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On the structure of space-time and linear evolution equations

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Abstract

Let $\{T_v : v \in [0, c)\}$ the one-parameter semi-module of Lorentz operators on $\mathbb{R}^4$ with infinitesimal generator $\mathcal{T}$, where c is the constant speed of light in vacuum. In this paper the space-time of special theory of relativity is considered as affine and ordered vector space. We study the topology of space-time and we consider the velocity differential evolution equation $\left(\frac{d}{dv} - \mathcal{A}\right) y(v) = f(v)$ where $f : [0, c) \to \mathbb{R}^4$ is a given function and the corresponding linear time differential evolution equation, where $\mathcal{A}$ is the corresponding matrix of the infinitesimal generator $\mathcal{T}$.

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1 Introduction

In special theory of relativity if $\mathfrak{F}$ and $\mathfrak{F}'$ are two inertial frames of reference

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and $\mathfrak{F}'$ is moving relative to $\mathfrak{F}$ with constant velocity υ among the direction of x-axis, then the coordinates (x, y, z, t) and (x', y', z', t') of a pointevent relative to $\mathfrak{F}$ and $\mathfrak{F}'$ respectively are connected through the Lorentz transformation

$$x' = \frac{x-\upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-\frac{\upsilon}{c^2}x}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

with $|\upsilon| < c$, where c is the speed of light in vacuum.

As is well-known the above transformations are linear and consist an one-parameter group (the identity transformation is given for $\upsilon = 0$ and the inverse transformation replacing υ by $-\upsilon$).

Assuming $0 \leq \upsilon < c$, for the Lorentz transformations, we have a semi-module structure.

A semi-module $(\mathfrak{G}, \circ)$ is an assosiative and commutative groupoid, i.e. (i) for every $(a, b) \in \mathfrak{G} \times \mathfrak{G}$ there is a unique element $a \circ b \in \mathfrak{G}$, (ii) $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in \mathfrak{G}$ and (iii) $a \circ b = b \circ a$ for all $a, b \in \mathfrak{G}$.

In section 2 we describe the geometrical and ordering structure of space-time.

Following Weyl [1] an affine space is defined as a triple $(\mathfrak{S}, \mathfrak{V}, \varphi)$, where $\mathfrak{S}$ is a non-empty set, $\mathfrak{V}$ is a real (linear) vector space and $\varphi : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{V} : (x, y) \to \varphi(x, y)$ is a map such that: (i) for each $x \in \mathfrak{S}$ the map $\varphi_x : \mathfrak{S} \to \mathfrak{V} : y \to \varphi_x(y) := \varphi(x, y)$ is bijection and (ii) for all $x, y, z \in \mathfrak{S}$ it follows $\varphi(x, y) + \varphi(y, z) + \varphi(z, x) = 0 \in \mathfrak{V}$ (triangle property).

For more details in affine geometry and special relativity, and axiomatic approaches of space-time we refer to Efimov [2], Weyl [1] and Wilson and Lewis [3]; Carathéodory [4], Einstein et al. [5] and Reichenbach [6].

In general a non-empty set $\mathfrak{A}$ is called partially ordered if there is a relation $\emptyset \neq \prec \subseteq \mathfrak{A} \times \mathfrak{A}$ with the properties: (i) $x \prec x$ (reflexive), (ii) $x \prec y$ and $y \prec x$ implies $x = y$ (antisymmetric) and (iii) $x \prec y$ and $y \prec z$ implies $x \prec z$ (transitivity).

Let $\mathfrak{A}$ be a real vector space and $\mathfrak{P} \subseteq \mathfrak{A}$. The subset $\mathfrak{P}$ is called cone of vertex 0 if $\lambda\mathfrak{P} \subseteq \mathfrak{P}$ for all $\lambda > 0$. In particular $\mathfrak{P}$ is called convex cone of vertex 0 if $\mathfrak{P} + \mathfrak{P} \subseteq \mathfrak{P}$ and $\lambda\mathfrak{P} \subseteq \mathfrak{P}$ for all $\lambda > 0$. A convex cone of vertex 0 is proper if $\mathfrak{P} \cap (-\mathfrak{P}) = \{0\}$. Then the elements of the cone are called positive and the cone $\mathfrak{P}$ is positive cone of the ordered vector space $\mathfrak{A}$. When the cone $\mathfrak{P}$ is proper a partial order is defined as: $a \prec b$ if and only if $(b - a) \in \mathfrak{P}$.

We note that if $\mathfrak{A}$ is a normed vector space, then a cone $\mathfrak{P}$ is normal if satisfies the property: exists $\delta > 0$ such that for all $u, v \in \mathfrak{P}$, $\|u+v\| \geq \delta \cdot max\{\|u\|, \|v\|\}$.

Also in case of $\mathbb{R}^\kappa$ the dual cone $\mathfrak{P}^*$ of $\mathfrak{P}$ by Riesz's representation theorem is $\mathfrak{P}^* = \{u \in \mathbb{R}^\kappa : \langle u, x\rangle \geq 0, \text{ for all } x \in \mathfrak{P}\}$, where $\langle u, x\rangle := \sum_{j=0}^{\kappa-1} u_j x_j$ is the usual inner product in $\mathbb{R}^\kappa$.

For details on the theory of partially ordered spaces we refer to Krein and Rutman [7], Peressini [8] and Vulikh [9].

Let $\mathfrak{E}$ be a Banach space and $\mathcal{L}(\mathfrak{E}, \mathfrak{E})$ the space of all linear operators with domain $\mathfrak{E}$ and range $\mathfrak{E}$. If $\{T_t : t \geq 0\} \subseteq \mathcal{L}(\mathfrak{E}, \mathfrak{E})$ satisfy the conditions: (i) $T_t \circ T_s = T_{t+s}$ for $t, s \geq 0$, (ii) $T_0 = I$ and (iii) $\underset{t \to t_0}{\ell im}\, T_t(x) = T_{t_0}(x)$ for each $t_0 \geq 0$ and each $x \in \mathfrak{E}$, then $\{T_t : t \geq 0\}$ is called a semi-group of class (C_0). The notion of the infinitesimal generator $\mathcal{T}$ of the family $\{T_t : t \geq 0\}$ is defined by $\mathcal{T} := \underset{h \to 0^+}{\ell im} \frac{1}{h}(T_h - I)$. In case of semi-module of operators we have analogue notions (cf. Hille and Phillips [10]).

The semi-module property of the family of operators $\{T_\upsilon : 0 \leq \upsilon < c\}$ that we are intrested in, is given by $T_{\upsilon_1} \circ T_{\upsilon_2} = T_{\upsilon_1 \oplus \upsilon_2}$, where $\upsilon_1 \oplus \upsilon_2 := \frac{\upsilon_1 + \upsilon_2}{1 + \frac{\upsilon_1 \upsilon_2}{c^2}}$, with $\upsilon_1, \upsilon_2 \in [0, c)$. We recall that the subset $[0, c)$ of the real numbers $\mathbb{R}$ with the above operation $\oplus$ is usually refered as Einstein's numbers and the triple $((-c, c), \oplus, \odot)$ consist an algebraic field consernig operation $\oplus$ on $(-c, c)$ and a second operation $\odot$ on $(-c, c)$, defined by the formula $\upsilon_1 \odot \upsilon_2 := c \cdot tanh\left[tanh^{-1}\left(\frac{\upsilon_1}{c}\right) \cdot tanh^{-1}\left(\frac{\upsilon_2}{c}\right)\right]$ (cf. Baker [11]).

In section 3 we establish solution for the velocity differential evolution equation $\left(\frac{d}{d\upsilon} - \mathcal{A}\right) y(\upsilon) = f(\upsilon)$, where $f : [0, c) \to \mathbb{R}^4$ is a given function.

We conclude studying the corresponding time differential evolution equation $\left(\frac{d}{dt} - (\lambda + \mu\phi^2(t))\mathcal{A}\right) y(\phi(t)) = (\lambda + \mu\phi^2(t)) f(\phi(t))$ with ϕ a real linear map, λ, μ real constants and $\mathcal{A}$ is the corresponding matrix of the infinitesimal generator $\mathcal{T}$.

For classical notions and more details on functional analysis, semi-groups, semi-modules and differential evolution equations we refer to Hille and Phillips [10] and Yosida [12].

2 Geometrical and Topological Structure of space-time

The space-time can be concidered as affine space as well as a space with quadratic form. Indeed, Lorentz transformation leaves invariant the quadratic form $Q : \mathbb{R}^4 \to \mathbb{R}$, with $Q(x, y, z, t) := c^2t^2 - x^2 - y^2 - z^2$ and conversely assuming that the above quadratic form is invariant under linear transformations and combining the physical postulates of special relativity we find the Lorentz transformation. Thus space-time is affine and Minkowski space. Combining this structures we can introduce a partial order in space-time and therefore space-time is also a partially ordered vector space.

Let the real vector space $\mathbb{R}^4$ and the map $\varphi : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ defined by the formula $\varphi\left((x', y', z', t'), (x, y, z, t)\right) := \left(\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, y, z, \frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}\right) = (x', y', z', t')$, where $|v| < c$.

The map φ is well-defined and we consider the partial map $\varphi_{(x',y',z',t')} : \mathbb{R}^4 \to \mathbb{R}^4 : (x, y, z, t) \to \varphi_{(x',y',z',t')}(x, y, z, t) := \varphi((x', y', z', t'), (x, y, z, t))$. This map is injection and linear. Also its kernel is trivial. Thus its image coincidence with $\mathbb{R}^4$ and therefore the partial map is also surjection.

Let $(x_i, y_i, z_i, t_i) \in \mathbb{R}^4$, $i = 1, 2, 3$, $\chi := x_1 + x_2 + x_3$, $\psi := y_1 + y_2 + y_3$, $\zeta := z_1 + z_2 + z_3$, $\tau := t_1 + t_2 + t_3$ and $k := \sqrt{1 - \frac{v^2}{c^2}}$. The triangle property follows since the linear system $\begin{cases} \frac{1}{k}\chi - \frac{v}{k}\tau + 0\psi + 0\zeta = 0 \\ 0\chi + 0\tau + 1\psi + 0\zeta = 0 \\ 0\chi + 0\tau + 0\psi + 1\zeta = 0 \\ -\frac{v}{kc^2}\chi + \frac{1}{k}\tau + 0\psi + 0\zeta = 0 \end{cases}$ is homogeneous with nonzero determinant.

Hence, the triple $(\mathbb{R}^4, \mathbb{R}^4, \varphi)$ is an affine space.

The affine Klein geometry of space-time is $((\mathbb{R}^4, +), \mathbb{R}^4, \alpha)$, where the addition group of $\mathbb{R}^4$ is acting strictly transitivity to the carrier $\mathbb{R}^4$ of the above affine space, through the (right) action map $\alpha : \mathbb{R}^4 \times (\mathbb{R}^4, +) \to \mathbb{R}^4 : ((x', y', z', t'), (x, y, z, t)) \to \alpha((x', y', z', t'), (x, y, z, t)) := \ell_{(x,y,z,t)}(x', y', z', t') := \varphi^{-1}_{(x',y',z',t')}(x, y, z, t)$, where

$\ell_{(x,y,z,t)} : \mathbb{R}^4 \to \mathbb{R}^4 : (x', y', z', t') \to \ell_{(x,y,z,t)}(x', y', z', t') := \varphi^{-1}_{(x',y',z',t')}(x, y, z, t)$ belongs in $Aut(\mathbb{R}^4)$.

Let the Minkowski space $\mathfrak{M} := (\mathbb{R}^4, Q)$, with $Q : \mathbb{R}^4 \to \mathbb{R}$ the quadratic form $Q(x) := x_0^2 - x_1^2 - x_2^2 - x_3^2$, $x = (x_0, x_1, x_2, x_3)$, and the subset $\mathfrak{K} := \{y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4 : y_0 \geq 0 \text{ and } y_0^2 - y_1^2 - y_2^2 - y_3^2 \geq 0\}$ of $\mathbb{R}^4$.

Then $(\mathbb{R}^4, \succ)$ is a partially ordered vector space with set $\mathfrak{K}$ to be convex, proper, selfdual, closed and normal cone; and the partial order $\succ$ in $\mathbb{R}^4$ is defined as $y \succ x$ if and only if $(y - x) \in \mathfrak{K}$ if and only if $\big(y_0 \geq x_0$ and $(y_0 - x_0)^2 - \sum_{i=1}^3 (y_i - x_i)^2 \geq 0\big)$.

The assertion follows by classical set theoritic arguments and the Cauchy - Schwartz inequality. For example, let as prove that the cone is normal. If $y, x \in \mathfrak{K}$, then $y_0 x_0 \geq \sqrt{\sum_{i=1}^3 y_i^2}\sqrt{\sum_{i=1}^3 (-x_i)^2} \geq |\sum_{i=1}^3 y_i(-x_i)| \geq \sum_{i=1}^3 y_i(-x_i)$. Thus $y_0 x_0 + \sum_{i=1}^3 y_i x_i \geq 0$. Hence, $\sqrt{\sum_{i=0}^3 (y_i + x_i)^2} \geq \sqrt{\sum_{i=0}^3 y_i^2}$ and thus $||y + x|| \geq ||y||$. Similarly $||y + x|| \geq ||x||$. Hence exists $\delta = 1 > 0$ such that for all $y, x \in \mathfrak{K}$, $||y + x|| \geq 1 \cdot max\,\{||y||, ||x||\}$.

In general a cone $\mathfrak{K}$ is normal in a space $\mathfrak{E}$, if and only if for all functionals $f \in \mathfrak{E}^*$ it follows $f = g - h$, where $g, h \in \mathfrak{K}^*$ (cf. Krein [7]). In our case we have the space $\mathbb{R}^4$, the cone $\mathfrak{K}$ is normal and selfdual ($\mathfrak{K} = \mathfrak{K}^*$).

Thus, space-time of special theory of relativity is generating partially ordered space, i.e. for all $z \in \mathbb{R}^4$ there exist $x, y \in \mathfrak{K}$ such that $z = x - y$.

The element $e_1 := (1, 0, 0, 0) \in \mathbb{R}^4$ is also an element in the positive cone $\mathfrak{K}$. Then the open sphere $S(e_1, \frac{1}{3}) \subseteq \mathfrak{K}$, and therefore the interior set of the positive cone $\mathfrak{K}$ is not empty, i.e. $Int(\mathfrak{K}) \neq \emptyset$. Hence space-time is Krein space. Also because of the previous properties of the positive cone $\mathfrak{K}$ it follows that exists strong order unit, i.e. exists $e \in \mathfrak{K}$ such that for all $x \in \mathbb{R}^4$ exists $\lambda > 0$ such that $-\lambda e \leq x \leq \lambda e$. Actually every positive element is order unit since in a Krein space the concepts of a strong unit and of a strongly positive element are equivalent (cf. Vulikh [9]).

Let $\mathfrak{E}$ be an ordered vector space. The order topology τ_0 on $\mathfrak{E}$ is the finest locally convex topology τ on $\mathfrak{E}$ for which every order bounded subset of $\mathfrak{E}$ is τ-bounded. A neighborhood basis of 0 for τ_0 is the class of all convex, circled subsets of $\mathfrak{E}$ that absorb all order bounded subsets of $\mathfrak{E}$. If $\mathfrak{E}$ is a Banach space with strong topology τ_s ordered by a τ_s-closed positive cone $\mathfrak{P}$ containing an order unit e, then $\tau_s = \tau_0$ if and only if $\mathfrak{P}$ is normal (cf. Peressini [8]).

$\mathbb{R}^4$ is a Banach space with strong topology τ_d ordered by the closed positive and normal cone $\mathfrak{K}$ and is a Krein space. Hence if τ_0 is the corresponding order topology we have $\tau_d = \tau_0$. Hence, in space-time of special theory of relativity the order topology and the strong topology are coincidence.

The above results can be extend in the case of $\mathbb{R}^\kappa$, $\kappa \in \mathbb{N}$, $\kappa > 4$, with

$Q(x) := x_0^2 - \sum_{i=1}^{\kappa-1} x_i^2$. Also the same results hold in the Hilbert space of square summable sequences $\ell^2 := \left\{x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < +\infty\right\}$, with positive cone the subset $\mathfrak{K} := \{x \in \ell^2 : x_1 \geq 0 \text{ and } x_1^2 - \sum_{n=2}^{\infty} x_n^2 \geq 0\}$ (see Krein and Rutman [7] and Peressini [8]). Furthermore, for the order structure on space-time we refer in our paper [26]. This is where we studied the mean ergodic theorem for Lorentz operators introducing methods of functional analysis in space-time.

All the topologies suggested, since Zeeman suggested the fine topology, for Minkowski space-time $\mathfrak{M} := (\mathbb{R}^4, Q)$ of special relativity, with the exception of the order topology, are finer than the Euclidean topology and therefore Hausdorff. The fine topology on $\mathfrak{M}$ is defined to be the finest topology satisfying the property: "The topology on $\mathfrak{M}$ induces the 3-dimensional Euclidean topology on every space axis and the 1-dimensional Euclidean topology on every time axis". The fine topology enjoys the properties: (i) The fine topology is not locally homogeneous, and the light cone through any point can be deduced from the fine topology and (ii) The group of all homeomorphisms of the fine topology is generated by the inhomogeneous Lorentz group and dilatations. Also the fine topology is Hausdorff, is finer than the Euclidean topology, it is not normal; and although it is connected and locally connected it is not locally compact, nor does any point have a countable base of neighbourhouhds (cf. Zeeman [14]). The order topology is the one generated by the positive cone at the origin and its translates. More precisely a positive cone at the origin is defined by $\mathfrak{K} = \{x \in \mathfrak{M} : Q(x) > 0,\ x_0 > 0\}$ which gives rise to the usual order on $\mathfrak{M}$ as $x > y$ if and only if $(x - y) \in \mathfrak{K}$. Another positive cone at the origin is defined by $\mathfrak{L} = \{x \in \mathfrak{M} : Q(x) \geq 0,\ x_0 > 0\}$ which again defines another partial order on $\mathfrak{M}$ by $x >> y$ if and only if $(x - y) \in \mathfrak{L}$. For each $x \in \mathfrak{M}$ the translates of the positive cones $\mathfrak{K}$ and $\mathfrak{L}$ are $\mathfrak{K} + x = \{y + x : y \in \mathfrak{K}\}$ and $\mathfrak{L} + x = \{y + x : y \in \mathfrak{L}\}$. Also for each $x \in \mathfrak{M}$ the cones $\mathfrak{K}_x^* := (\mathfrak{K} + x) \cup \{x\}$ and $\mathfrak{L}_x^* := (\mathfrak{L} + x) \cup \{x\}$ are defined. The cones $\{\mathfrak{K}_x^* : x \in \mathfrak{M}\}$ generate a topology on $\mathfrak{M}$, the order topology or the Zeeman-order topology on $\mathfrak{M}$. This topology is not Hausdorff, it is not $\mathcal{T}_1$ but it is $\mathcal{T}_0$. It is also not compact though it is locally compact. It is connected and locally connected (cf. Nanda and Panda [15]). Also, we refer to Williams [16], Whiston [17], Zeeman [13], Nanda [18]-[21], Dossena [22], Agrawal and Shrivastava [23], Bălan [24], Bochers and Hegerfeldt [25] for other topologies (A-topology, t-topology and s-topology), and results and remarks for those topologies on space-time.

Finally, we note that the order topology we introduced earlier is not the same with the above order topology (Zeeman's order topology) and we should avoid confusion in terminology.

3 Differential Evolution Equations

Let the family of operators $T_\upsilon : \mathbb{R}^4 \to \mathbb{R}^4$, $0 \leq \upsilon < c$, corresponds to Lorentz transformations defined by the formula:

$$T_\upsilon(x,y,z,t) := \left(\frac{x-\upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}}, y, z, \frac{t-\frac{\upsilon}{c^2}x}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \right) = (x',y',z',t')$$

Note that operator T_υ is related with the partial map $\varphi_{(x',y',z',t')}$ introduced in the proof of the affine structure of space time.

Operator T_υ is positivity preserving for $x-\upsilon t \geq 0$, i.e. $(x,y,z,t) \in \mathfrak{K}$ and $x-\upsilon t \geq 0$ implies $T_\upsilon(x,y,z,t) = (x',y',z',t') \in \mathfrak{K}$.

Also T_υ is linear and relative to the usual orthocanonical base $\bar{e}$ of $\mathbb{R}^4$ it has coresponding matrix $A_\upsilon = (T_\upsilon : \bar{e}) = \begin{pmatrix} \frac{1}{k} & 0 & 0 & \frac{-\upsilon}{k} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\upsilon}{c^2 k} & 0 & 0 & \frac{1}{k} \end{pmatrix}$, where $k := \sqrt{1-\frac{\upsilon^2}{c^2}}$.
As is well-known if $\mathfrak{V}$ is a vector space with $dim_{\mathbb{R}}(\mathfrak{V}) < \infty$ then every linear operator $T : \mathfrak{V} \to \mathfrak{V}$ is continuous (equivelantly bounded). Therefore operator T_υ is continuous, and $||T_\upsilon|| := sup\,\{||T_\upsilon(u)||_{\mathbb{R}^4} : ||u||_{\mathbb{R}^4} \leq 1\} < +\infty$ is the norm of the bounded operator T_υ.

The eigenvalues of matrix A_υ are $\mu_0 = 1$ (double), $\mu_1 = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\left(1-\frac{\upsilon}{c}\right)$ and $\mu_2 = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\left(1+\frac{\upsilon}{c}\right)$, i.e. $\mu_j \in \mathbb{R}$, $j = 0,1,2$. Also the matrix norm of A_υ is $\|A_\upsilon\|_\infty := max\left\{\sum_{j=1}^{4} |a_{ij}| : 1 \leq i \leq 4\right\} = \frac{(1+\upsilon)c}{\sqrt{c^2-\upsilon^2}} < +\infty$.

The infinitesimal generator $\mathcal{T}$ of the one-parameter semi-module of class

(C_0) of Lorentz transformations has corresponding matrix

$$\mathcal{A} := \underset{h\to 0^+}{\ell im}\,\frac{1}{h}(A_h - I) = \underset{h\to 0^+}{\ell im}\,\frac{1}{h}\begin{pmatrix} \frac{1}{\sqrt{1-\frac{h^2}{c^2}}} - 1 & 0 & 0 & \frac{-h}{\sqrt{1-\frac{h^2}{c^2}}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-h}{c^2\sqrt{1-\frac{h^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1-\frac{h^2}{c^2}}} - 1 \end{pmatrix}.$$

Calculating the limits by L'Hôspital's rule we find $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{c^2} & 0 & 0 & 0 \end{pmatrix}$ and $\mathcal{A}^2 = \mathcal{A}\cdot\mathcal{A} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c^2} \end{pmatrix}$.

Inductively we conclude $\mathcal{A}^\kappa = \begin{cases} \frac{1}{c^{\kappa-1}}\mathcal{A}, \text{ if } \kappa \text{ is odd} \\ \\ \frac{1}{c^{\kappa-2}}\mathcal{A}^2, \text{ if } \kappa \text{ is even} \end{cases}$ (cf. [26]).

The eigenvalues of matrix $\mathcal{A}$ are $\lambda_0 = 0$ (double), $\lambda_1 = -\frac{1}{c}$ and $\lambda_2 = \frac{1}{c}$. Thus $\lambda_j \in \mathbb{R}$, $j = 0, 1, 2$, and of $\mathcal{A}^2$ are $\nu_0 = 0$ (double) and $\nu_1 = \frac{1}{c^2}$ (double).

Also by elementary calculation we conferm that $\mathcal{A}\cdot(x,y,z,t)^{tr} = \mathcal{T}(x,y,z,t)$, for all $(x,y,z,t) \in \mathbb{R}^4$.

3.1 Velocity Differential Evolution Equation

Let the differential evolution equation

$$\left(\frac{d}{dv} - \mathcal{A}\right) y(v) = f(v), \tag{3.1}$$

where f is a given $\mathbb{R}^4$-valued function in $[0, c) \subseteq \mathbb{R}^+$.

Let the functions $f, y : [0, c) \to \mathbb{R}^4$, with $f(v) := (f_1(v), f_2(v), f_3(v), f_4(v))^{tr}$ and $y(v) := (y_1(v), y_2(v), y_3(v), y_4(v))^{tr}$ are sufficiently continuous, differentiable and integrable respectively.

Firstly we consider the homogeneous differential equation

$$\frac{d}{dv}y(v) = \mathcal{A}y(v)$$

$$\Leftrightarrow \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{c^2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \Leftrightarrow \begin{cases} \dot{y}_1 = -y_4 \\ \dot{y}_2 = 0 \\ \dot{y}_3 = 0 \\ \dot{y}_4 = -\frac{1}{c^2} y_1 \end{cases}$$

where for simplicity we have replace $\frac{d}{d\upsilon}$ by the notation of a simple dot and we have omit the argument υ.

From the second and third equation immediately we have $y_2 = \xi_1$ and $y_3 = \xi_2$, with constants $\xi_1, \xi_2 \in \mathbb{R}$.

From the fourth equation applying $\frac{d}{d\upsilon}$ in both sides we get $\ddot{y}_4 = -\frac{1}{c^2}\dot{y}_1$. Thus from the first equation we have $\ddot{y}_4 - \frac{1}{c^2} y_4 = 0$ which is a second order homogeneous with constant coefficients. Calculating the characteristic polynomial we find that $e^{\frac{1}{c}\upsilon}$ and $e^{-\frac{1}{c}\upsilon}$ are two linear independent solutions and therefore $y_4 = \xi_3 e^{-\frac{1}{c}\upsilon} + \xi_4 e^{\frac{1}{c}\upsilon}$, with $\xi_3, \xi_4 \in \mathbb{R}$. Then by the fourth equation follows $y_1 = -c^2 \dot{y}_4$ and thus $y_1 = \xi_3 c e^{-\frac{1}{c}\upsilon} - \xi_4 c e^{\frac{1}{c}\upsilon}$.

Therefore,

$$y_{hom}(\upsilon) = \begin{pmatrix} \xi_3 c e^{-\frac{1}{c}\upsilon} - \xi_4 c e^{\frac{1}{c}\upsilon} \\ \xi_1 \\ \xi_2 \\ \xi_3 e^{-\frac{1}{c}\upsilon} + \xi_4 e^{\frac{1}{c}\upsilon} \end{pmatrix} = \xi_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \xi_3 e^{-\frac{1}{c}\upsilon} \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix} + \xi_4 e^{\frac{1}{c}\upsilon} \begin{pmatrix} -c \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{R}$.

Now for the non-homogeneous equation we have

$$\frac{d}{d\upsilon} y(\upsilon) - \mathcal{A} y(\upsilon) = f(\upsilon)$$

$$\Leftrightarrow \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{c^2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \Leftrightarrow \begin{cases} \dot{y}_1 + y_4 = f_1 \\ \dot{y}_2 = f_2 \\ \dot{y}_3 = f_3 \\ \dot{y}_4 + \frac{1}{c^2} y_1 = f_4 \end{cases}$$

Applying the well-known method of variation of parameters we find a special solution of the above system and the general solution is given by

$$y(\upsilon) = y_{hom}(\upsilon) + y_{spec}(\upsilon) = (y_1(\upsilon), y_2(\upsilon), y_3(\upsilon), y_4(\upsilon))^{tr}$$

with

$$y_1(\upsilon) = \xi_1 c e^{-\frac{1}{c}\upsilon} - \xi_2 c e^{\frac{1}{c}\upsilon} - \frac{1}{2c} \int \left[e^{-\frac{1}{c}\upsilon} \int e^{\frac{1}{c}\upsilon} f_1(\upsilon) d\upsilon - e^{\frac{1}{c}\upsilon} \int e^{-\frac{1}{c}\upsilon} f_1(\upsilon) d\upsilon \right] d\upsilon$$

$$-\frac{1}{2}\int\left[e^{-\frac{1}{c}\upsilon}\int e^{\frac{1}{c}\upsilon}f_4(\upsilon)d\upsilon+e^{\frac{1}{c}\upsilon}\int e^{-\frac{1}{c}\upsilon}f_4(\upsilon)d\upsilon\right]d\upsilon+\int f_1(\upsilon)d\upsilon,$$

$$y_2(\upsilon)=\int f_2(\upsilon)d\upsilon,\quad y_3(\upsilon)=\int f_3(\upsilon)d\upsilon,$$

$$y_4(\upsilon)=\xi_1e^{-\frac{1}{c}\upsilon}+\xi_2e^{\frac{1}{c}\upsilon}+\frac{1}{2c}\left[e^{-\frac{1}{c}\upsilon}\int e^{\frac{1}{c}\upsilon}f_1(\upsilon)d\upsilon-e^{\frac{1}{c}\upsilon}\int e^{-\frac{1}{c}\upsilon}f_1(\upsilon)d\upsilon\right]$$
$$+\frac{1}{2}\left[e^{-\frac{1}{c}\upsilon}\int e^{\frac{1}{c}\upsilon}f_4(\upsilon)d\upsilon+e^{\frac{1}{c}\upsilon}\int e^{-\frac{1}{c}\upsilon}f_4(\upsilon)d\upsilon\right].$$

For the final form of the above expressions we have also apply integration by parts.

Similarly we can solve the differential evolution equations $\frac{d}{d\upsilon}y(\upsilon)=\mathcal{A}^2y(\upsilon)$ and $\frac{d}{d\upsilon}y(\upsilon)=\mathcal{A}^2y(\upsilon)+f(\upsilon)$. In this case calculations are made easier and we find $y_{hom}(\upsilon)=\begin{pmatrix}\frac{\xi_1}{c^2}e^{\upsilon}\\ \xi_2\\ \xi_3\\ \frac{\xi_4}{c^2}e^{\upsilon}\end{pmatrix}$ and $y(\upsilon)=\begin{pmatrix}e^{\frac{1}{c^2}\upsilon}\left(\int e^{-\frac{1}{c^2}\upsilon}f_1(\upsilon)d\upsilon+\xi_1\right)\\ \int f_2(\upsilon)d\upsilon\\ \int f_3(\upsilon)d\upsilon\\ e^{\frac{1}{c^2}\upsilon}\left(\int e^{-\frac{1}{c^2}\upsilon}f_4(\upsilon)d\upsilon+\xi_4\right)\end{pmatrix}$, with constants $\xi_1,\xi_2,\xi_3,\xi_4\in\mathbb{R}$, respectively.

Hence we can solve the equation $\frac{d}{d\upsilon}y(\upsilon)=\mathcal{A}^{\kappa}y(\upsilon)+f(\upsilon)$, for all $\kappa\in\mathbb{N}$.

If we concider x',y',z',t' as functions of υ and keep the initial variables x,y,z,t fixed, then $\frac{dx'}{d\upsilon}=-\frac{1}{1-\frac{\upsilon^2}{c^2}}t'$, $\frac{dy'}{d\upsilon}=0$, $\frac{dz'}{d\upsilon}=0$ and $\frac{dt'}{d\upsilon}=-\frac{\frac{1}{c^2}}{1-\frac{\upsilon^2}{c^2}}x'$. Setting $\frac{d}{d\zeta}:=\left(1-\frac{\upsilon^2}{c^2}\right)\frac{d}{d\upsilon}$ we deduce $\frac{dx'}{d\zeta}=-t'$, $\frac{dy'}{d\zeta}=0$, $\frac{dz'}{d\zeta}=0$ and $\frac{dt'}{d\zeta}=-\frac{1}{c^2}x'$. Thus, $\frac{d}{d\zeta}\begin{pmatrix}x'\\ y'\\ z'\\ t'\end{pmatrix}=\begin{pmatrix}0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -\frac{1}{c^2}&0&0&0\end{pmatrix}\begin{pmatrix}x'\\ y'\\ z'\\ t'\end{pmatrix}$, and therefore we have a physical explonation for the homogeneus evolution equation coresponds to (3.1). In particular, setting $\upsilon=c\cdot sin\vartheta$ then $\frac{d\upsilon}{d\vartheta}=c\cdot cos\vartheta$ and from $\frac{d\upsilon}{d\zeta}=1-\frac{\upsilon^2}{c^2}$ it follows $\ell og\left(\frac{1+\frac{\upsilon}{c}}{1-\frac{\upsilon}{c}}\right)=\frac{2}{c}\zeta+\eta$, with constant $\eta\in\mathbb{R}$. If we consider initial value $\upsilon=\upsilon_0$ at $\zeta=0$, then $\eta=\ell og\left(\frac{1+\frac{\upsilon_0}{c}}{1-\frac{\upsilon_0}{c}}\right)$ and therefore $\frac{1+\frac{\upsilon}{c}}{1-\frac{\upsilon}{c}}=e^{\frac{2}{c}\zeta}\left(\frac{1+\frac{\upsilon_0}{c}}{1-\frac{\upsilon_0}{c}}\right)$. Hence $\upsilon=c\cdot tanh\left(\frac{1}{c}\zeta\right)$, $0\leq\upsilon<c$. Also from equations $\frac{dx'}{d\zeta}=-t'$ and $\frac{dt'}{d\zeta}=-\frac{1}{c^2}x'$ we find the solutions $x'(\zeta)=A_1sinh\left(\frac{1}{c}\zeta\right)+A_2cosh\left(\frac{1}{c}\zeta\right)$ and $t'(\zeta)=-\frac{1}{c}\left(A_1cosh\left(\frac{1}{c}\zeta\right)+A_2sinh\left(\frac{1}{c}\zeta\right)\right)$, with constants $A_1,A_2\in\mathbb{R}$. Since for $\upsilon=0$ it is $x=x'$ and $t=t'$ we have $A_1=-ct$ and $A_2=x$. Thus we find the well-known pseudo-euclidean rotation $x'(\zeta)=-cxsinh\left(\frac{1}{c}\zeta\right)+xcosh\left(\frac{1}{c}\zeta\right)$

and $t'(\zeta) = -\frac{1}{c}xsinh\left(\frac{1}{c}\zeta\right) + tcosh\left(\frac{1}{c}\zeta\right)$; and finally $cosh\left(\frac{1}{c}\zeta\right) = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ and $sinh\left(\frac{1}{c}\zeta\right) = \frac{\frac{\upsilon}{c}}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$. Combining the last results inverse follows the Lorentz transformation (see Landau and Lifshitz in [27]).

3.2 Time Differential Evolution Equation

In this section we shall deal the time differential evolution equation corresponds to velocity differntial evolution equation (3.1).

Writing $\frac{d}{d\upsilon} = \frac{d}{dt}\frac{dt}{d\upsilon}$ and since $t = \frac{t'+\frac{\upsilon}{c^2}x'}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ we have $\frac{dt}{d\upsilon} = \frac{\frac{1}{c^2}}{1-\frac{\upsilon^2}{c^2}}x$. Therefore $\frac{d}{d\upsilon} = \frac{\frac{1}{c^2}}{1-\frac{\upsilon^2}{c^2}}x\frac{d}{dt}$.

Also from $t = \frac{t'+\frac{\upsilon}{c^2}x'}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ it follows $\left(x'^2 + c^2t^2\right)\upsilon^2 + 2c^2x't'\upsilon + c^4\left(t'^2 - t^2\right) = 0$.

The discriminant of the above quadratic equation with respect to υ is equal to $4c^4t^2\left[x'^2 - c^2\left(t'^2 - t^2\right)\right]$ and since $x'^2 - c^2t'^2 = x^2 - c^2t^2$ we find that the discriminant equals to $4c^4t^2x^2 \geq 0$, for all t and for all x. Hence $\upsilon = \frac{c^2(-x't'\pm xt)}{x'^2+c^2t^2}$.

Then, equation (3.1) becomes

$$\left(\frac{d}{dt} - \frac{c^2 - \psi^2(t)}{x}\mathcal{A}\right)y(\psi^2(t)) = \frac{c^2 - \psi(t)}{x}f(\psi(t))$$

with $\psi(t) = c^2\frac{-x't'\pm xt}{x^2+c^2t'^2}$ a linear map with respect to t and $\mathcal{A}$ is the matrix of the infinitesimal generator of the semi-module of Lorentz transforamtions.

Thus we shall study the differential equation

$$\left(\frac{d}{dt} - (\lambda + \mu\phi^2(t))\mathcal{A}\right)y(\phi(t)) = (\lambda + \mu\phi^2(t))f(\phi(t))$$

with $\phi(t)$ a linear map.

Setting $\phi(t) = \tau$ and since ϕ is linear it follows $\frac{d\phi(t)}{dt} = \tilde{\tau} = const.$, and thus $\frac{d}{dt} = \tilde{\tau}\frac{d}{d\tau}$.

Hence

$$\left(\frac{d}{d\tau} - (a_0 + b_0\tau^2)\mathcal{A}\right)y(\tau) = (a_0 + b_0\tau^2)f(\tau), \tag{3.2}$$

where $a_0 = \frac{\lambda}{\tilde{\tau}}$, $b_0 = \frac{\mu}{\tilde{\tau}}$ are constants.

Then, if the functions $f, y : \mathbb{R} \to \mathbb{R}^4$, $f(\tau) := (f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau))^{tr}$ and $y(\tau) := (y_1(\tau), y_2(\tau), y_3(\tau), y_4(\tau))^{tr}$ are sufficiently continuous, differentiable and integrable respectively, for the linear equation (3.2) we have

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & (a_0+b_0\tau^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_0+b_0\tau^2}{c^2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = (a_0+b_0\tau^2) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

$$\Leftrightarrow \left\{ \begin{aligned} \dot{y}_1 + (a_0+b_0\tau^2) y_4 &= (a_0+b_0\tau^2) f_1 \\ \dot{y}_2 &= (a_0+b_0\tau^2) f_2 \\ \dot{y}_3 &= (a_0+b_0\tau^2) f_3 \\ \dot{y}_4 + \tfrac{a_0+b_0\tau^2}{c^2} y_1 &= (a_0+b_0\tau^2) f_4 \end{aligned} \right.$$

where for simplicity we have replace $\frac{d}{d\tau}$ by the notation of a simple dot and we have omit the argument τ.

Applying $d/d\tau$ in both sides of the first equation and subtitution of y_4 by the first and of $\dot{y}_4$ by the fourth equation yields

$$\ddot{y}_1 - \frac{b_0}{a_0+b_0\tau^2}\dot{y}_1 - \frac{(a_0+b_0\tau^2)^2}{c^2} y_1 = (a_0+b_0\tau^2)\dot{f}_1 - (a_0+b_0\tau^2)^2 f_4 \qquad (3.3)$$

The above differential equation (3.3) with nonconstant coefficients can be transformed into a differential equation with constant coefficients.

Indeed, for $p(\tau) = -\frac{2b_0}{a_0+b_0\tau^2}$ and $q(\tau) = -\frac{(a_0+b_0\tau^2)^2}{c^2}$ the quantity $-\frac{q'(\tau)+2p(\tau)q(\tau)}{2\sqrt{-q(\tau)}}$ is constant. Thus the homogeneus differential equation can be written in the form $\frac{d^2 y_1}{d\xi} - y_1 = 0$ with $\xi = \int \sqrt{-q(\tau)}d\tau = \frac{a_0}{c}\tau + \frac{b_0}{3c}\tau^3$.

Finally, the solution of the system is

$$y_1(\tau) = \left\{ \zeta_1 + \frac{1}{2c}\int e^{-\frac{a_0}{c}\tau - \frac{b_0}{3c}\tau^3}[f_1'(\tau) - (a_0+b_0\tau^2)f_4(\tau)]d\tau \right\} e^{\frac{a_0}{c}\tau + \frac{b_0}{3c}\tau^3}$$

$$+ \left\{ \zeta_2 + \frac{1}{2c}\int e^{\frac{a_0}{c}\tau + \frac{b_0}{3c}\tau^3}[f_1'(\tau) - (a_0+b_0\tau^2)f_4(\tau)]d\tau \right\} e^{-\frac{a_0}{c}\tau - \frac{b_0}{3c}\tau^3}$$

$$y_2(\tau) = \int (a_0+b_0\tau^2) f_2(\tau) d\tau, \quad y_3(\tau) = \int (a_0+b_0\tau^2) f_3(\tau) d\tau,$$

$$y_4(\tau) = \int (a_0+b_0\tau^2)\Bigg[f_4(\tau) - \left\{ \zeta_1 + \frac{1}{2c}\int e^{-\frac{a_0}{c}\tau - \frac{b_0}{3c}\tau^3} \right.$$

$$\left. \cdot [f_1'(\tau) - (a_0+b_0\tau^2)f_4(\tau)]d\tau \right\} e^{\frac{a_0}{c}\tau + \frac{b_0}{3c}\tau^3} + \left\{ \zeta_2 + \frac{1}{2c}\int e^{\frac{a_0}{c}\tau + \frac{b_0}{3c}\tau^3} \right.$$

$$\left. \cdot [f_1'(\tau) - (a_0+b_0\tau^2)f_4(\tau)]d\tau \right\} e^{-\frac{a_0}{c}\tau - \frac{b_0}{3c}\tau^3} \Bigg] d\tau$$

with ζ_1, ζ_2 real constants, and we note that for the matrix $A(\tau) := (a_0 + b_0\tau^2) \cdot \mathcal{A}$ holds the commutative property $A(\tau_1)A(\tau_2) = A(\tau_2)A(\tau_1)$, for all τ_1, τ_2 and we have apply the method of variation of parameters.

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