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# A New Approach for a Class of Nonlinear Optimal Control Problems Using Linear Combination Property of Intervals

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## Abstract

Here, we consider a class of optimal control problems (OCP) containing nonlinear dynamical systems with the quadratic functionals of state variables. The major part of our technique is based upon linear combination property of intervals (LCPI) such that using this property the nonlinear dynamical system is converted to a linear one. And, the major difference of our approach from a large number of direct approaches for solving OCPs is that we finally solve a convex (linear or quadratic) programming problem which its optimal solution is global. We also extend our technique to a class of optimal control problems governed by differential inclusions (DI). The proposed idea is illustrated by numerical examples. Moreover, a comparison is made with a discretization method.

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## 1 Introduction

Although, many approaches have been developed and proposed for solving optimal control problems (OCP) (see [1]-[4], [6], [7] and references therein), modification of the existing methods and development of new techniques should yet be explored to obtain accurate solutions successfully.

The methods to numerical solutions of optimal control problems may be divided into two major classes: the indirect methods and the direct methods. The indirect methods are based on the Pontryagin Maximum Principle (PMP) and require the numerical solution of boundary value problems that result from the necessary conditions of optimal control. For many practical optimal control problems, these boundary value problems are quite difficult to solve. In fact, the manner in which PMP is used differs so significantly from one type of problem to another that no standard solution procedure can be devised. Therefore, we must use direct computational algorithms such as [4], [6], [7] to solve OCPs.

However direct approaches are more easy to handle and eliminate the requirement of solving a two point boundary value problem (2PBVP) in contrast to indirect approaches, they suffer from this fact that their solutions may not be global, because these approaches finally solve a Nonlinear Programming Problem (NLP), which for any initial point, they give different optimal (local) solution or even may have not global solutions.

Thus, we consider a general class of nonlinear dynamical systems that can be changed into a linear one together with a quadratic functional of state variables in the following form

$$\text{Minimize} \quad \int_{t_0}^{t_f} (x^T(t)Q(t)x(t) + c^T(t)x(t))dt \quad (1)$$

Subject to

$$\dot{x}(t) = A(t)x(t) + h(t, u(t)), \quad t \in [t_0, t_f], \quad u(.) \in U, \quad (2)$$

with the boundary conditions  $x(t_0) = \alpha$  and  $x(t_f) = \eta$ .

It is assumed that  $A(t), Q(t) \in R^{n \times n}$  and  $c(\cdot)$ ,  $\alpha, \eta \in R^n$  are known, whereas  $x(t) \in R^n$  and  $u(t) \in R^m$  are the unknown state and control variables respectively.

We although suppose that  $U$  is a compact and connected subset of  $R^m$  and  $h : [t_0, t_f] \times U \rightarrow R^n$  is a smooth or non-smooth continuous function. Moreover, there exists a pair of state and control variables  $(x(t), u(t))$  such that satisfies (2) and boundary conditions  $x(t_0) = \alpha$  and  $x(t_f) = \eta$  and  $Q(t)$  is a positive definite matrix. Here, we use the linear combination property of intervals (LCPI) to convert the nonlinear dynamical system (2) to an equivalent linear system. After this step, the control  $u(t)$  is replaced by the associated control  $\lambda(t)$ . The new optimal control problem with this linear dynamical system is transformed to a discrete-time problem that could be solved by quadratic programming methods [5].

This paper is organized as follows. Section 2, transforms the nonlinear function  $h(t, u(t))$  to a corresponding function that is linear with respect to the associated control variable. In Section 3, the new problem is converted to a discrete-time problem via discretization. In Section 4, we extend our approach to a general class of optimal control problems governed by differential inclusions (DI). In Section 5, numerical examples are presented to illustrate the effectiveness of the proposed method. In the latter section, we show that our strategy acquire better solutions, that attained in fewer time, than a discretization method [1] through several simplistic examples, which comparison of the solutions is included in the first two examples. Finally conclusions are given in Section 6.

## 2 Linearization of the Dynamical System

In this section, we use LCPI for changing the nonlinear dynamical system (2) to a linear one. The LCPI states that every uniform continuous function with a compact and connected domain can be written as a convex linear combination of its maximum and minimum. In other words, if  $\mu$  and  $\nu$  are the maximum and minimum of the uniform continuous function  $h(x)$ , one can write

$$h(x) = \lambda\mu + (1 - \lambda)\nu = h(\lambda), \quad \text{with } 0 \leq \lambda \leq 1,$$

where  $\mu = \text{Max}_x\{h(x) : x \in D\}$ ,  $\nu = \text{Min}_x\{h(x) : x \in D\}$  and  $D$  is a

compact and connected set. Here, for using the above-mentioned property (i.e., LCPI) we need to the following theorems.

**Theorem 2.1.** *Let  $h_i : [t_0, t_f] \times U \longrightarrow R$  for  $i = 1, 2, \dots, n$  be a continuous function where  $U$  is a compact and connected subset of  $R^m$ , then for any arbitrary (but fixed)  $t \in [t_0, t_f]$  the set  $\{h_i(t, u(t)) : u(t) \in U\}$  is a closed interval in  $R$ .*

**Proof.** Assume that  $t \in [t_0, t_f]$  be given. Let  $\psi_i(u) = h_i(t, u(t))$  for  $i = 0, 1, \dots, n$ . Obviously  $\psi_i(u)$  is a continuous function on  $U$ . Since continuous functions preserve compactness and connectedness properties,  $\{\psi_i(u) : u(t) \in U\}$  is compact and connected in  $R$ . Therefore  $\{h_i(t, u(t)) : u(t) \in U\}$  is a closed interval in  $R$ .  $\square$

Now, for any  $t \in [t_0, t_f]$  suppose that the lower and upper bounds of the closed interval  $\{h_i(t, u(t)) : u(t) \in U\}$  are  $g_i(t)$  and  $w_i(t)$ , respectively. Thus, for  $i = 0, 1, \dots, n$

$$g_i(t) \leq h_i(t, u(t)) \leq w_i(t), \quad t \in [t_0, t_f]. \quad (3)$$

In other words

$$g_i(t) = \text{Min}_u \{h_i(t, u(t)) : u \in U\}, \quad t \in [t_0, t_f], \quad (4)$$

$$w_i(t) = \text{Max}_u \{h_i(t, u(t)) : u \in U\}, \quad t \in [t_0, t_f]. \quad (5)$$

**Theorem 2.2.** *Let functions  $g_i(t)$  and  $w_i(t)$  for  $i = 0, 1, \dots, n$  be defined by relations (4) and (5). Then they are uniformly continuous on  $t \in [t_0, t_f]$ .*

**Proof.** We will show that  $g_i(t)$  for  $i = 0, 1, \dots, n$  is uniformly continuous. It is sufficient to show that for any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $s_1 \in N_\delta(s_2)$  then  $|g_i(s_1) - g_i(s_2)| < \varepsilon$ , where  $N_\delta(z)$  is a  $\delta$ -neighborhood of  $z$ . Since any continuous function on a compact domain is uniformly continuous, The function  $h_i(t, u(t))$  on the compact set  $[t_0, t_f] \times U$  is uniformly continuous, i.e., for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $(s_1, u) \in N_\delta(s_2, u)$ , then  $|h_i(s_1, u) - h_i(s_2, u)| < \varepsilon$ . Thus  $h_i(s_1, u) < h_i(s_2, u) + \varepsilon$ . In addition, by (4),  $g_i(s_1) < h_i(s_1, u)$  and so,  $g_i(s_1) < h_i(s_2, u) + \varepsilon$ . Now, by taking infimum on the right hand side of the latter inequality  $g_i(s_1) < g_i(s_2) + \varepsilon$ . By a similar argument we have also  $g_i(s_2) - g_i(s_1) < \varepsilon$ . Thus  $|g_i(s_1) - g_i(s_2)| < \varepsilon$ . The proof of uniform continuity of  $w_i(t)$  for  $i = 0, 1, 2, \dots$  is similar.  $\square$

By linear combination property of intervals and relations (4)-(5), for any  $t \in [t_0, t_f]$

$$h_i(t, u(t)) = \beta_i(t)\lambda_i(t) + g_i(t), \quad \lambda_i(t) \in [0, 1] \quad (6)$$

where  $\beta_i(t) = w_i(t) - g_i(t)$  for  $i = 0, 1, 2, \dots, n$ . Thus, we transform problem (1)-(2) by relations (3), (4) and (5) to the following problem.

$$\text{Minimize} \quad \int_{t_0}^{t_f} (x^T(t)Q(t)x(t) + c^T(t)x(t))dt \quad (7)$$

$$\text{Subject to} \quad \begin{cases} \dot{x}_k(t) = (\sum_{r=1}^n a_{kr}(t)x_r(t)) + \beta_k(t)\lambda_k(t) + g_k(t), & t \in [t_0, t_f] \\ 0 \leq \lambda_k(t) \leq 1, & k = 1, 2, \dots, n, \quad t \in [t_0, t_f] \\ x(t_0) = \alpha & \text{and} \quad x(t_f) = \eta \end{cases}$$

where  $a_{kr}(t)$  is the  $k^{\text{th}}$  row and  $r^{\text{th}}$  column component of matrix  $A(t)$ . Note that in the problem (7), which is a linear quadratic optimal control problem,  $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))^T$  is the associated control vector. Next section, converts the latter problem to a corresponding discrete-time problem.

### 3 Discrete-time Problem

Now, discretization method enables us transforming continuous problem (7) to a corresponding discrete form. Consider equidistance points  $t_0 = s_0 < s_1 < s_2 < \dots < s_N = t_f$  on  $[t_0, t_f]$  which defined as  $s_j = t_0 + j\delta$  for all  $j = 0, 1, \dots, N$  with step-length  $\delta = \frac{t_f - t_0}{N}$  where  $N$  is a given large number. We use the trapezoidal approximation in numerical integration and the following approximations to change problem (7) to a corresponding discrete form:

$$\begin{aligned} \dot{x}_k(s_j) &\approx \frac{x_k(s_{j+1}) - x_k(s_j)}{\delta}, \quad \text{and} \\ \dot{x}_k(s_N) &\approx \frac{x_k(s_N) - x_k(s_{N-1})}{\delta}, \quad \text{for } k = 1, 2, \dots, n, \quad \text{and } j = 1, 2, \dots, N-1. \end{aligned}$$

Thus we have the following discrete-time problem with unknown variables  $x_{kj}$  and  $\lambda_{kj}$  for  $k = 1, 2, \dots, n$  and  $j = 0, 1, \dots, N-1$ .

$$\text{Minimize} \quad C + \sum_{j=1}^{N-1} \sum_{k=1}^n (c_{kj}x_{kj} + (\sum_{i=1}^n Q_{ikj}x_{kj}x_{ij})) \quad (8)$$

Subject to

$$\begin{cases} x_{k,j+1} - (1 + \delta a_{kkj})x_{kj} - (\sum_{r=1, r \neq k}^n \delta a_{krj}x_{rj}) - \delta \beta_{kj}\lambda_{kj} = \delta g_{kj}, & (0 \leq j \leq N-1) \\ (1 - \delta a_{kkN})x_{kN} - x_{k,N-1} - (\sum_{r=1, r \neq k}^n \delta a_{krN}x_{rN}) - \delta \beta_{kN}\lambda_{kN} = \delta g_{kN} \\ 0 \leq \lambda_{kj} \leq 1, \quad x_{k0} = \alpha_k, \quad x_{kN} = \eta_k, \quad \text{for } j = 0, 1, \dots, N, \quad k = 1, 2, \dots, n \end{cases}$$

where  $Q_{ikj} = Q_{ik}(s_j)$ ,  $x_{kj} = x_k(s_j)$ ,  $c_{kj} = c_k(s_j)$ ,  $a_{krj} = a_{kr}(s_j)$ ,  $\lambda_{kj} = \lambda_k(s_j)$ ,  $g_{kj} = g_k(s_j)$ ,  $\beta_{kj} = \beta_k(s_j)$ , for all  $j = 0, 1, \dots, N$ , and  $k = 1, 2, \dots, n$  while  $C$  is a constant.

By solving problem (8), which is a quadratic programming problem, we are able to obtain optimal solutions  $\lambda_{kj}^*$  and  $x_{kj}^*$  for all  $j = 1, 2, \dots, N$  and  $k = 1, 2, \dots, n$ . Note that, for evaluating the control variable  $u^*(t)$ , we must use the following system

$$h(t, u^*(t)) = \beta(t)\lambda^*(t) + g(t) \quad (9)$$

## 4 Extend to Optimal Control Problems Governed by Differential Inclusions

In this section, we apply our approach for solving a class of continuous-time problems involving differential inclusion constraints which are as follow

$$\text{Minimize} \quad \int_{t_0}^{t_f} (x^T(t)Q(t)x(t) + c^T(t)x(t))dt \quad (10)$$

Subject to

$$\dot{x}(t) \in H(t), \quad t \in [t_0, t_f], \quad (x(t_0), x(t_f)) \in S$$

where  $H(t)$  is a set of continuous functions on  $[t_0, t_f]$  and  $S$  is a set which containing boundary points of state variable  $x(\cdot)$ . By the approach of this paper, we can convert the problem (10) to an equivalent quadratic programming (QP) problem. Here we assume that

$$H(t) = \{h(t, u(t)) : u(t) \in U\}, \quad t \in [t_0, t_f], \quad (11)$$

where  $U \subset R^m$  is a compact set and  $h(t) = (h_1(t), h_2(t), \dots, h_n(t))^T$  is a continuous function on  $[t_0, t_f] \times U$ . By attention to the linearization idea of Section 2, the problem (10) is equivalent to the following problem

$$\text{Minimize} \quad \int_{t_0}^{t_f} (x^T(t)Q(t)x(t) + c^T(t)x(t))dt \quad (12)$$

Subject to

$$\begin{cases} \dot{x}_k(t) = \lambda_k(t)\beta_k(t) + g_k(t), & 0 \leq \lambda_k(t) \leq 1, \quad (x(t_0), x(t_f)) \in S \\ \text{for all } t \in [t_0, t_f] \text{ and } k = 1, 2, \dots, n \end{cases}$$

where  $\beta_i(t) = w_i(t) - g_i(t)$  for  $i = 1, 2, \dots, n$  and functions  $g_i(t)$  and  $w_i(t)$  satisfy the relations (4) and (5). Now by the discretization method of Section 3, we obtain the following problem for approximating problem (10)

$$\text{Minimize} \quad C + \sum_{j=1}^{N-1} \sum_{k=1}^n (c_{kj}x_{kj} + \left(\sum_{i=1}^n Q_{ikj}x_{kj}x_{ij}\right)) \quad (13)$$

Subject to

$$\begin{cases} x_{k,j+1} - (1 + \delta a_{kkj})x_{kj} - \delta \beta_{kj}\lambda_{kj} = \delta g_{kj}, & (0 \leq j \leq N-1) \\ (1 - \delta a_{kkN})x_{kN} - x_{k,N-1} - \delta \beta_{kN}\lambda_{kN} = \delta g_{kN} \\ 0 \leq \lambda_{kj} \leq 1, \quad x_{k0} = \alpha_k, \quad x_{kN} = \eta_k, \quad (x_{k0}, x_{kN}) \in S \\ \text{for } j = 0, 1, \dots, N, \text{ and } k = 1, 2, \dots, n. \end{cases}$$

Again, by solving problem (13), which is a quadratic programming (QP) problem, we are able to obtain optimal solutions  $x_{kj}^*$  and  $\lambda_{kj}^*$  for all  $j = 1, 2, \dots, N$  and  $k = 1, 2, \dots, n$ . Note that, for evaluating the optimal control variable  $u^*(t)$ , we can use the system (9).

## 5 Numerical Examples

Here, we use our approach to obtain approximate optimal solutions of the following three nonlinear optimal control problems by solving problems (8) and/or (13) which is a linear programming (LP) or quadratic programming (QP) problem in MATLAB software. Moreover, comparisons of our solutions with the method that argued in [1] are included in Tables 1 and 2 respectively for two first examples.

**Example 5.1.** Consider the following nonlinear optimal control problem

$$\text{Minimize} \quad J = \frac{1}{2} \int_0^1 (\exp(-t)x(t) - 2tx^2(t))dt \quad (14)$$

Subject to

$$\begin{cases} \dot{x}(t) = tx(t) - t \ln(u(t) + t + 2), & t \in [0, 1], \quad u(t) \in [-1, 1] \\ \text{with BV conditions } x(0) = 0.9 \text{ and } x(1) = 0.2. \end{cases}$$

Table 1: comparison of  $J^*$  and CPU time between methods of Ex. 5.1

N=100	Our Method	Discretization Method [1]
Objective function ( $J^*$ )	0.00065	0.00676
CPU Time (sec)	24.090	28.318

By relations (4) and (5),

$$g(t) = \text{Min}_u \{-\ln(u(t) + t + 2) : u(t) \in [-1, 1]\} = -\ln(t + 3),$$

$$w(t) = \text{Max}_u \{-\ln(u(t) + t + 2) : u(t) \in [-1, 1]\} = -\ln(t + 1)$$

and hence

$$\beta(t) = w(t) - g(t) = \ln\left(\frac{t+3}{t+1}\right), \quad \text{for } t \in [0, 1].$$

Let  $N = 100$ , then  $\delta = s_j = \frac{1}{100}$ , for all  $j = 0, 1, \dots, 100$ . We obtain optimal solutions  $x_j^*$  and  $\lambda_j^*$ ,  $j = 0, 1, \dots, 100$  of this problem by solving corresponding problem (8), which is illustrated in Figures 1 and 2 respectively. In addition, the corresponding  $u_j^*$  for this example is

$$u_j^* = \exp(-\beta(s_j)\lambda_j^* - g(s_j)) - s_j - 2, \quad j = 0, 1, \dots, 100.$$

The optimal control  $u_j^*$ ,  $j = 0, 1, \dots, 100$  of problem (8) is shown in Figure 3. Here, The value of optimal solution of objective function is 0.00065.

**Example 5.2.** Consider the following nonlinear optimal control problem

$$\text{Minimize } J = \int_0^1 (|\sin(2\pi t)| - \exp(-t))x(t)dt \quad (15)$$

Subject to

$$\begin{cases} \dot{x}(t) = (t^5 - t^2 + t)x(t) - |u(t)|^3 \exp(\sin(2\pi t)), & t \in [0, 1], u(t) \in [-1, 1] \\ \text{with BV conditions } x(0) = 0.9 \text{ and } x(1) = 0.4. \end{cases}$$

By relations (4) and (5),

$$g(t) = \text{Min}_u \{-|u(t)|^3 \exp(\sin(2\pi t)) : u(t) \in [-1, 1]\} = -\exp(\sin(2\pi t)),$$

$$w(t) = \text{Max}_u \{-|u(t)|^3 \exp(\sin(2\pi t)) : u(t) \in [-1, 1]\} = 0$$

and hence,

$$\beta(t) = w(t) - g(t) = \exp(\sin(2\pi t)), \quad \text{for } t \in [0, 1].$$



Table 2: comparison of  $J^*$  and CPU time between methods of Ex. 5.2

N=100	Our Method	Discretization Method [1]
Objective function ( $J^*$ )	-0.0434	-0.0261
CPU Time (sec)	0.078	6.680

Let  $N = 100$ , then  $\delta = s_j = \frac{1}{100}$  for all  $j = 0, 1, \dots, 100$ . We obtain optimal solutions  $x_j^*$  and  $\lambda_j^*$ ,  $j = 0, 1, \dots, 100$  of this problem by solving corresponding problem (8) which is illustrated in Figures 4 and 5 respectively. In addition, the corresponding  $u_j^*$  for this example is

$$u_j^* = (-(\beta(s_j)\lambda_j^* + g(s_j)) \exp(-\sin(2\pi t)))^{\frac{1}{3}}, \quad j = 0, 1, \dots, 100.$$

The optimal controls  $u_j^*$  for  $j = 0, 1, \dots, 100$  is shown in Figure 6. Here, The value of optimal solution of objective function is  $-0.0435$ .

**Example 5.3.** Consider the following optimal control problem governed by differential inclusion:

$$\text{Minimize } J = \int_0^1 \sin(3\pi t)x(t)dt \quad (16)$$

Subject to

$$\begin{cases} \dot{x}(t) \in \{-\tan(\frac{\pi}{8}u^3(t) + t) : u(t) \in [0, 1]\} & t \in [0, 1], \\ \text{with BV conditions } x(0) = 1 \text{ and } x(1) = 0. \end{cases}$$

Here,

$$h(t, u(t)) = -\tan(\frac{\pi}{8}u^3(t) + t) \quad \text{and} \quad c(t) = \sin(3\pi t),$$

for  $(t, u(t)) \in [0, 1] \times [0, 1]$ . Thus by (4) and (5)

$$\begin{aligned} g(t) &= \text{Min}_u \{-\tan(\frac{\pi}{8}u^3(t) + t) : u(t) \in [0, 1]\} = -\tan(\frac{\pi}{8} + t) \\ w(t) &= \text{Max}_u \{-\tan(\frac{\pi}{8}u^3(t) + t) : u(t) \in [0, 1]\} = -\tan(t), \end{aligned}$$

and hence,

$$\beta(t) = w(t) - g(t) = \tan(\frac{\pi}{8} + t) - \tan(t).$$

Let  $N = 100$ , then  $\delta = s_j = \frac{1}{100}$ , for all  $j = 0, 1, \dots, 100$ . We obtain optimal solutions  $x_j^*$  and  $\lambda_j^*$ ,  $j = 0, 1, \dots, 100$  of this problem by solving

corresponding problem (13) which is illustrated in Figures 7 and 8 respectively. In addition, the corresponding  $u_j^*$  for this example is

$$u_j^* = \left(\frac{8}{\pi}(\tan^{-1}(-\beta(s_j)\lambda_j^* - g(s_j)) - s_j)\right)^{\frac{1}{3}}, \quad j = 0, 1, \dots, 100.$$

The optimal controls  $u_j^*$ , for  $j = 0, 1, \dots, 100$  is shown in Figure 9. Here, The value of optimal solution of objective function is 0.0235.

## 6 Conclusions

In this paper, we proposed a different approach for solving a class of nonlinear optimal control problems which have quadratic functionals and nonlinear dynamical systems. In our approach, the linear combination property of intervals (LCPI) is used to obtain the new corresponding problem which is a linear-quadratic optimal control problem. The new problem can be converted to a QP Problem by a simple discretization method. Also, our approach extended for solving a class of optimal control problems governed by differential inclusions (DI). By the approach of this paper we may solve a wide class of nonlinear optimal control problems.

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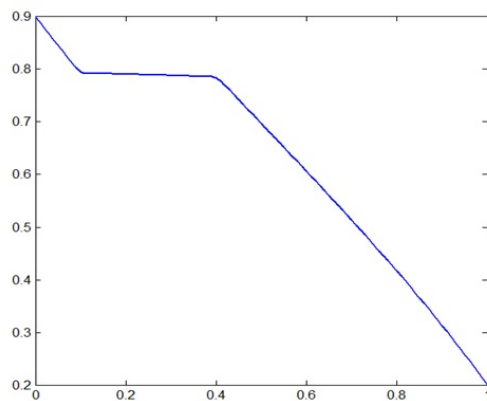


Figure 1. Optimal state of Ex. 5.1.

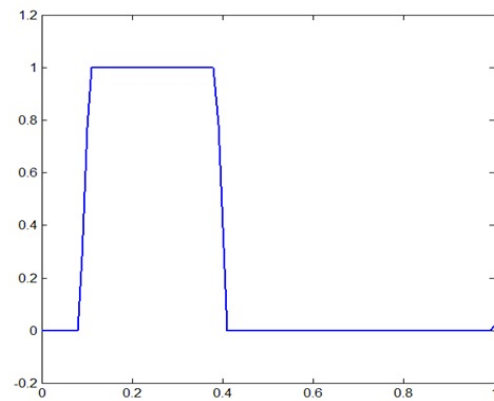


Figure 2. Optimal associated control of Ex. 5.1.

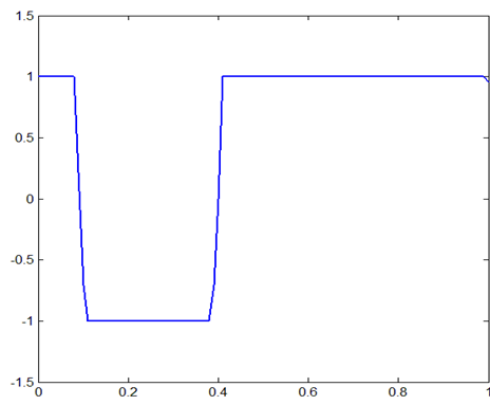


Figure 3. Optimal control of Ex. 5.1.

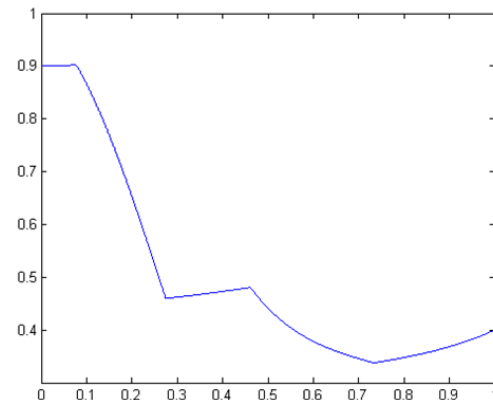


Figure 4. Optimal state of Ex. 5.2.

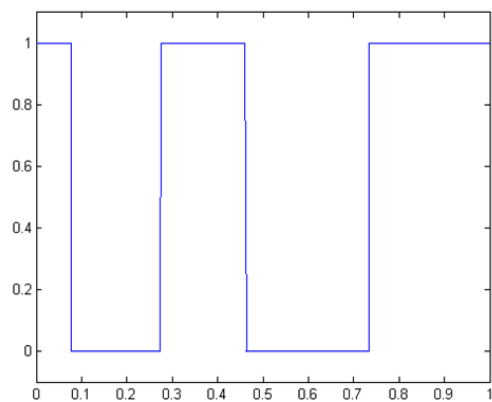


Figure 5. Optimal associated control of Ex. 5.2.

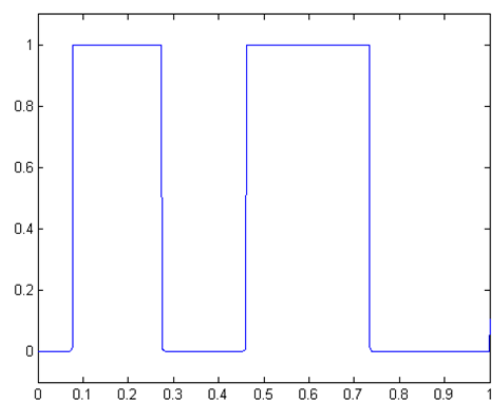


Figure 6. Optimal control of Ex. 5.2

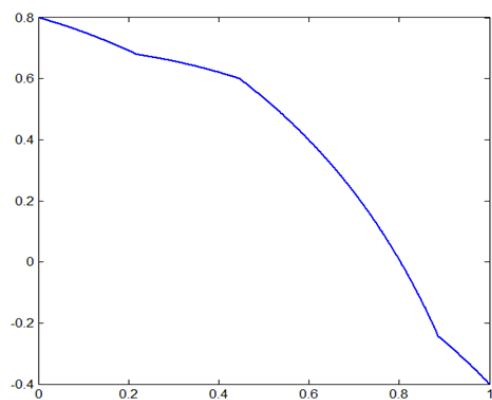


Figure 7. Optimal state of Ex. 5.3.

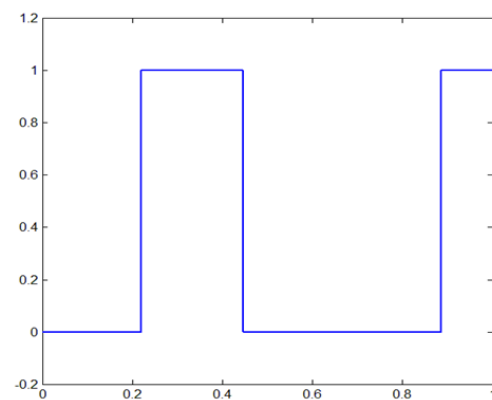


Figure 8. Optimal associated control of Ex. 5.3.

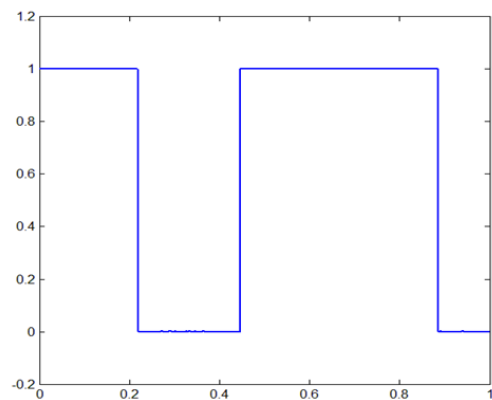


Figure 9. Optimal control of Ex. 5.3.