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A comparative study of a class of implicit multi-derivative methods for the numerical solution of second-order ordinary differential equations

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Abstract

In this paper, we considered the development, analysis and implementation of a class of multi-derivative methods for solving second order ordinary differential equations.

These multi derivatives methods incorporated more analytical properties of the differential equations into the conventional implicit linear multistep method. The step -size (k) and the order of the derivatives (l) methods have been varied to ensure accuracy and efficiency in the methods. The basic properties of these methods were analyzed and the results show that methods are accurate, consistent and zero stable. Comparative studies of the methods were carried out to determine the effect of increasing the step size (k) and the order of the derivatives (l) using some second order ODES.

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The result shows that the methods are more efficient and accurate when the step-size (k) and order of derivative are increased.

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1 Introduction

The formulation of physical phenomenon in engineering, sciences and economics often to lead to differential equations. Though these problems exists in theory and principles, their mathematical formulation leads to differential equations .According to Ross, Auzinger et'al, and Courant in [1-3]; the objects involves obeys certain physical and chemical laws that has to do with rate of change. Awoyemi [4] posited that differential equations occur in connection with mathematical description of some problems that arise in various branches of science and social science such as Mechanics, Chemistry, Biology and Economics .Differential equations constitute a large and important aspect of today's mathematics. Ordinary differential equation involves the derivatives of one or more of the dependent variables with respect to a single independent variable, Ross [5]. Only a few of these differential equations have analytical solution hence the resort to numerical method or approximation.

A differential equation together with an initial condition is called initial value problem (IVP)

The general first order initial value problem is of the form:

$$y' = f(x, y), \, y(x_0) = y_0, \; x \in [a, b] \tag{1}$$

1.1 Linear Multistep Method LMM

According to Lambert (see [6, 7]), the general linear k-step Multistep Method (LMM) for first order ordinary differential equation is given as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (2)$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$

where α_j and β_j are the parameters to be determined and α_j and $\beta_j \neq 0$

When$\beta_k = 0$, the method is explicit and implicit if $\beta_k \neq 0$.

In our work, we shall consider the development of methods for the solution general second order ordinary differential equations when k = 2, 3, 4.

The LMM for the integration of second order ordinary differential equation is of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (3)$$

We shall investigate the inclusion of more analytical properties of the differential equation by way of considering more derivative properties of the differential equation. The study also attempts to determine the effect of increasing the order of the derivatives as well as varying the step-size of the Linear Multistep Method. This we did by reformulating the method (3) in the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = \sum_{i=0}^{l}h^i\sum_{j=0}^{k}\beta_{ij} y_{n+j}^{(i)}, \qquad \alpha_k = 1 \qquad (4)$$

where l is the order of derivatives.

The LMM (3) involves more derivatives properties of the differential equations. The aim of the study is to compare the accuracy and stability of some implicit multiderivative multistep methods.

2 Methodology

The local truncation error of the method (3) is of the form

$$T_{n+k} = \sum_{j=0}^{k} \alpha_j y_{n+j} - \sum_{i=2}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} \tag{5}$$

where l is the order of the derivatives of y_{n+j}.

Adopting Taylor series expansion of the variables $y_{n+j}^{(i)}$, $j = 0(1)k$ and $i = o(1)L$ given as

$$y_{n+j}^{(i)} = \sum_{j=0}^{\infty} \frac{(jh)^r y^{r+i}}{r!}, \, j = 1(1)m \tag{6}$$

Expanding equation (4) and combining terms in equal powers of h , we have:

$$\begin{aligned} T_{n+k} &= C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + ... \\ &+ C_p h^p y^p(x_n) + C_{p+1} h^{p+1} y^{p+1}(x_n) + C_{p+2} h^{p+2} y^{p+2}(\xi) \end{aligned}$$

where

$$\begin{aligned} C_q = \frac{1}{q!}\left[-\sum_{j=0}^{k} j^q \alpha_j \right] + \frac{1}{(q-2)!} \sum_{j=0}^{k} j^{q-2} \beta_{2j} \\ + \frac{1}{(q-3)!} \sum_{j=0}^{k} j^{q-3} \beta_{3j} + \frac{1}{(q-4)!} \sum_{j=4}^{k} j^{q-4} \beta_{4j} \end{aligned} \tag{7}$$

The method (3) may be written in the form

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = \sum_{i=i_0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^i \tag{8}$$

Then we have

$$y_{n+k} = \sum_{i=i_0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^i - \sum_{j=0}^{k-1} \alpha_j y_{n+j} \tag{9}$$

If β_{ij} =0 method is explicit and implicit if $\beta_{ij} \neq 0$.

2.1 Derivation of Specific methods

2.1.1 Two step second derivative method ($\boldsymbol{k=2,l=2}$)

Equation (8) becomes

$$y_{n+2} = \sum_{i=i_0}^{2} h^i \sum_{j=0}^{2} \beta_{ij} y_{n+j}^{2} - \sum_{j=0}^{1} \alpha_j y_{n+j} \tag{10}$$

$$y_{n+2} = h^2\left[\beta_{20} y_n^{(2)} + \beta_{21} y_{n+1}^{(2)} + \beta_{22} y_{n+2}^{(2)}\right] - \alpha_0 y_n - \alpha_1 y_{n+1} \tag{11}$$

Adopting Taylor's expansion in (9) and substituting the results and collecting in equal powers of h yield the following system of equations given as

$$-\alpha_0 - \alpha_1 = 1, \quad -\alpha_1 = 2,$$

$$\frac{-\alpha_1}{2} + \beta_{20} + \beta_{21} + \beta_{22} = 2 \quad , \quad \frac{-\alpha_1}{6} + \beta_{21} + 2\beta_{22} = 4/3,$$

$$\frac{-\alpha_1}{24} + \frac{\beta_{21}}{2} + 2\beta_{22} = 2/3, \qquad \frac{-\alpha_1}{120} + \frac{\beta_{21}}{6} + \frac{4\beta_{22}}{3} = 4/15 \tag{12}$$

Solving the above set of equations yields

$$\beta_{20} = 1/12,\ \beta_{21} = 5/6,\quad \beta_{22} = 1/12,\ \ \alpha_0 = 1 \ ,\ \alpha_1 = -2$$

Substituting the above values into equation (11) and simplifying we obtained the two step second derivative method of the form

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{12}\left(y_{n+2}^{(2)} + 10y_{n+1}^{(2)} + y_n^{(2)}\right) \tag{13}$$

2.1.2 Two- step third derivative method ($\boldsymbol{k=2,l=3}$)

From (9) we have,

$$\begin{aligned} y_{n+2} &= h^2\left[\beta_{20} y_n^{(2)} + \beta_{21} y_{n+1}^{(2)} + \beta_{22} y_{n+2}^{(2)}\right] \\ &\quad + h^3\left[\beta_{30} y_n^{(3)} + \beta_{31} y_{n+1}^{(3)} + \beta_{32} y_{n+2}^{(3)}\right] - \alpha_0 y_n - \alpha_1 y_{n+1} \end{aligned} \tag{14}$$

By adopting Taylor's series expansion of y_{n+j}, $y_{n+j}^{(i)}$ in (14) and combining terms in equal powers of h to obtain the following set of equations

$$-\alpha_0-\alpha_1=1,\quad -\alpha_1=2,\quad \frac{-\alpha_1}{2}+\beta_{20}+\beta_{21}+\beta_{22}=2$$

$$\frac{-\alpha_1}{6}+\beta_{21}+2\beta_{22}+\beta_{30}+\beta_{31}+\beta_{32}=4/3$$

$$\frac{-\alpha_1}{24}+\frac{\beta_{21}}{2}+2\beta_{22}+\beta_{31}+2\beta_{32}=2/3$$

$$\frac{-\alpha_1}{120}+\frac{\beta_{21}}{6}+\frac{4\beta_{22}}{3}+\frac{\beta_{31}}{2}+2\beta_{32}=4/15$$

$$\frac{-\alpha_1}{720}+\frac{\beta_{21}}{24}+\frac{2\beta_{22}}{3}+\frac{\beta_{31}}{6}+\frac{4}{3}\beta_{32}=4/45$$

$$\frac{-\alpha_1}{5040}+\frac{\beta_{21}}{120}+\frac{4\beta_{22}}{15}+\frac{\beta_{31}}{24}+\frac{2}{3}\beta_{32}=8/315 \tag{14}$$

Solving the above set of equations gives

$$\alpha_0=1\ ,\ \alpha_1=-2,$$

$$\beta_{20}=2/15,\beta_{21}=11/15,\beta_{22}=2/15,\beta_{30}=1/40,\beta_{31}=0,\beta_{32}=-1/40$$

Substituting these values into (14) yields the scheme

$$y_{n+2}=2y_{n+1}-y_n+\frac{h^2}{15}\left(2y_{n+2}^{(2)}+11y_{n+1}^{(2)}+2y_n^{(2)}\right)-\frac{h^3}{40}\left(y_{n+2}^{(3)}-y_n^{(3)}\right) \tag{15}$$

2.1.3 Two-step fourth-derivative method $(k=2,l=4)$

Using (9), we have

$$\begin{aligned}y_{n+2}&=h^2\left[\beta_{20}y_n^{(2)}+\beta_{21}y_{n+1}^{(2)}+\beta_{22}y_{n+2}^{(2)}\right]+h^3\left[\beta_{30}y_n^{(3)}+\beta_{31}y_{n+1}^{(3)}+\beta_{32}y_{n+2}^{(3)}\right]\\&\quad+h^4\left[\beta_{40}y_n^{(4)}+\beta_{41}y_{n+1}^{(4)}+\beta_{42}y_{n+2}^{(4)}\right]-\alpha_0y_n-\alpha_1y_{n+1}\end{aligned} \tag{16}$$

Again, by adopting Taylor's series expansion on (16) and combining terms in equal powers of h , we obtained the following set of equations

$$-\alpha_0-\alpha_1=1,\quad -\alpha_1=2,$$

$$\frac{-\alpha_1}{2}+\beta_{20}+\beta_{21}+\beta_{22}=2$$

$$\frac{-\alpha_1}{6} + \beta_{21} + 2\beta_{22} + \beta_{30} + \beta_{31} + \beta_{32} = 4/3$$

$$\frac{-\alpha_1}{24} + \frac{1}{2}\beta_{21} + 2\beta_{22} + \beta_{31} + 2\beta_{32} + \beta_{40} + \beta_{41} + \beta_{42} = 2/3$$

$$\frac{-\alpha_1}{120} + \frac{1}{6}\beta_{21} + \frac{4}{3}\beta_{22} + \frac{1}{2}\beta_{31} + 2\beta_{32} + \beta_{41} + 2\beta_{42} = 4/15$$

$$\frac{-\alpha_1}{720} + \frac{1}{24}\beta_{21} + \frac{2}{3}\beta_{22} + \frac{1}{6}\beta_{31} + \frac{4}{3}\beta_{32} + \frac{1}{2}\beta_{41} + 2\beta_{42} = 4/45$$

$$\frac{-\alpha_1}{5040} + \frac{1}{120}\beta_{21} + \frac{4}{15}\beta_{22} + \frac{1}{24}\beta_{31} + \frac{2}{3}\beta_{32} + \frac{1}{6}\beta_{41} + \frac{4}{3}\beta_{42} = 8/315$$

$$\frac{-\alpha_1}{40320} + \frac{1}{720}\beta_{21} + \frac{4}{45}\beta_{22} + \frac{1}{120}\beta_{31} + \frac{4}{15}\beta_{32} + \frac{1}{120}\beta_{41} + \frac{2}{3}\beta_{42} = 2/315 \qquad (17)$$

$$\frac{-\alpha_1}{3628800} + \frac{1}{40320}\beta_{21} + \frac{2}{315}\beta_{22} + \frac{1}{5040}\beta_{31} + \frac{8}{315}\beta_{32} + \frac{1}{5040}\beta_{41} + \frac{4}{45}\beta_{42} = 4/141755$$

Solving the above set of equations gives

$\beta_{20} = 19/210$, $\beta_{21} = 86/105$, $\beta_{22} = 19/210$, $\beta_{30} = 59/3360$,

$\beta_{31} = 0$, $\beta_{32} = -59/3360$

$\beta_{40} = 11/10080$, $\beta_{41} = 13/504$, $\beta_{42} = 11/10080$, $\alpha_0 = 1$, $\alpha_1 = -2$,

On substituting the above values into (16) and simplifying we obtain

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{210}\left[19y_{n+2}^{(2)} + 172y_{n+1}^{(2)} + 19y_n^{(2)}\right] - \frac{59h^3}{3360}\left[y_{n+2}^{(3)} - y_n^{(3)}\right] + \frac{h^4}{10080}\left[11y_{n+2}^{(4)} + 260y_{n+1}^{(4)} + 11y_n^{(4)}\right] \qquad (18)$$

2.1.4 Three- step second-derivative method $(k = 3, l = 2)$

Again (9), we have

$$y_{n+3} = \sum_{i=2}^{l} h^i \sum_{j=0}^{k} \beta_{2j} y_{n+j}^{(2)} - \sum_{j=0}^{k-1} \alpha_j y_{n+j}$$

$$y_{n+3} = h^2 \left[\beta_{20} y_n^{(2)} + \beta_{21} y_{n+1}^{(2)} + \beta_{22} y_{n+2}^{(2)} + \beta_{23} y_{n+3}^{(2)} \right] - \alpha_0 y_n - \alpha_1 y_{n+1} - \alpha_2 y_{n+2} \quad (19)$$

By adopting Taylor's series expansion on (19) and combining terms in equal powers of h, we obtained the following set of equations:

$$-\alpha_0 - \alpha_1 - \alpha_2 = 1, \quad -\alpha_1 - 2\alpha_2 = 3,$$

$$-\frac{1}{2}\alpha_1 - 2\alpha_2 + \beta_{20} + \beta_{21} + \beta_{22} + \beta_{23} = \frac{9}{2},$$

$$-\frac{1}{6}\alpha_1 - \frac{8}{6}\alpha_2 + \beta_{21} + 2\beta_{22} + 3\beta_{23} = \frac{27}{6},$$

$$-\frac{1}{24}\alpha_1 - \frac{16}{24}\alpha_2 + \frac{1}{2}\beta_{21} + 2\beta_{22} + \frac{9}{2}\beta_{23} = \frac{81}{24}$$

$$-\frac{1}{120}\alpha_1 - \frac{32}{120}\alpha_2 + \frac{1}{6}\beta_{21} + \frac{8}{6}\beta_{22} + \frac{27}{6}\beta_{23} = \frac{243}{120}$$

$$-\frac{1}{720}\alpha_1 - \frac{64}{720}\alpha_2 + \frac{1}{24}\beta_{21} + \frac{16}{24}\beta_{22} + \frac{81}{24}\beta_{23} = \frac{729}{720} \quad (20)$$

Solving the above set of equations gives

$$\alpha_0 = -1, \alpha_1 = 3, \alpha_2 = -3, \alpha_3 = 1, \beta_{20} = -\frac{1}{12}, \beta_{21} = -\frac{3}{4}, \beta_{22} = \frac{3}{4}, \beta_{23} = \frac{1}{12}$$

Substituting the above value in (19) and re-arranging, we have the method

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = h^2 \left[\frac{1}{12} y_{n+3}^{(2)} + \frac{3}{4} y_{n+2}^{(2)} - \frac{3}{4} y_{n+1}^{(2)} - \frac{1}{12} y_n^{(2)} \right] \quad (21)$$

2.1.5 Four- step second-derivative method $(k = 4, l = 2)$

Applying equation (9)

$$y_{n+k} = \sum_{i=2}^{2} h^i \sum_{j=0}^{4} \beta_{2j} y_{n+j}^{(2)} - \sum_{j=0}^{3} \alpha_j y_{n+j} \tag{22}$$

$$\begin{aligned} y_{n+4} = h^2 \Big[& \beta_{20} y_n^{(2)} + \beta_{21} y_{n+1}^{(2)} + \beta_{22} y_{n+2}^{(2)} + \beta_{23} y_{n+3}^{(2)} + \beta_{24} y_{n+4}^{(2)} \Big] \\ & - \alpha_0 y_n - \alpha_1 y_{n+1} - \alpha_2 y_{n+2} - \alpha_3 y_{n+3} \end{aligned} \tag{23}$$

$$-\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 1$$

$$-\alpha_1 - 2\alpha_2 - 3\alpha_2 = 4$$

$$-\frac{1}{2}\alpha_1 - 2\alpha_2 - \frac{9}{2}\alpha_3 + \beta_{20} + \beta_{21} + \beta_{22} + \beta_{23} + \beta_{24} = 8$$

$$-\frac{1}{6}\alpha_1 - \frac{8}{6}\alpha_2 - \frac{27}{6}\alpha_3 + \beta_{21} + 2\beta_{22} + 3\beta_{23} + 3\beta_{24} = \frac{64}{6}$$

$$-\frac{1}{24}\alpha_1 - \frac{16}{24}\alpha_2 - \frac{81}{24}\alpha_2 + \frac{1}{2}\beta_{21} + 2\beta_{22} + \frac{9}{2}\beta_{23} + 8\beta_{24} = \frac{256}{24}$$

$$-\frac{1}{120}\alpha_1 - \frac{32}{120}\alpha_2 - \frac{243}{120}\alpha_3 + \frac{1}{6}\beta_{21} + \frac{8}{6}\beta_{22} + \frac{27}{6}\beta_{23} + \frac{64}{6}\beta_{24} = \frac{1024}{120}$$

$$-\frac{1}{720}\alpha_1 - \frac{64}{720}\alpha_2 - \frac{729}{720}\alpha_3 + \frac{1}{24}\beta_{21} + \frac{16}{24}\beta_{22} + \frac{81}{24}\beta_{23} + \frac{256}{24}\beta_{24} = \frac{4096}{720}$$

$$-\frac{1}{5040}\alpha_1 - \frac{128}{5040}\alpha_2 - \frac{2187}{5040}\alpha_3 + \frac{1}{120}\beta_{21} + \frac{32}{120}\beta_{22} + \frac{243}{120}\beta_{23} + \frac{1024}{120}\beta_{24} = \frac{16384}{5040}$$

$$\begin{aligned} -\frac{1}{40320}\alpha_1 - \frac{256}{40320}\alpha_2 - \frac{6561}{40320}\alpha_3 + \frac{1}{720}\beta_{21} + \frac{64}{720}\beta_{22} + \frac{729}{720}\beta_{23} \\ + \frac{4096}{720}\beta_{24} = \frac{65536}{40320} \end{aligned}$$

Solving the above set of equations gives

$$\alpha_0 = 1, \alpha_1 = \frac{122}{31}, \alpha_2 = -\frac{306}{31}, \alpha_3 = \frac{122}{31}, \alpha_4 = 1,$$
$$\beta_{20} = \frac{187}{3720}, \beta_{21} = \frac{679}{465}, \beta_{22} = \frac{3047}{620}, \beta_{23} = \frac{679}{465}, \beta_{24} = \frac{187}{3720}$$

Substituting the values above into equation (22) yields

$$y_{n+4}+\frac{122}{31}y_{n+3}-\frac{306}{31}y_{n+2}+\frac{122}{31}y_{n+1}+y_n=$$
$$h^2\left[\frac{187}{3720}y_{n+4}^{(2)}+\frac{679}{465}y_{n+3}^{(2)}+\frac{3047}{620}y_{n+2}^{(2)}+\frac{679}{465}y_{n+1}^{(2)}+\frac{187}{3720}y_n^{(2)}\right] \tag{24}$$

3 Basic Properties of the Methods

3.1 Order of accuracy

The local truncation error of our method in (2.2) is given as

$$T_{n+k}=\sum_{j=0}^{k}\alpha_j y_{n+j}-\sum_{i=2}^{l}h^i\sum_{j=0}^{k}\beta_{ij}y_{n+j}^{(i)}\ ,\quad i,j\geq k\ \text{ and }\ T_{n+k}\approx 0 \tag{25}$$

Adopting Taylor series expansion of (25) and collect terms in equal powers of h, we have

$$\begin{aligned}T_{n+k}&=C_0 y(x_n)+C_1 hy'(x_n)+C_2h^2y''(x_n)+\ldots\\&\quad+C_ph^py^p(x_n)+C_{p+1}h^{p+1}y^{p+1}(x_n)+C_{p+2}h^{p+2}y^{p+2}(\xi)\end{aligned} \tag{26}$$

where $x_{n+k}<x\leq x_{n+k+1}$ and

$$\begin{aligned}C_q&=\frac{1}{q!}\left[1-\sum_{j=0}^{k}j^q\alpha_j\right]-\frac{1}{(q-2)!}\sum_{j=2}^{k}j^{q-2}\beta_{2j}\\&\quad-\frac{1}{(q-3)!}\sum_{j=3}^{k}j^{q-3}\beta_{3j}-\frac{1}{(q-4)!}\sum_{j=4}^{k}j^{q-4}\beta_{4j},\qquad q=0,1,2,\ldots,p+2\end{aligned} \tag{27}$$

3.1.1 Two-step second derivative methods (k=2,l=2)

From the lte

$$T_{n+2}=C_0y_n+C_1hy_n^{(1)'}+C_2h^2y_n^{(2)}+C_3h^3y_n^{(3)}+C_4h^4y_n^{(4)}+C_5h^5y_n^{(5)}+O(h^6) \tag{28}$$

where $C_0=\alpha_0+\alpha_1+\alpha_2,\quad C_1=\alpha_1+2\alpha_2,\quad C_2=\frac{1}{2}\alpha_1+2\alpha_2-\beta_{20}-\beta_{21}-\beta_{22},$

$C_3=\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2-\beta_{21}-2\beta_{22},\qquad C_4=\frac{1}{24}\alpha_1+\frac{2}{3}\alpha_2-\frac{1}{2}\beta_{21}-2\beta_{22},$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 - \frac{1}{24}\beta_{21} - \frac{2}{3}\beta_{22}$$

By imposing an accuracy of order 4, we have

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \quad \text{and} \quad C_6 = C_{p+2} = \frac{1}{240} \neq 0$$

hence the method (12) is of order 4.

3.1.2 Two-step third derivative method (k=2, l=3)

Let the lte be given by

$$T_{n+2} = C_0 y_n + C_1 h y_n^{(1)'} + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + C_4 h^4 y_n^{(4)} + C_5 h^5 y_n^{(5)} + C_6 h^6 y_n^{(5)} + C_7 h^7 y_n^{(5)} + O(h^8) \quad (29)$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 \ , \ C_1 = \alpha_1 + 2\alpha_2 \ , \ C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 - \beta_{20} - \beta_{21} - \beta_{22}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 - \beta_{21} - 2\beta_{22} - \beta_{30} - \beta_{31} - \beta_{32}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 - \frac{1}{2}\beta_{21} - 2\beta_{22} - \beta_{31} - 2\beta_{32}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 - \frac{1}{6}\beta_{21} - \frac{4}{3}\beta_{22} - \frac{1}{2}\beta_{31} - 2\beta_{32}$$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 - \frac{1}{24}\beta_{21} - \frac{2}{3}\beta_{22} - \frac{1}{6}\beta_{31} - \frac{4}{3}\beta_{32}$$

$$C_7 = \frac{1}{5040}\alpha_1 + \frac{8}{315}\alpha_2 - \frac{1}{120}\beta_{21} - \frac{4}{15}\beta_{22} - \frac{1}{120}\beta_{31} - \frac{2}{3}\beta_{32}$$

$$C_8 = \frac{1}{40320}\alpha_1 + \frac{2}{315}\alpha_2 - \frac{1}{720}\beta_{21} - \frac{4}{45}\beta_{22} - \frac{1}{720}\beta_{31} - \frac{4}{15}\beta_{32} \quad (30)$$

By imposing an accuracy of order 6 , we have that

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = 0 \ , \ C_8 = \frac{-29}{302400} \neq 0 \ .$$

Hence the scheme is order 6 and error constant $C_{p+2} = \frac{-29}{302400}$.

3.1.3 Two-step fourth derivative method $(k = 2, l = 4)$

The lte for the method is given by

$$T_{n+2} = C_0 y_n + C_1 h y_n^{(1)'} + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + C_4 h^4 y_n^{(4)} + C_5 h^5 y_n^{(5)} + C_6 h^6 y_n^{(5)} + C_7 h^7 y_n^{(7)} + C_8 h^8 y_n^{(8)} + C_9 h^9 y_n^{(9)} + C_{10} h^{10} y_n^{(10)} + C_{11} h^{11} y_n^{(11)} + O(h^{12}) \quad (31)$$

By imposing an accuracy of order 10, we have that

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = C_{10} = C_{11} = 0$$

$$C_{12} = 3.042x10^{-9}$$

Hence the scheme is of order 10 and error constant

$$C_{p+2} = 3.042x10^{-9}.$$

3.1.4 Three-step second-derivative method $(k = 3, l = 2)$

The lte for the method (21) is given by

$$T_{n+3} = C_0 y_n + C_1 h y_n^{(1)'} + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + C_4 h^4 y_n^{(4)} + C_5 h^5 y_n^{(5)} + C_6 h^6 y_n^{(5)} + C_7 h^7 y_n^{(7)} + O(h^8) \quad (31)$$

By imposing an accuracy of order 5, we have that

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0, C_7 \neq 0$$

Hence, the method (21) has order p = 5, error constant

$$C_{p+2} = -4.1667 \times 10^{-3}.$$

3.1.5 Four-step second-derivative method $(k = 4, l = 2)$

Using similar approach, the order of the method (24) is p = 6 and the error constant is

$$C_{p+2} = -9.9206 \times 10^{-5}.$$

Table 0: Coefficients and order of the methods

k	l	α	α_1	α_2	α_3	α_4	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{30}	β_3	β_{32}	p
2	2	1	-2	1			$\frac{1}{12}$	$\frac{5}{6}$	$\frac{1}{12}$						5
2	3	1	-2	1			$\frac{2}{15}$	$\frac{11}{15}$	$\frac{2}{15}$			$\frac{1}{40}$	0	$\frac{-1}{40}$	6
2	4	1	-2	1			$\frac{19}{210}$	$\frac{86}{105}$	$\frac{19}{210}$			$\frac{59}{3360}$	0	$\frac{-59}{3360}$	10
3	2	-1	3	-3	1		$\frac{-1}{12}$	$\frac{-3}{4}$	$\frac{3}{4}$	$\frac{1}{12}$					5
4	2	1	$\frac{122}{31}$	$\frac{-30}{31}$	$\frac{-122}{31}$	1	$\frac{187}{3720}$	$\frac{679}{465}$	$\frac{3047}{620}$	$\frac{679}{465}$	$\frac{187}{3720}$				6

3.2 Consistency

In Lambert and Awoyemi [6-8], a linear multistep method of the type (3) for the solution of nth order ordinary differential equation is consistent if and only if the following conditions are satisfied

(i) $p \geq 0$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\sum_{j=0}^{k} \frac{j^n}{n} \alpha_j = \sum_{j=0}^{k} \beta_{ij}$

3.2.1 Two-step second derivative method

(i) Since $p > 2$ in method (12) ,then the first condition is satisfied

(ii) $\sum_{j=0}^{k}\alpha_j = \alpha_0 + \alpha_1 + \alpha_2 = 1 - 2 + 1 = 0$ .thus the second condition is satisfied

(iii) $\sum_{j=0}^{k}\frac{j^n}{n}\alpha_j = \sum_{j=0}^{k}\beta_{ij}$

LHS = $0\alpha_0 + \frac{1}{2}\alpha_1 + 2\alpha_2 = 0 - \frac{2}{2} + 2 = 1$

RHS = $\beta_{20} + \beta_{21} + \beta_{22} = \frac{1}{12} + \frac{5}{6} + \frac{1}{12} = 1.$

Thus satisfying the third condition. Hence the method is consistent.

3.2.2 Two-step third derivative method

(i) The implicit scheme (15) is of order 6, then the first condition is satisfied

(ii) $\sum_{j=0}^{k}\alpha_j = \alpha_0 + \alpha_1 + \alpha_2 = 1 - 2 + 1 = 0$. Hence the second condition is satisfied

(iii) LHS: $0\alpha_0 + \frac{1}{2}\alpha_1 + 2\alpha_2 = 0 - \frac{2}{2} + 2 = 1$,

RHS=$\frac{2}{15} + \frac{11}{15} + \frac{2}{15} + \frac{1}{40} + 0 - \frac{1}{40} = 1$

LHS = RHS=1.

Hence the third condition is also satisfied. The method is therefore consistent.

3.2.3 Two-step fourth derivative method

(i) $P > 2$, hence the first condition is satisfied

(ii) $\sum_{j=0}^{k}\alpha_j = \alpha_0 + \alpha_1 + \alpha_2 = 1 - 2 + 1 = 0$. Again the second condition is satisfied

(iii) LHS= $0\alpha_0 + \frac{1}{2}\alpha_1 + 2\alpha_2 = 0 - \frac{2}{2} + 2 = 1$

(iv) RHS = $\beta_{20} + \beta_{21} + \beta_{22} = \frac{19}{210} + \frac{86}{105} + \frac{19}{210} = 1$

LHS = RHS =1.

The third condition is also satisfied .Hence the method is consistent.

3.3 Zero stability

Given the linear k-step method (2), we consider its first and second characteristics polynomial as

$$\rho(\xi) = \sum_{j=0}^{k}\alpha_j \xi^j, \qquad \sigma(\xi) = \sum_{j=0}^{k}\beta_j \xi^j \tag{2.20}$$

where, as before $\alpha_0 \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$.

Theorem 3.1 (Root condition) A linear multistep method is zero – stable for any initial value problem of the form (1) where f satisfies the hypothesis of Picard's Theorem, if and only if, all the roots of the first characteristics polynomial of the method are inside the closed unit disc in the complex plane, with any which lie on the unit circle being simple , Endre and Mayers (see [9]).

Definition 3.1 Jain and Iyengar [10], The linear multistep method (3) is said to satisfy the root condition if the root of the equation $\rho(\xi) = 0$ lie inside the unit circle in the complex plane, and are simple if it lie on the unit circle.

The main consequence of zero stability is to control the propagation of the errors as the integration progresses Fatunla [11].

Definition 3.2 [10]

A linear multistep method of the type (3) is said to be zero stable if the root of the first characteristics polynomial $\rho(\xi)=0$ lie inside the unit circle in the complex plane, and are of multiplicity not exceeding at most two if they lie in unit circle Therefore for the methods (13), (15), (18), the first characteristic polynomials are all of the forms

$$\rho(r)=r^{n+2}-2r^{n+1}+r^{n}=r^{n}(r^{2}-2r+1)=r^{n}(r-1)^{2}=0.$$

Solving this gives $r=0,1,1$. Hence the roots are within unit circle, the methods are therefore zero-stable.

For the methods (21) and (24), all the roots lie on the unit circle and none of the roots has modulus greater than one. Hence they all zero stable.

3.4 Region of Absolute Stability of Method

3.4.1 Two –step second derivative method.

The method is given as

$$y_{n+2}=2y_{n+1}-y_{n}+\frac{h^{2}}{12}\left(y_{n+2}^{(2)}+10y_{n+1}^{(2)}+y_{n}^{(2)}\right)$$

The first and second characteristics polynomial is:

$$\rho(r)=r^{2}-2r+1 \quad , \quad \sigma(r)=\frac{1}{12}(r^{2}+10r+1)$$

Using the boundary locus method

$$h(r)=\frac{\rho(r)}{\sigma(r)}=\frac{12(r^{2}-2r+1)}{r^{2}+10r+1}$$

where $r=e^{i\theta}=\cos\theta+i\sin\theta$.

$$\bar{h}(\theta)=\frac{12\left[\cos 2\theta+i\sin 2\theta-2\cos\theta-i\sin\theta+1\right]}{\cos 2\theta+i\sin 2\theta+10\cos\theta+i10\sin\theta+1}.$$

Therefore setting imaginary $y(\theta)=0$

and $$x(\theta)=\frac{12(\cos 2\theta-2\cos\theta+1)}{\cos 2\theta+10\cos\theta+1}, \quad \text{with } 0\le\theta\le 180^{\circ}$$

$$x(\theta)=(-6,0)$$

Hence the region of absolute stability of the Two-step second derivative method is (-6, 0).

3.4.2 Two-step third derivative method

The first and second characteristics polynomials are:

$$\rho(r)=r^2-2r+1, \quad \sigma(r)=\frac{1}{15}(2r^2+11r+2).$$

Again by applying the boundary locus method

$$h(r)=\frac{\rho(r)}{\sigma(r)}=\frac{15(r^2-2r+1)}{2r^2+11r+2}\text{ , where } r=e^{i\theta}=\cos\theta+i\sin\theta$$

Therefore,

$$\bar{h}(\theta)=\frac{15\left[\cos 2\theta+i\sin 2\theta-2\cos\theta-i\sin\theta+1\right]}{2\cos 2\theta+i2\sin 2\theta+11\cos\theta+i11\sin\theta+2}.$$

Considering only the real part:

$$x(\theta)=\frac{15(\cos 2\theta-2\cos\theta+1)}{2\cos 2\theta+11\cos\theta+2}\text{ , and with } 0\le\theta\le 180^{\circ}$$

$$x(\theta)=(-8.57,0).$$

3.4.3 Two –step fourth derivative methods

The first and second characteristics polynomial of the method are :

$$\rho(r)=r^2-2r+1, \quad \sigma(r)=\frac{1}{210}(19r^2+172r+19).$$

Using the boundary locus method

$$h(r)=\frac{\rho(r)}{\sigma(r)}=\frac{210(r^2-2r+1)}{19r^2+172r+19} \quad \text{,where} \quad r=e^{i\theta}=\cos\theta+i\sin\theta$$

Therefore,

$$\bar{h}(\theta)=\frac{210\left[\cos 2\theta+i\sin 2\theta-2\cos\theta-i\sin\theta+1\right]}{19\cos 2\theta+i19\sin 2\theta+172\cos\theta+i172\sin\theta+19}.$$

Considering only the real part:

$$x(\theta)=\frac{210\left(\cos 2\theta-2\cos\theta+1\right)}{19\cos 2\theta+172\cos\theta+19}, \text{ and with } 0\le\theta\le 180^\circ$$

$$x(\theta)=(-6.27,0).$$

4 Numerical Examples

Example 1

$y''=y \quad , \quad y(0)=1, y'(0)=0 \quad , \quad h=0.1$

Exact solution: $y(x)=\frac{1}{2}(e^x+e^{-x})$

Example 2

A highly oscillatory problem

$y''=\lambda^2 y \quad , \quad \lambda=2 \ , \ y(0)=1, \quad y'(0)=2, h=0.01$

Exact solution: $y(x)=\cos 2x+\sin 2x$.

Example 3

$y''-x(y')^2=0 \quad , \quad y(0)=1 \ , \quad y'(0)=\frac{1}{2} \quad , \ h=0.1$

Exact solution: $y(x)=1+\frac{1}{2}\ln(\frac{2+x}{2-x})$

Table I: Results of problem 1, h=0.1

		Computed Results for:				
X	Exact Solution	K=2,L=2	K=2,L=3	K=2,L=4	K=3, L=2	K=4, L=2
0.1	1.005004168	1.0050041667	1.0050041680	1.0050041681	1.005004176	1.005004161
0.2	1.020066756	1.0200666667	1.0200667556	1.0200667556	1.020066756	1.020066755
0.3	1.045338514	1.0453383403	1.0453385141	1.0453385141	1.045338513	1.045338513
0.4	1.081072372	1.0810721042	1.0810723718	1.0810723728	1.081072356	1.081072378
0.5	1.127625965	1,1276255417	1,1276259649	1,1276259652	1.127625868	1.127625926
0.6	1.185465218	1.1854644028	1.1854652166	1.1854652182	1.185464800	1.185465432
0.7	1.255169006	1.2551671042	1.2551689978	1.2551690056	1.255167568	1.255167749
0.8	1.337434946	1.3374302292	1.3374349165	1.3374349463	1.337430756	1.337442102
0.9	1.433086385	1.4330750278	1.4330862887	1.4330863854	1.433075613	1.433044487
1.0	1.543080635	1.5430549167	1.5430803571	1.5430806348	1.543058556	1.543318962

Table II: Errors in the computed Results for Problem 1, h=0.1

	Errors in the Computed Results for:				
X	K=2,L=2	K=2,L=3	K=2,L=4	K=3, L=2	K=4, L=2
0.1	1.3891370276E-009	2.4824586831E-013	2.2204460493E-016	2.482458683E-013	2.482458683E-013
0.2	8.8952409216E-008	1.4837020501E-012	8.8817841970E-016	6.352030013E-011	6.352030013E-011
0.3	1.7385108242E-007	5.2498005942E-012	1.1102230246E-015	1.628859714E-009	1.628860379E-009
0.4	2.6767178785E-007	3.5677683030E-011	1.5543122345E-015	1.628289725E-008	5.750255161E-009
0.5	4.2353971352E-007	2.8045521461E-010	2.2204460493E-015	9.715082094E-008	3.882476273E-008
0.6	8.1546448927E-007	1.6867125474E-009	9.9920072216E-015	4.182422608E-007	2.135768764E-007
0.7	1.9014642758E-006	7.8351103383E-009	7.3496764230E-014	1.437575377E-006	1.256609340E-006
0.8	4.7171381774E-006	2.9762252884E-008	4.7228887468E-013	4.190749275E-006	7.155541700E-006
0.9	1.1351670996E-005	9.6715442099E-008	2.4824586831E-012	1.077294876E-005	4.150958687E-005
1.0	2.5718148576E-005	2.7771802280E-007	1.0969891662E-011	2.507925966E-005	2.383271489E-004

Table III: Results of problem 2, h= 0.01

X	Exact Solution	K=2,L=2	K=2,L=3	K=2,L=4	K=3 ,L=2	K=4, L=2
0.01	0.10197987E+01	0.101979E+01	0.101979E+01	0.101979E+01	0.101979E+01	0.101979 E+01
0.02	0.10391894E+01	0.103919E+01	0.103918E+01	0.103919E+01	0.103919E+01	0.103919 E+01
0.03	0.10581646E+01	0.105816E+01	0.105817E+01	0.105817E+01	0.105817E+01	0.105816 E+01
0.04	0.10767164E+01	0.107672E+01	0.107672E+01	0.107672E+01	0.107672E+01	0.107671 E+01
0.05	0.10948376E+01	0.109484E+01	0.109484E+01	0.109484E+01	0.109484E+01	0.109484 E+01
0.06	0.11125208E+01	0.111252E+01	0.111252E+01	0.111252E+01	0.111252E+01	0.111252 E+01
0.07	0.11297591E+01	0.112976E+01	0.112976E+01	0.112976E+01	0.112976E+01	0.112976 E+01
0.08	0.11465455E+01	0.114655E+01	0.114655E+01	0.114655E+01	0.114655E+01	0.114654 E+01
0.09	0.11628733E+01	0.116287E+01	0.116287E+01	0.116287E+01	0.116287E+01	0.116287 E+01
0.1	0.11787359E+01	0.117874E+01	0.117874E+01	0.117874E+01	0.117874E+01	0.117873 E+01

Table IV: Comparison of Errors in the results problem 2, h=0.01

	Errors in the Computed results for:				
	K=2,L=2	K=2,L=3	K=2,L=4	K=3, L=2	K=4, L=2
0.01	2.6577E-11	4.4408E-16	2.2204E-16	8.9262E-14	4.44089210E-16
0.02	5.3390E-11	8.8818E-16	6.6611E-16	5.7212E-12	3.26405569E-14
0.03	7.9982E-11	1.3323E-15	1.3322E-15	4.3430E-12	5.51558799E-13
0.04	1.0379E-10	1.3322E-15	1.7763E-15	8.4632E-12	1.84985360E-12
0.05	1.1596E-10	8.8818E-16	2.4424E-15	1.9333E-11	1.25917055E-11
0.06	9.3451E-11	1.2656E-14	3.1086E-15	3.6497E-11	7.00373093E-11
0.07	1.3794E-11	5.7288E-14	3.5527E-15	5.8766E-11	4.06566336E-10
0.08	3.0168E-10	1.9385E-13	4.9968E-15	8.3445E-11	2.33935693E-09
0.09	9.3806E-10	5.5822E-13	4.2188E-15	1,0513E-10	1.34819704E-08
0.1	2.1970E-09	1.4308E-12	4.6629E-15	1.1373E-10	7.76816826E-08

Table V: Results of problem 3, h=0.1

X-Value	Exact Solution	K=2,L=2	K=2,L=3	K=2,L=4	K=3 ,L=2
0.1	1.050041730	1.050041667	1.050041701	1.050041766	1.050041666
0.2	1.100335349	1.100333333	1.100335265	1.100335333	1.100333335
0.3	1.151140438	1.151125002	1.151140800	1.151140999	1.151125002
0.4	1.202732557	1.202666670	1.202732110	1.202732670	1.202666670
0.5	1.255412816	1.255208330	1.255413807	1.255414336	1.255208337
0.6	1.309519609	1.309000000	1.309513000	1.309516990	1.309000005
0.7	1.365443760	1.364291673	1.365439683	1.365449833	1.364291673
0.8	1.423648937	1.421333333	1.423639335	1.423639876	1.421333340
0.9	1.484700287	1.480375008	1.484688898	1.484712008	1.480375008
1.0	1.549306154	1.541666676	1.549166676	1.549356676	1.541666676

Table VI: Comparison of errors in the results of problem 3, h=0.1

X-Value	Errors in			
	K=2,L=2	K=2,L=3	K=2,L=4	K=3,L=2
0.1	6.26118E-08	2.90000E-08	1.00000E-09	6.2612E-08
0.2	2.01440E-07	5.80220E-08	1.60000E-08	2.0144E-06
0.3	1.54459E-05	3.27100E-07	3.90000E-08	1.5436E-05
0.4	6.58874E-05	4.47013E-07	1.13000E-07	6.5887E-05
0.5	2.04479E-04	9.91200E-07	1.52000E-06	2.0441E-04
0.6	5.19604E-04	3.60912E-06	2.61000E-06	5.1960E-04
0.7	1.15209E-03	4.07700E-06	6.07310E-06	1.1521E-03
0.8	2.31561E-03	9.60231E-06	9.06100E-06	2.3156E-03
0.9	4.32529E-03	1.13890E-05	1.17210E-05	4.3253E-03
1.0	7.63945E-03	1.39478E-04	5.05220E-05	7.6395E-03

5 Conclusion

In this study, we have developed a class of implicit multi-derivative linear multistep methods for the numerical solution of general second order ordinary differential equations. Analysis of the basic properties showed that the methods are consistent, zero-stable, convergent and absolutely stable. The results displayed in tables 1-3 shows that there is a remarkable improvement in accuracy if the order of the derivatives (l) is increased rather than the steps. Also it was observed that the order of the methods increases when the derivative is increased. This can be seen in table 0 above.

References

[1] S.L Ross, *Introduction to Ordinary Differential Equations*, John Wiley And Sons Inc, Singapore, 1989.

[2] W. Auzinger, R. Frank and G. Kiringer, A note on the Convergence Concepts of Stiff Problems. Journal of Computation and Applied Math. **44**, (1990), 197-208.

[3] F. Courant-Friedrichs, *Numerical differential Equations Explicit and Implicit Methods*, https://n.wikipedia.org/wiki, 2007.

[4] D.O. Awoyemi, *On Some Continuous Linear Multistep Methods for Initial Value Problems*, Ph.D Thesis (unpublished), University of Ilorin, Nigeria, 2007.

[5] S.L Ross, *A course in elementary differential equations*, John Wiley And Sons Inc., New York, 1974.

[6] J.D. Lambert, *Computational Methods in* ODES, John Wiley & Sons, New York, 1973.

[7] D.O Awoyemi, A class of Continuous Methods for General Second Initial Value Problems in Ordinary Differential Equations, *Inter. Journal of Computer Math*. **72**, (1999), 29-39.

[8] D.O Awoyemi, A new Sixth – order Algorithms for General Second Order Ordinary Differential Equations, *Inter. Journal of Computer Math*, **77**, (2001), 117-124.

[9] Endre Suli and David, T. Mayers, *An introduction to Numerical Analysis*, Press Sydicate of the University of Cambridge, The Pitt Building Trumpington Street, Cambridge, U.K., 2003.

[10] M.K Jain, S.R.K Iyengar and R.K Jain, *Numerical Methods for Scientific and Engineering Computations,* fifth edition, New Age International Publishers Limited, 2008.

[11] S.O. Fatunla, *Numerical Methods for Initial Value Problems in Ordinary Differential Equations*, Academic Press Inc. Harcourt Brace, Jovanovich Publishers, New York, 1988.