - 2P_x(t) * d\mu(t)) - \frac{3x^3}{R^5} P_x(t) \right]. \quad (22)$$

Comparing (22) and the first equation in (13), we have:

$$\begin{cases} 2P_x(t) * d\mu(t) = G_1 * d\psi(t), & \text{in time-domain, or} \\ 2sP_x^*(s)\mu^*(s) = sG_1^*(s)\psi^*(s), & \text{in frequency-domain} \end{cases} \quad (23)$$

If the above relationship holds, then the solution from correspondence principle will be the same as the solution from the method used in this paper. To verify this equation, firstly we need to obtain the expression of $\psi(t)$ from the two determining conditions (10) and (12). We rewrite them in (24):

$$\begin{cases} G_1 * d\varphi(t) = 2P_x(t) \\ (G_1 + 2G_2) * d(\varphi(t) - 2\psi(t)) = 6P_x(t) \end{cases} \quad (24)$$

Taking Laplace transform to (24), we have:

$$\begin{cases} G_1^*(s)[s\varphi^*(s)] = 2P_x^*(s) \\ (G_1^*(s) + 2G_2^*(s)) * [s(\varphi^*(s) - 2\psi^*(s))] = 6P_x^*(s) \end{cases} \quad (25)$$

Solving (25) for $s\psi^*$ and multiplying by $G_1^*(s)$, we have:

$$sG_1^*(s)\psi^*(s) = \frac{2P_x^*(s)[G_2^*(s) - G_1^*(s)]}{G_1^*(s) + 2G_2^*(s)} \quad (26)$$

Comparing (23) with (26), we should have the following relationship:

$$\mu^*(s) = \frac{G_2^*(s) - G_1^*(s)}{G_1^*(s) + 2G_2^*(s)} \quad (27)$$

Relation (27) gives the expression of time-dependent Poisson's ratio by shear and bulk relaxations $G_1(t)$ and $G_2(t)$ for a homogenous and isotropic viscoelastic material.

In another hand, for a homogeneous and isotropic viscoelastic material, the relationships among the relaxation modulus $E(t)$, shear modulus $G(t)$ and bulk modulus $K(t)$ can be expressed as (28) (J. Kim, H.S. Lee and N.P. Kim, 2010):

$$\begin{cases} G^*(s) = \frac{E^*(s)}{2[1+s\mu^*(s)]} \\ K^*(s) = \frac{E^*(s)}{3[1-2s\mu^*(s)]} \end{cases} \quad (28)$$

Eliminating $E^*(s)$ in (28), we have:

$$\mu^*(s) = \frac{3K^*(s) - 2G^*(s)}{2[G^*(s) + 3K^*(s)]} \quad (29)$$

Comparing (27) and (29), we have two expressions of the time-dependent Poisson's ratio in Laplace domain. Notice different material functions are used, a translation (29) is necessary (Christensen, R. M. 1971):

$$\begin{cases} G^*(s) = \frac{1}{2}G_1^*(s) \\ K^*(s) = \frac{1}{3}G_2^*(s) \end{cases} \quad (30)$$

Bringing relation (30) into (29), we can verify that equation sets (27) and (29) are actually the same. Therefore, the results from the method in this paper are verified by the elastic-viscoelastic correspondence principle.

6 Conclusions

Firstly, the solution to the stress and displacement distributions for Cerruti's viscoelastic problem was presented in this paper. Based on the elastic-viscoelastic corresponding principle, the solution to the displacements of Cerruti's elastic problem was used as the displacement solution for Cerruti's viscoelastic problem. Based on the equilibrium equations and boundary conditions of a viscoelastic system, we obtained the solution of the two time-dependent determining functions via the Volterra integral equations of the second kind. Furthermore, Cerruti's viscoelastic problem with a tangential force pointing to the y -axis was solved

based on a frame rotation method. Finally, by combining the solutions to Boussinesq's viscoelastic problem, we get the results for the generalized case, where an arbitrary force with three non-zero components in the x , y , and z directions, is applied. These results could be further used to solve half-space viscoelastic problem under a distributive force by taking the integral over the area where the distributive force is applied.

References

- [1] J. Boussinesq, *Application des Potentiels a l'Etude de l'Equilibre et du Mouvement des Solides Elastiques*, Gauthier-Villars, Paris, 1885.
- [2] V. Cerruti., *Ricerche intorno all'equilibrio dei corpi elastici isotropi*, Reale Accademia dei Lincei, Roma, 1882.
- [3] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Fourth edition, Cambridge University Press, 1927.
- [4] L.K. Talybly, Boussinesq's viscoelastic problem on normal concentrated force on a half-space surface, *Mechanics of Time-Dependent Materials*, **3**, (2010), 253-259.
- [5] Z. Xu, *Elasticity Mechanics (Tan Xing Li Xue)*, 5th Edition, Higher-level Education Publication, in Chinese, 1996.
- [6] C.Y. Zhang, *Viscoelastic Fracture Mechanics (Nian Tan Xing Duan Lie Li Xue)*, Huazhong University of Science and Technology Press, in Chinese, 1994.
- [7] D. Zhou and G. McMurray, Slicing Cuts on Food Materials Using Robotic Controlled Razor Blade, *Modelling and Simulation in Engineering*, **2011**, (2011).
- [8] R.M. Christensen, *Theory of viscoelasticity; an introduction*, New York, New York, Academic Press, 1971.