

Some Mathematical Problems Arising in Biological Models: A Predator-Prey Model Fish-Plankton

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Abstract

The aim of this work is to develop and analyze a mathematical model for a predator-2 preys system arising in marine environments. We are interested in the evolution of fish population depending of plankton. We first look at the model; this yields a singular system of ordinary differential equations having interesting dynamical futures, such as finite time extinction and persistence of populations. In addition, the mathematical analysis permits to isolate extinction conditions in finite or infinite time. Finally, the numerical simulations permits to establish the effect of the fishing on the evolution of fish population in spite of abundance resource and it's permit to know if area is severely exploited or not.

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1 Introduction

We are interested in some deterministic mathematical population dynamics model motivated by biological problems in the marine environment. The problem of the management of bio-diversity resources in general and particularly, the resources halieutic management, interest many researchers. Almost 47% of the world fish stocks commercial are maximum exploited [6, 11].

To contribute to resolve the fish resource preservation, here, we develop a deterministic model which govern the dynamic of the fish and her diet (plankton).

In previous works [6, 8, 9], a mathematical model has been developed to model the interactions between a native prey and alien predators and preys in insular environments, taking into account the spatial heterogeneity in the habitat. Problems of persistence and extinction of species have been studied in [3, 6], and invasion processes have been analyzed.

In this paper, we develop a mathematical model to better understand interactions between diet (plankton) and fish population in a marine environment, taking into account heterogeneities in the distribution of the species. Some modeling work in this direction concerning unstructured populations has been previously developed in [1, 7] for two and three species.

We extended the unstructured model of Courchamp and Sugihara [2, 3, 6] to the case of fish-plankton. Along the same line, we derive a predator (fish)-2prey (zooplankton and phytoplankton) model based on Courchamp and al, [5]. The models study in the present work use predator-2prey ratio instead of prey density. It's about to develop Fish-zooplankton-phytoplankton deterministic model.

First, we consider fish and phytoplankton population dynamic, so F-B model. After we consider fish and zooplankton system and zooplankton and phytoplankton system for to develop F-R and R-B models. In end, combining of these dynamic, we develop the final model B-R-P of the system phytoplankton-zooplankton-fish. One novelty of these models, involving sys-

tems of ordinary differential equations, is the possibility for species to go extinct in finite and infinite time; This feature is closely related to a specific mathematical difficulty: the occurrence of singularities i.e vanishing determinators, in the systems of ordinary differential equations (ODE).

The paper is organized as follows. In section 2, we present the different mathematical models of our problem. Section 3 provides the mathematical analysis of the final model. Computational simulations are performed in section 4 and finally, in the last section, the section 5 we end with some conclusions remarks and future works.

2 Mathematical models

In this first section, we briefly describe a mathematical models devised in F. Courchamp and Sugihara and Courchamp et al [3] on the Predator-preys models.

2.1 Fish-Phytoplankton model

If we consider the Fish-Phytoplankton system without the Zooplankton, the dynamic system is governed by the following Figure 1.

State variables are: B , the phytoplankton density and P , the fish density. Let

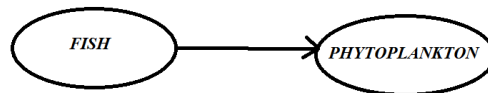


Figure 1: Fish-Phytoplankton model

μ_b , the annual individual intake of phytoplankton per fish, q the catchability coefficient fish, E the fishing effort. The carrying capacity of the introduced fish population is $\frac{B}{\mu_b}$, [4, 8] and the fishing mortality rate is qE .

Then according to the figure 1, we obtain the following system of two differ-

ential equations singular at $B = 0$.

$$\begin{cases} \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \mu_b P, & B(0) = B_0 > 0 \\ \frac{dP}{dt} = r_p P \left(1 - \frac{\mu_b P}{B}\right) - q_p P, & P(0) = P_0 > 0 \end{cases} \quad (2.1)$$

where

r_p is the natural growth-rate of the fish population

r_b is the net increase-rate of the phytoplankton quantity.

q_p is the fishing mortality rate

K_b is the carrying capacity of the phytoplankton

2.2 Fish-Zooplankton model

If we consider the Fish-Zooplankton system without the phytoplankton, the dynamic system is governed by the following Figure 2, [10].



Figure 2: Fish-Zooplankton model

State variables are: R , the zooplankton density and P , the fish density. Let μ_r , the annual individual intake of the zooplankton per fish, the carrying capacity of the introduced fish population is $\frac{R}{\mu_r}$, [9].

Then according to the Figure 2, we obtain the following system of two differential equations singular at $R = 0$.

$$\begin{cases} \frac{dR}{dt} = r_r R \left(1 - \frac{R}{K_r}\right) - \mu_r P, & R(0) = R_0 > 0 \\ \frac{dP}{dt} = r_p P \left(1 - \frac{\mu_r P}{R}\right) - q_p P, & P(0) = P_0 > 0 \end{cases} \quad (2.2)$$

Where

r_p is the natural growth-rate of the fish population

r_r is the growth-rate of the zooplankton population.

q_p is the fishing mortality rate.

K_r is the carrying capacity of the zooplankton.

2.3 Zooplankton-Phytoplankton model

If we consider the Zooplankton-Phytoplankton system without the fish, the dynamic system is governed by the following Figure 3, [10, 11].

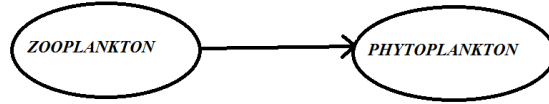


Figure 3: Zooplankton-Phytoplankton model

State variables are: B , the phytoplankton density and R , the zooplankton density. Let δ_b , the annual individual intake of phytoplankton per zooplankton, the carrying capacity of the introduced zooplankton population is $\frac{B}{\delta_b}$. Then according to the Figure 3, we obtain the following system of two differential equations singular at $B = 0$.

$$\begin{cases} \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \delta_b R, & B(0) = B_0 > 0 \\ \frac{dR}{dt} = r_r R \left(1 - \frac{\delta_b R}{B}\right), & R(0) = R_0 > 0 \end{cases} \quad (2.3)$$

where

r_r is the natural growth-rate of the zooplankton population

r_b is the net increase-rate of the phytoplankton quantity.

K_b is the carrying capacity of the phytoplankton.

2.4 Phytoplankton-Zooplankton-Fish model

If we consider the Phytoplankton-Zooplankton-Fish system, the dynamic system is governed by the following Figure 4, [8, 9].

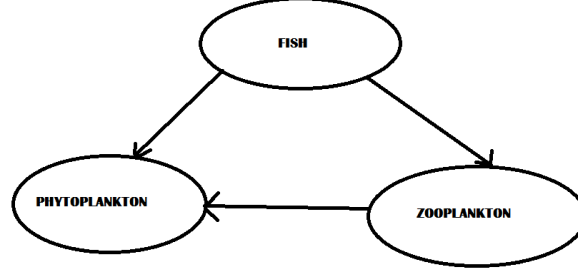


Figure 4: -Phytoplankton-Zooplankton-Fish model

Let μ_b and μ_r the annual individual intake of preys phytoplankton and zooplankton per individual fish, so that the carrying capacity of the fish is $\frac{B}{\mu_b} + \frac{R}{\mu_r}$. Let $\beta > 0$, the preference coefficient for the phytoplankton and zooplankton and μ_ζ the annual individual intake of prey phytoplankton per individual zooplankton .

Then according to the Figure 4, we obtain the following system of three differential equations singular at $B = R = 0$.

$$\left\{ \begin{array}{l} \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \mu_\zeta R B - \frac{\beta B}{\beta B + R} \mu_b P, \quad B(0) = B_0 > 0 \\ \frac{dR}{dt} = r_r R \left(1 - \frac{R}{K_r}\right) - \frac{R}{\beta B + R} \mu_r P, \quad R(0) = R_0 > 0 \\ \frac{dP}{dt} = r_p P \left(1 - \mu_b \mu_r \frac{P}{\mu_r B + \mu_b R}\right) - q_p P, \quad P(0) = P_0 > 0 \end{array} \right. \quad (2.4)$$

3 Mathematical analysis

3.1 Two species model

We study a mathematical problem arising in modeling Fish-Phytoplankton interaction in marine environment. The dynamic is presented by the following

system:

$$\begin{cases} \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \mu_b P, & B(0) = B_0 > 0 \\ \frac{dP}{dt} = r_p P \left(1 - \frac{\mu_b P}{B}\right) - q_p P, & P(0) = P_0 > 0 \end{cases} \quad (3.1)$$

Herein B is the phytoplankton species density with growth-rate, r_b , and carrying capacity, K_b . P is the Fish density with growth-rate r_p , while, μ_b , is the annual intake of phytoplankton per individual fish, [3, 8].

Hypothesis 1: All parameters in the system (3.1) are positives constants.

The mathematical analysis simplifies upon introducing a new state variable $F = \frac{P}{B}$, yielding a non singular locally Lipschitz continuous system of ODEs. Sure enough, we have

$$\begin{cases} \frac{dB}{dt} = \left[r_b \left(1 - \frac{B}{K_b}\right) - \mu_b F \right] B, & B(0) = B_0 > 0 \\ \frac{dF}{dt} = \left[r_p - r_b + r_b \frac{B}{K_b} - \mu_b \left(r_p - 1 - q_p\right) F \right] F, & F(0) = F_0 > 0 \end{cases} \quad (3.2)$$

System (3.1) has a solution if and only if system (3.2) has a solution.

Proposition 3.1. *System (3.2) has unique componentwise positive solution (B, F) , defined on a maximal existence interval, $[0, T_{max}(B_0, F_0)[$.*

Proof: Sure enough, the system (3.2) can be written by the following form

$$\frac{dB}{dt} = \varphi_1(B, F)B, \quad \frac{dF}{dt} = \varphi_2(B, F)F.$$

The functions φ_1 and φ_2 are C^∞ , so according to the Cauchy-Lipschitz, system (3.2) has an unique componentwise positive solution, (B, F) , defined on a maximal existence interval, $[0, T_{max}[$ if $B_0 > 0$ and $F_0 > 0$.

Proposition 3.2. $(0, 0)$ (unstable), $(K_b, 0)$ (unstable) and $\left(0, \frac{1}{\mu_b} \frac{r_p - r_b}{r_p - q_p - 1}\right)$ (stable) if and only if $(r_p - r_b)(r_p - q_p - 1) > 0$ are the stationaries states of the System (3.2).

Therefore we have a third on with positive components

$$\left(B^* = \frac{r_b - 1}{r_b} K_b, F^* = \frac{1}{\mu_b} \right)$$

if and only if

$$r_b - 1 > 0.$$

Proof: A direct calculating gave the four states.

Concerning to the stability and instability, we consider the Jacobian matrix of the system.

Sure enough, we have

$$J(B, F) = \begin{pmatrix} r_b - 2\frac{r_b B}{K_b} - \mu_b F & -\mu_b B \\ r_b \frac{F}{K_b} & r_p - r_b + \frac{r_b B}{K_b} - 2\mu_b(r_p - q_p - 1)F \end{pmatrix}$$

Let $specter(J)$ the set of eigenvalues of the Jacobian matrix.

- The Jacobian matrix at $(0, 0)$ and at $(K_b, 0)$ show that, these states are always unstable.
- The Jacobian matrix at $\left(0, \frac{1}{\mu_b} \frac{r_p - r_b}{r_p - q_p - 1}\right)$ if $1 > r_p$, is

$$\begin{pmatrix} \frac{r_p(1 - r_b)}{1 - r_p} & 0 \\ \frac{r_b(r_b - r_p + q_p)}{K_b \mu_b (1 - r_p)} & r_b - r_p + q_p \end{pmatrix}$$

So, this state is asymptotically stable if $(r_p - r_b)(r_p - q_p - 1) > 0$ ie $r_b < 1 < r_p - q_p$

- The Jacobian matrix at state $\left(B^* = \frac{r_b - 1}{r_b} K_b, F^* = \frac{1}{\mu_b}\right)$ is

$$\begin{pmatrix} 1 - r_b & -\mu_b K_b \frac{r_b - 1}{r_b} \\ \frac{r_b}{K_b \mu_b} & 1 - r_p + q_p \end{pmatrix}$$

Equally, the state is stable if $1 > r_b$

Concerning to the global solution existence, we assume the following assumption, [3, 5, 8]:

Hypothesis 2: $r_p - q_p > 1$.

Proposition 3.3. *According to the assumption 2, system (3.2) has global solution (B, F) , defined on interval, $[0, +\infty)$.*

Proof: We have

$$\frac{dF}{dt} \leq \left(r_p - q_p + r_b \max\left(\frac{B_0}{K_b}, 1\right) - \mu_b(r_p - q_p - 1)F \right) F$$

for all $t \in [0, T_{max}]$ and then, according to the hypothesis 2 and by integrating, we obtain

$$0 \leq F(t) \leq \max\left(F_0, \frac{r_p - q_p r_b + r_b \max\left(\frac{B_0}{K_b}, 1\right)}{\mu_b(r_p - q_p - 1)} \right), \text{ so } T_{max} = +\infty$$

3.2 Three species model

We are interested of the mathematical analysis of Fish-Phytoplankton-Zooplankton model, [3, 5, 8]. The mathematical model associate is the following differential system singular at $B = R = 0$.

$$\begin{cases} \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \mu_\zeta R B - \frac{\beta B}{\beta B + R} \mu_b P, & B(0) = B_0 > 0 \\ \frac{dR}{dt} = r_r R \left(1 - \frac{R}{K_r}\right) - \frac{R}{\beta B + R} \mu_r P, & R(0) = R_0 > 0 \\ \frac{dP}{dt} = r_p P \left(1 - \mu_b \mu_r \frac{P}{\mu_r B + \mu_b R}\right) - q_p P, & P(0) = P_0 > 0 \end{cases} \quad (3.3)$$

3.2.1 Global analysis

Concerning to the analysis of the system (3.3), we rescale the state variables, [1, 2, 3].

Indeed, let take $\widehat{B} = \beta B$, $\widehat{K}_b = \beta K_b$, $\widehat{\mu}_b = \beta \mu_b$ and assume $\beta = 1$. Then, using $H = R + B$ (the total ressource),

$$\phi = \frac{R}{B + R} \text{ (the proportion of zooplankton within the total resource quantity)}$$

and

$$Q = \frac{P}{R + B} \text{ (the ratio fish/resource).}$$

Our aim is to reduce the nonsingular system in the following form

$$\begin{cases} \frac{dH}{dt} = g_1(H, \phi, Q), \\ \frac{d\phi}{dt} = g_2(H, \phi, Q), \\ \frac{dQ}{dt} = g_3(H, \phi, Q), \end{cases} \quad (3.4)$$

Under the hypothesis $0 \leq \phi \leq 1$, we obtain

$$\frac{dH}{dt} = \frac{dR}{dt} + \frac{dB}{dt} = r_b B \left(1 - \frac{B}{K_b}\right) - \mu_\zeta R B - \frac{\beta B}{\beta B + R} \mu_b P + r_r R \left(1 - \frac{R}{K_r}\right) - \frac{R}{\beta B + R} \mu_r P.$$

When we develop and we replace H , ϕ and Q by their values, we get that

$$\frac{dH}{dt} = \left[\left(r_b - \mu_b Q \right) (1 - \phi) + (r_r - \mu_r Q) \phi - r_b \frac{(1 - \phi)^2}{K_b} - r_r \frac{\phi^2}{K_r} + \mu_\zeta \Phi (1 - \phi) \right] H.$$

Likewise, we have:

$$\frac{d\phi}{dt} = \phi \left(r_r - \frac{r_r \phi H}{K_r} - \mu_r Q + r_p Q - r_b - \mu_\zeta \phi H \right) (1 - \phi) + (1 - \phi)^2 \phi H \frac{r_b}{K_b}$$

and

$$\frac{dQ}{dt} = \left[r_p - r_r \phi - r_b (1 - \phi) + \frac{r_b}{K_b} (1 - \phi)^2 H + \mu_\zeta \phi (1 - \phi) H + \frac{r_r}{K_r} \phi^2 H - \frac{r_p \mu_b \mu_r Q - (\mu_r \phi + \mu_b (1 - \phi)) (\mu_r (1 - \phi) + \mu_b \phi) Q}{\mu_r (1 - \Phi) + \mu_b \phi} - q_p \right] Q.$$

By setting :

$$\left\{ \begin{array}{l} G_1(\phi, Q) = (r_b - \mu_b Q)(1 - \phi) + (r_r - \mu_r Q)\phi, \\ G_2(\phi) = r_b \frac{(1 - \phi)^2}{K_b} + r_r \frac{\phi^2}{K_r} + \mu_\zeta \phi(1 - \phi), \\ L(\phi) = (1 - \phi) \phi \frac{r_b}{K_b}, \\ V_1(\phi) = r_p - r_r \phi - r_b(1 - \phi), \\ V_2(\phi) = \frac{r_b}{K_b} (1 - \phi)^2 + \mu_\zeta \phi(1 - \phi) + \frac{r_r \phi^2}{K_r}, \\ V_3(\phi) = \frac{r_p \mu_b \mu_r - (\mu_r \phi + \mu_b(1 - \phi))(\mu_r(1 - \phi) + \mu_b \phi)}{\mu_r(1 - \phi) + \mu_b \phi}. \end{array} \right.$$

we obtain the following system regular if $0 \leq \phi \leq 1$.

$$\left\{ \begin{array}{l} \frac{dH}{dt} = [G_1(\phi, Q) - G_2(\phi)H]H = g_1(H, \phi, Q), \\ \frac{d\phi}{dt} = \left[r_r \left(1 - \frac{\phi H}{K_r} \right) - (\mu_r \mu_b)Q - r_b \mu_\zeta \phi H \right] \phi(1 - \phi) + HL(\phi) = g_2(H, \phi, Q), \\ \frac{dQ}{dt} = [V_1(\phi) + V_2(\phi)H - V_3(\phi)Q]Q = g_3(H, \phi, Q), \end{array} \right. \quad (3.5)$$

Solutions of model system (3.3) exist if and only if the solutions of model system (3.5) exist, [6, 7].

Proposition 3.4. *The system (3.5) admit one maximal unique solution*

$$(H(t), \phi(t), Q(t))$$

definite on a $[0, T_{max}[$. More, the set $\{H \geq 0, 0 \leq \phi \leq 1, Q \geq 0\}$ is positively invariant for the system (3.5).

Proof: The local existence of system (3.5) solution result from the fact that this system is locally Lipschitz.

According to the system (3.5), we obtain

$$\left\{ \begin{array}{l} g_1(0, \phi, Q) = 0, \quad \text{for } 0 \leq \phi \leq 1 \quad \text{and} \quad Q \geq 0, \\ g_2(H, \phi = 0, Q) = 0, \quad \text{for } H \geq 0 \quad \text{and} \quad Q \geq 0, \\ g_2(H, \phi = 1, Q) = 0, \quad \text{for } H \geq 0 \quad \text{and} \quad Q \geq 0, \\ g_3(0, \phi, Q) = 0, \quad \text{for } 0 \leq \phi \leq 1 \quad \text{and} \quad H \geq 0. \end{array} \right.$$

we conclude that $\{H \geq 0, 0 \leq \phi \leq 1, Q \geq 0\}$ is positively invariant for the system (3.5).

Proposition 3.5. *A state variable Q of system (3.5) explode in finite time under hypothesis: $1 > r_p > \max(r_r, r_b) > 0$, i.e that exist $T_{max} < +\infty$ such as $Q(t) \rightarrow +\infty$ when $t \rightarrow +\infty$. The Cauchy problem admit one global unique solution at time $(H > 0, 0 \leq \phi \leq 1, Q > 0)$ i.e $T_{max} = +\infty$ under hypothesis $\frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r} < r_p$.*

Proof: We have

- $V_2 > 0$ for $0 \leq \phi \leq 1$.
- again, we obtain $V_3(\phi) < 0$ when $r_p < 1$
- and $V_3(\phi) \geq 0$ when $r_b < r_p$ and $r_r < r_p$.

In this case $Q(t) \rightarrow +\infty$ when $t \rightarrow T_{max}$ ($T_{max} < +\infty$), i.e that Q explodes in the finite time if $1 > r_p > \max(r_r, r_b) > 0$. Inversely, always under hypothesis $0 \leq \phi \leq 1$, we have $V_2 > 0$. On the other hand $V_3(\phi) > 0$ means that

$$\frac{r_p\mu_b\mu_r - (\mu_r\phi + \mu_b(1 - \phi))(\mu_r(1 - \phi) + \mu_b\phi)}{\mu_r(1 - \phi) + \mu_b\phi} > 0$$

If we set or define

$$\lambda(\phi) = \frac{\mu_b\mu_r - (\mu_r\phi + \mu_b(1 - \phi))(\mu_r(1 - \phi) + \mu_b\phi)}{\mu_r\mu_b},$$

we obtain $V_3(\phi) > 0$ if $r_p > \lambda(\phi)$.

Under condition $0 \leq \phi \leq 1$, the function $\lambda(\phi)$ admit a minimum in

$$\phi = \frac{1}{2} \quad \text{and} \quad \lambda\left(\frac{1}{2}\right) = \frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r}.$$

so $V_3(\phi) > 0$ if

$$r_p > \lambda\left(\frac{1}{2}\right) = \frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r}.$$

If we proceed in the same way, we can said that $Q(t)$ can not tighten to infinity when

$t \longrightarrow T_{max}(T_{max} < +\infty)$ with the condition $\frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r} < r_p$.

When, $0 \leq \phi \leq 1$ we necessary obtain $V_2 \geq 0$. It result of this last result and expression of V_2 that state variable H is narrow.

3.2.2 Equilibrium stability analysis

3.2.2.1 Stationaries states

Proposition 3.6. *The stationaries states of system (3.3) are:*

$$(E_1) : \left(B = K_b, R = 0, P = 0 \right),$$

$$(E_2) : \left(B = 0, R = K_r, P = 0 \right),$$

$$(E_3) : \left(B = \frac{r_b - \mu_\zeta K_r}{r_b} K_b, R = K_r, P = 0 \right) \text{ eligible if only if } r_b > \mu_\zeta K_r,$$

$$(E_4) : \left(B = 0, R = \frac{r_p r_r - r_p + q_p}{r_p r_r} K_r, P = \frac{r_r r_p - r_p + q_p}{r_r r_p^2 \mu_r} K_r (r_p - q_p) \right) \text{ eligi-}$$

ble if only if $r_r > \frac{r_p - q_p}{r_p}$,

$$(E_5) : \left(B = \frac{r_p r_b - r_p + q_p}{r_p r_b} K_b, R = 0, P = \frac{r_b r_p - r_p + q_p}{r_b r_p^2 \mu_b} K_b (r_p - q_p) \right) \text{ feasi-}$$

ble if only if $r_b > \frac{r_p - q_p}{r_p}$.

Proof: We search the equilibrium states with positive or zero components.

For this, we must solve the following system:

$$\begin{cases} r_b B \left(1 - \frac{B}{K_b}\right) - \mu_\zeta R B - \frac{\beta B}{\beta B + R} \mu_b P = 0, \\ r_r R \left(1 - \frac{R}{K_r}\right) - \frac{R}{\beta B + R} \mu_r P = 0, \\ r_p P \left(1 - \mu_b \mu_r \frac{P}{\mu_r B + \mu_b R}\right) - q_p P = 0, \end{cases} \quad (3.6)$$

So we get:

- if $P = R = 0$ and $B \neq 0$, we obtain $B = K_b$
- if $P = B = 0$ and $R \neq 0$, we have $R = K_r$

Now, we consider the states with one nil component. So we have if $B = 0$, then R and P verify the following equation:

$$r_r R \left(1 - \frac{R}{K_r}\right) - \mu_r P = 0 \text{ and } r_p P \left(1 - \frac{\mu_r P}{R}\right) - q_p P = 0.$$

As we search $P \neq 0$, pulling the value of R in the second equation of system (3.6) we find $R = \frac{r_p \mu_r P}{r_p - q_p}$ and we obtain

$$R = \frac{r_p r_r - r_p + q_p}{r_p r_r} \mu_r K_r \text{ and } P = \frac{r_r r_p - r_p + q_p}{r_r r_p^2} K_r (r_p - q_p)$$

which exist if and only if $r_r > \frac{r_p - q_p}{r_p}$.

In the similar way, supposing that $R = 0$ we get

$$B = \frac{r_p r_b - r_p + q_p}{r_p r_b} \mu_b K_b \text{ and } P = \frac{r_b r_p - r_p + q_p}{r_b r_p^2} K_b (r_p - q_p)$$

which exist if and only if $r_b > \frac{r_p - q_p}{r_p}$.

Finally, for $C = 0$ the states variable R and B must verify the following equation

$$r_b B \left(1 - \frac{B}{K_b}\right) - \mu_\zeta B R = 0 \text{ and } r_r R \left(1 - \frac{R}{K_r}\right) = 0.$$

Since we are looking for variables B and R strictly positive, we have $r_b - r_b \frac{B}{K_b} - \mu_\zeta R = 0$ and $R = K_r$. As a result there will $B = \frac{r_b - \mu_\zeta K_r}{r_b} K_b$ which exist if and only if $r_b > \mu_\zeta K_r$.

Proposition 3.7. *The states E_1, E_2 and E_3 are always stable according to the Lyapunov theorem. Equally, according to the **Lyapunov** theorem, state E_4 is locally asymptotically stable if and only if*

$$r_b - \frac{\mu_\zeta K_r}{r_p r_r} (r_p r_r - r_p + q_p) - \frac{\beta \mu_b}{r_p \mu_r} (r_p - q_p) < 0, r_p + r_r + q_p \left(\frac{2}{r_p} - 1 \right) > 2 \text{ and} \\ r_p (r_r - 1) + q_p > 0$$

and state E_5 locally asymptotically stable if and only if

$$r_r - \frac{-\mu_r}{\beta \mu_b r_p (r_p - q_p)} < 0, r_p + r_b + q_p \left(\frac{2}{r_p} - 1 \right) > 2 \text{ et } r_p (r_b - 1) + q_p > 0$$

Proof: Let $J(B, R, P)$, the Jacobian matrix of the system (3.3), we have:

$$J(B, R, P) = \begin{pmatrix} r_b - 2 \frac{r_b}{K_b} B - \mu_\zeta R - \frac{\beta \mu_b R P}{(\beta B + R)^2} & -\mu_\zeta B - \frac{\beta \mu_b B P}{(\beta B + R)^2} & -\frac{\beta \mu_b B}{\beta B + R} \\ \frac{\beta \mu_r R P}{(\beta B + R)^2} & r_r - 2 \frac{r_r}{K_r} R - \frac{\beta \mu_r B P}{(\beta B + R)^2} & -\frac{\mu_r R}{\beta B + R} \\ \frac{r_p \mu_r^2 \mu_b P^2}{(\mu_r B + \mu_b R)^2} & \frac{r_p \mu_b^2 \mu_r P^2}{(\mu_r B + \mu_b R)^2} & r_p - \frac{2 r_p \mu_r \mu_b P}{\mu_r B + \mu_b R} - q_p \end{pmatrix}$$

Let $specter(J)$ set of the eigenvalues of the Jacobian matrix:

- we have $specter(J(E_1)) = \{-r_b, r_r, r_p - q_p\}$, so, the state (E_1) is unstable.
- we obtain also $specter(J(E_2)) = \{-r_r, r_b - \mu_\zeta K_r, r_p - q_p\}$, so this state is unstable.
- concerning to the state (E_3), we have $specter(J(E_3)) = \{-r_r, r_b - 2\mu_\zeta K_r, r_p - q_p\}$, this state is also unstable.
- for to the state (E_4), the eigenvalues of $J(E_4)$ are:

$$\gamma_1 = r_b - \frac{\mu_\zeta K_r}{r_p r_r} (r_p r_r - r_p + q_p) - \frac{\beta \mu_b}{r_p \mu_r} (r_p - q_p), \gamma_2 \text{ and } \gamma_3 \text{ satisfactory} \\ \gamma_2 + \gamma_3 = - \left(\frac{r_p r_r - 2r_p + 2q_p + r_p^2 - r_p q_p}{r_p} \right) \text{ and} \\ \gamma_2 \gamma_3 = \frac{-r_p r_r + r_p - q_p}{r_p} (q_p - r_p).$$

So state (E_4) is locally asymptotically stable if and only if $\gamma_1 < 0$, $\gamma_2 + \gamma_3 < 0$ and $\gamma_2\gamma_3 > 0$, i.e $r_b - \frac{\mu_r K_r}{r_p r_r} (r_p r_r - r_p + q_p) - \frac{\beta \mu_b}{r_p \mu_r} (r_p - q_p) < 0$, $r_p + r_r + q_p (\frac{2}{r_p} - 1) > 2$ and $r_p (r_r - 1) + q_p > 0$

- in the end, the eigenvalues of state $J(E_5)$ are:

$$\gamma_1 = r_r - \frac{-\mu_r}{\beta \mu_b r_p (r_p - q_p)}, \gamma_2 \text{ and } \gamma_3 \text{ satisfactory}$$

$$\gamma_2 + \gamma_3 = \frac{-r_p r_b + 2r_p - 2q_p - r_p^2 + r_p q_p}{r_p} \text{ and } \gamma_2 \gamma_3 = \frac{-r_p r_b + r_p - q_p}{r_p} (q_p - r_p).$$

So state (E_5) is locally asymptotically stable if and only if $\gamma_1 < 0$, $\gamma_2 + \gamma_3 < 0$ and $\gamma_2\gamma_3 > 0$, i.e

$$r_r - \frac{-\mu_r}{\beta \mu_b r_p (r_p - q_p)} < 0, r_p + r_b + q_p (\frac{2}{r_p} - 1) > 2 \text{ and } r_p (r_b - 1) + q_p > 0.$$

3.2.2.2 Component strictly equilibrium

Consider the functions π at ϕ define by:

$$\pi(\phi) = a_2 \phi^2 + a_1 \phi + a_0, \quad (3.7)$$

with

$$a_0 = \frac{r_b K_r}{\beta K_b} \left(\frac{\mu_r}{\beta \mu_b} - 1 \right) \frac{(r_p - q_p)}{r_p}$$

$$a_1 = \frac{(r_p - q_p)}{r_p} \left[r_b \left(1 - \frac{r_b K_r}{\beta K_b} \right) - K_r \left(\frac{r_b}{\beta K_b} \right) \left(1 - \frac{\mu_r}{r_r \beta \mu_b} \right) - 1 \right] \quad (3.8)$$

$$a_2 = \frac{\mu_r}{\beta \mu_b} \left[\frac{K_r}{r_r} \left(\frac{r_b}{\beta K_b} - \mu_r \right) + \frac{\beta \mu_b}{\mu_r} \right] \frac{(r_p - q_p)}{r_p}$$

Consider the affine functions h and l define at $[0, 1]$ to \mathbb{R} by:

$$h(\phi) = r_r - \frac{(r_p - q_p)}{r_p} \left(\phi + (1 - \phi) \frac{\mu_r}{\beta \mu_b} \right)$$

$$l(\phi) = r_b - \frac{(r_p - q_p)}{r_p} \left(\frac{\beta \mu_b}{\mu_r} + (1 - \phi) \right)$$

The following proposition gives us the necessary and sufficient conditions for the existence of strictly positive equilibrium, [6, 8, 9]:

Proposition 3.8. *There is a strictly positive component to equilibrium ($B > 0$, $R > 0$, $P > 0$) of system (3.3) exist if and only if there exists a solution $\phi \in]0, 1[$ of $\pi(\phi) = 0$ equation as one of the following assertions holds:*

$$(i) h(\phi) > 0 ,$$

$$(ii) l(\phi) > 0.$$

Proof: With the previous results we have the following equations:

$$r_b - r_b \frac{B}{K_b} - \mu_\zeta R - \frac{\beta}{\beta B + R} \mu_b P = 0, \quad (3.9)$$

$$r_r - r_r \frac{R}{K_r} - \frac{1}{\beta B + R} \mu_r P = 0, \quad (3.10)$$

$$r_p \beta \mu_b \mu_r P = (r_p - q_p) \beta (\mu_r B + \mu_b R), \quad (3.11)$$

By performing the scale change of variable:

$$\bar{\mu}_b = \beta \mu_b, \bar{B} = \beta B \text{ and } \bar{K}_b = \beta K_b,$$

we obtain the following equation:

$$r_b - r_b \frac{B}{K_b} - \mu_\zeta R - \frac{1}{\beta B + R} \mu_b P = 0, \quad (3.12)$$

$$r_r - r_r \frac{R}{K_r} - \frac{1}{B + R} \mu_r P = 0, \quad (3.13)$$

$$r_p \mu_b \mu_r P = (r_p - q_p) (\mu_r B + \mu_b R), \quad (3.14)$$

Then, we consider the states variable

- $H = R + B$ (the total resource),
- $\phi = \frac{R}{B + R}$ (proportion of zooplankton within the total resource quantity),
- $Q = \frac{P}{R + B}$ (ratio fish/ total resource).

When, we use them expression

$B = H(1 - \phi)$, $R = \phi H$, $P = QH$, the above equations become:

$$r_b - H \left(\frac{r_b}{K_b} (1 - \phi) + \mu_\zeta \phi \right) - \mu_b Q = 0, \quad (3.15)$$

$$r_r - \frac{r_r}{K_r} \phi H - \mu_r Q = 0, \quad (3.16)$$

$$r_p \mu_b \mu_r Q = (r_p - q_p) \left(\mu_r (1 - \phi) + \mu_b \phi \right), \quad (3.17)$$

Consequently, through these equations we have the expressions of H and Q

$$Q = \frac{(r_p - q_p)}{r_p \mu_b \mu_r} \left[\mu_r (1 - \phi) + \mu_b \phi \right] \quad \text{and} \quad H = \frac{K_r}{r_r \phi} \left[r_r - \frac{(r_p - q_p)}{r_p} \left(\frac{\mu_r}{\mu_b} (1 - \phi) + \phi \right) \right] \quad (3.18)$$

After a substitution in (3.17), we obtain:

$$r_b - \frac{K_r}{r_r \phi} \left[r_r - \frac{(r_p - q_p)}{r_p} \left(\frac{\mu_r}{\mu_b} (1 - \phi) + \phi \right) \right] \left(\frac{r_b}{K_b} (1 - \phi) + \mu_\zeta \phi \right) - \frac{(r_p - q_p)}{r_p \mu_r} \left[\mu_r (1 - \phi) + \mu_b \phi \right] = 0 \quad (3.19)$$

we get the following equation by multiplying (3.19) by ϕ

$$a_2 \phi^2 + a_1 \phi + a_0 = 0 \quad (3.20)$$

with

$$a_0 = \frac{r_b K_r}{\beta K_b} \left(\frac{\mu_r}{\beta \mu_b} - 1 \right) \frac{(r_p - q_p)}{r_p}$$

$$a_1 = \frac{(r_p - q_p)}{r_p} \left[r_b \left(1 - \frac{r_b K_r}{\beta K_b} \right) - K_r \left(\frac{r_b}{\beta K_b} \right) \left(1 - \frac{\mu_r}{r_r \beta \mu_b} \right) - 1 \right] \quad (3.21)$$

$$a_2 = \frac{\mu_r}{\beta \mu_b} \left(\frac{K_r}{r_r} \left(\frac{r_b}{\beta K_b} - \mu_\zeta \right) + \frac{\beta \mu_b}{\mu_r} \right) \frac{(r_p - q_p)}{r_p}.$$

We search the triplets (H, ϕ, Q) verifying $0 < \phi < 1$, $H > 0$ and $Q > 0$.

Consequently, it's about to find the root $0 < \phi < 1$ of function polynomial π .

Existence of this root $0 < \phi < 1$ give us the existence of equilibrium state at component strictly positive of the system (3.5), [9, 10, 11]. Indeed, suppose that ϕ has a root $0 < \phi < 1$. According to (3.18) we have $Q > 0$ and H which write

$$H = \frac{K_r}{r_r \phi} h(\phi) \quad \text{with} \quad h(\phi) = r_r - \frac{(r_p - q_p)}{r_p} \left(\frac{\mu_r}{\mu_b} (1 - \phi) + \phi \right). \quad (3.22)$$

In the same manner using the equations (3.15) and (3.16) we can write H in the following style.

$$H = \frac{1}{\left(\frac{r_b}{K_b} (1 - \phi) + \mu_\zeta \phi \right)} l(\phi) \quad \text{with} \quad l(\phi) = r_b - \left(\frac{r_p - q_p}{r_p} \right) \left(\frac{\mu_b}{\mu_r} \phi + (1 - \phi) \right) \quad (3.23)$$

As we have $0 < \phi < 1$, according to (3.22) $H > 0$ if and only if $h(\phi) > 0$. Likewise $H > 0$ in (3.21) if and only if $h(\phi) > 0$.

Remark 1. In practice, it is sufficient to check whether at least one of the following conditions is satisfied:

- (i) $h(0) \geq 0$ and $h(1) > 0$
- (ii) $h(0) > 0$ and $h(1) \geq 0$
- (iii) $h(0) > 0$ and $h(1) < 0$
- (iv) $h(0) < 0$ and $h(1) > 0$

Proposition 3.9. *The polynomial (3.7) has a single positive root strictly if and only if one of the following is true:*

- (a) $-\text{signe}(a_0) = \text{signe}(a_1) = \text{signe}(a_2)$
- (b) $a_1^2 = 4a_0a_2$ and $\text{signe}(a_0) = \text{signe}(a_2) = -\text{signe}(a_1)$
- (c) $\text{signe}(a_0) = -\text{signe}(a_1)$ and $a_2 = 0$
- (d) $\text{signe}(a_0) = -\text{signe}(a_1)$ and $a_2 = 0$
- (e) $\text{signe}(a_1) = -\text{signe}(a_2)$ and $a_0 = 0$

In fact, the polynomial (3.7) admit two strictly positive root if and only if $a_1^2 - 4a_0a_2 > 0$ and $\text{signe}(a_0) = \text{signe}(a_2) = -\text{signe}(a_1)$

Proof: First, we consider the case $a_0 \neq 0$, $a_2 \neq 0$ and $a_1 \neq 0$. Let $\Delta = a_1^2 - 4a_0a_2$.

- (i) When $\Delta > 0$ we have a following case:
 - a) Two roots $\chi_1 > 0$ and $\chi_2 > 0$ if and only if $\frac{a_0}{a_2} > 0$ and $\frac{a_1}{a_2} < 0$ i.e $-\text{signe}(a_0) = \text{signe}(a_1) = \text{signe}(a_2)$;
 - b) One strictly positive root $\chi_0 > 0$ if and only if $\frac{a_0}{a_2} < 0$ and $\frac{a_1}{a_2} > 0$ i.e $-\text{signe}(a_0) = \text{signe}(a_1) = \text{signe}(a_2)$;
- (ii) If $\Delta = 0$, we have an unique root $\chi = \frac{-a_1}{2a_2} > 0$ if and only if $-\text{signe}(a_0) = \text{signe}(a_1) = \text{signe}(a_2)$;

We look at now of the particular case:

- first, if $a_0 \neq 0$, $a_2 = 0$ and $a_1 \neq 0$, then we have only root $\chi = \frac{-a_0}{a_1} > 0$ if and only if $-\text{signe}(a_0) = \text{signe}(a_1)$;
- then, if $a_0 \neq 0$, $a_2 \neq 0$ and $a_1 = 0$, so we have one root $\chi = \sqrt{\frac{-a_0}{a_2}} > 0$ if and only if $\text{signe}(a_0) = -\text{signe}(a_2)$;
- at last, if $a_0 = 0$, $a_2 \neq 0$ and $a_1 \neq 0$, so we have one root $\chi = \frac{-a_1}{a_2} > 0$ if and only if $\text{signe}(a_1) = -\text{signe}(a_2)$.

4 Numerical experiments and results

In this section, we present a synthesis of results observing during our numerical simulations of these different mathematical models.

4.1 Numerical simulations of two species system

For our simulations, we use the following parameters, [8, 9, 10, 11].

Parameters	B_0	R_0	μ_r	μ_b	P_0	q
Value used	100000	10000	180	180	300	1

4.1.1 Fish-Phytoplankton(B-P)

In this simulation, we take $r_b = 1,52$, $r_p = 0,53$ and $E = 0$. We consider that the system evolve in no exploited area i.e there is no fishing in this area.

The Figure 5 show the existence of the orbits of solution. As overall result, we have the existence and dynamic stability in long time for the system fish-phytoplankton(B-P). So we can say that if the area is not exploited the dynamic of Fish-Phytoplankton system remains stable.

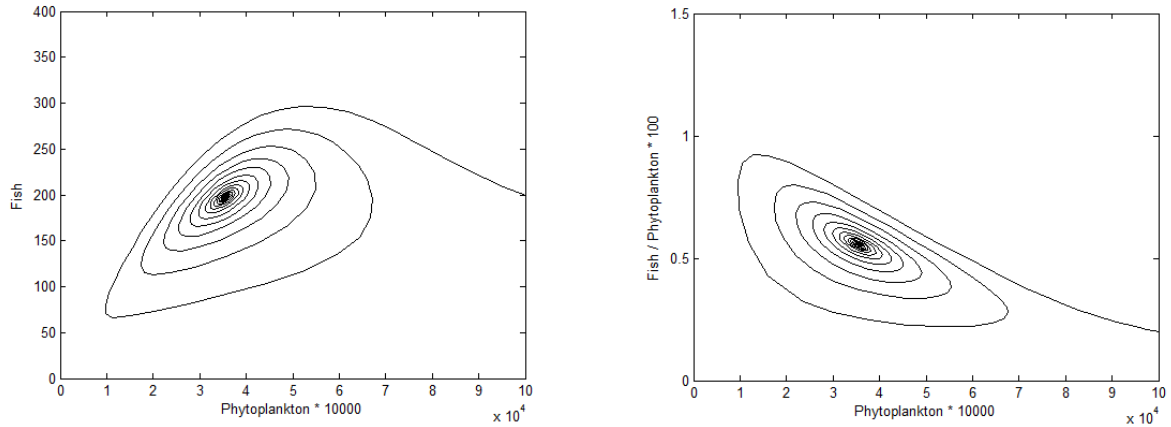


Figure 5: Fish-Phytoplankton(B-P) model with $E = 0$

Then, we consider that the system Fish-Phytoplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_b = 1,45$, $r_p = 0,5$ and $E = 0,5$.

The Figure 6 show the existence of the orbits of solution. As overall result, we have the existence and dynamic stability in long time for the system fish-phytoplankton (B-P). We observe a slight perturbation of this dynamic. We can say that if area

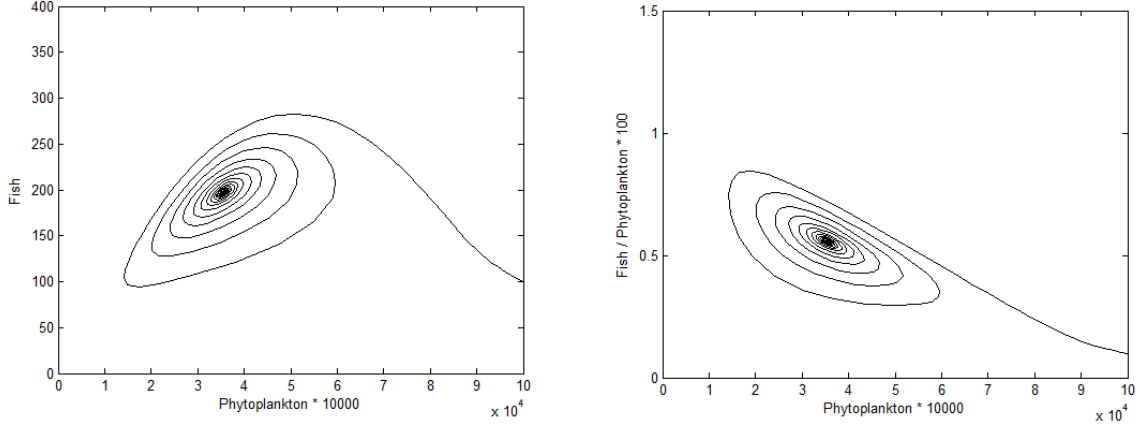


Figure 6: Fish-Phytoplankton(B-P) model with $E = 0,5$

is exploited with a fishing effort $E = 0,5$, the dynamic of Fish-Phytoplankton is stable. In this case, we speak about area normally exploited.

We continue the computational simulation and we consider that the system Fish-Phytoplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_b = 1,42$, $r_p = 0,58$ and $E = 2,3$.

The Figure 7 show the existence of the orbits of solution. As a consequence global we have extinction with large time of this dynamic. We observe a real disturbance of this dynamic. We can say that if, area is exploited with a fishing effort $E = 2,3$, the dynamic of Fish-Phytoplankton is unstable. In this case, we speak about the severely exploitation of the area.

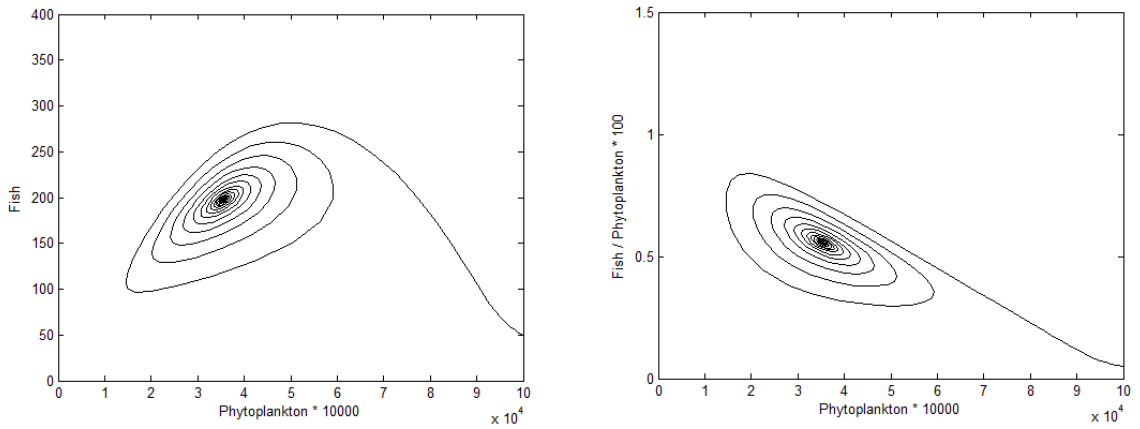


Figure 7: Fish-Phytoplankton(B-P) model with $E = 2,3$

4.1.2 Fish-Zooplankton(R-P)

We take $r_r = 1,52$, $r_p = 0,53$ and $E = 0$ for the simulation. We consider that the system evolve in no exploited area i.e that is no fishing in this area.

The Figure 8 show the existence of the orbits of solution. As a consequence global existence with large time stabilization of this dynamic is observed. We can say that if, area is not exploited, the dynamic of Fish-Zooplankton(R-P) is stable.

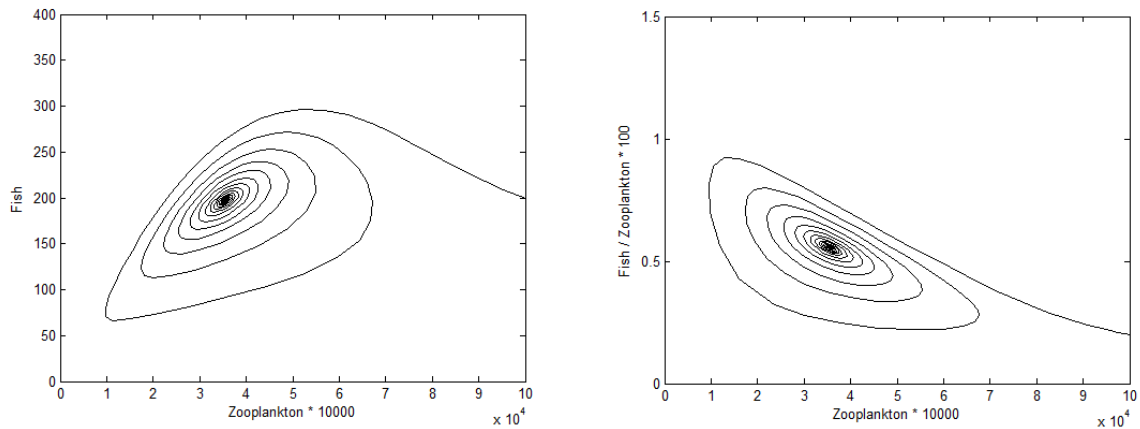


Figure 8: Fish-Zooplankton(P-R) model with $E = 0$

Then, we consider that the system Fish-Zooplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_r = 1,45$, $r_p = 0,5$ and $E = 0,5$.

The Figure 9 show the existence of the orbits of solution. As a consequence global existence with large time stabilization of this dynamic is observed. We observe also a slight perturbation of this dynamic. We can say that if, area is exploited with a fishing effort $E = 0,5$, the dynamic of Fish-Phytoplankton is stable. In this case, we say that the area is normally exploited.

We continue the computational simulation and we consider that the system Fish-Zooplankton evolve in exploited area. In this simulation, we take $r_b = 1,42$, $r_p = 0,58$ and $E = 2,3$.

The Figure 10 show the existence of the orbits of solution. We observe a real disturbance of this dynamic. We can say that if, area is exploited with a fishing effort $E = 2,3$, the dynamic of Fish-Phytoplankton is unstable. In this case, we say about severely exploited of area.

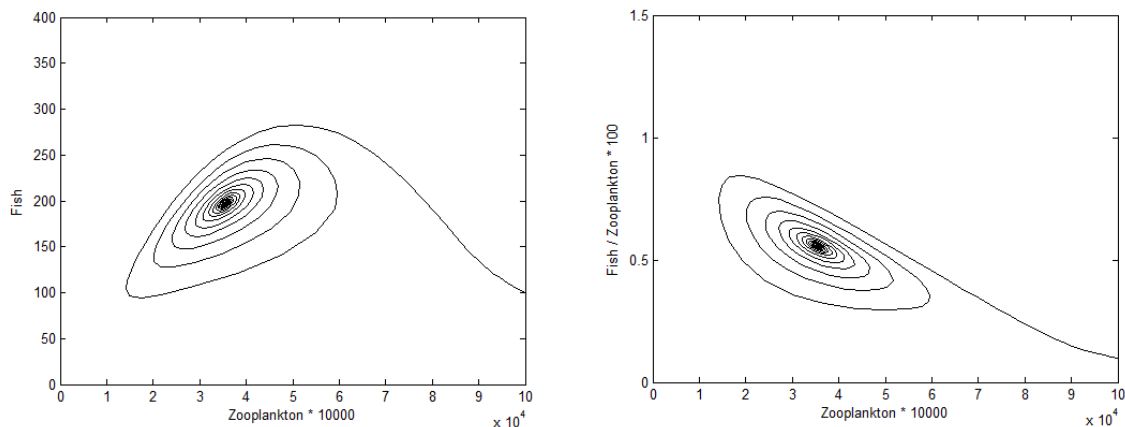


Figure 9: Fish-Zooplankton(P-R) model with $E = 0, 5$

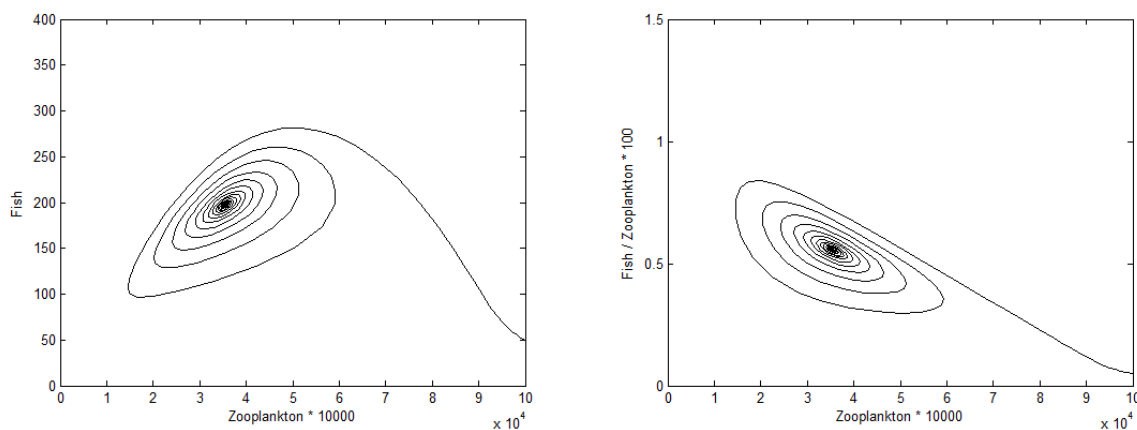


Figure 10: Fish-Zooplankton(P-R) model with $E = 2, 3$

4.1.3 Zooplankton-Phytoplankton(B-R)

The numerical simulation, show that for $r_b = 1, 53$, $r_r = 0, 53$, we have the existence of orbits of solution. As a consequence global we have the existence with large time stabilization of this dynamic. The Figure 11 show this observation.

4.2 Numerical simulations of three species: Phytoplankton-Zooplankton-Fish

In this subsection, for our simulations, the set of the demographic parameters

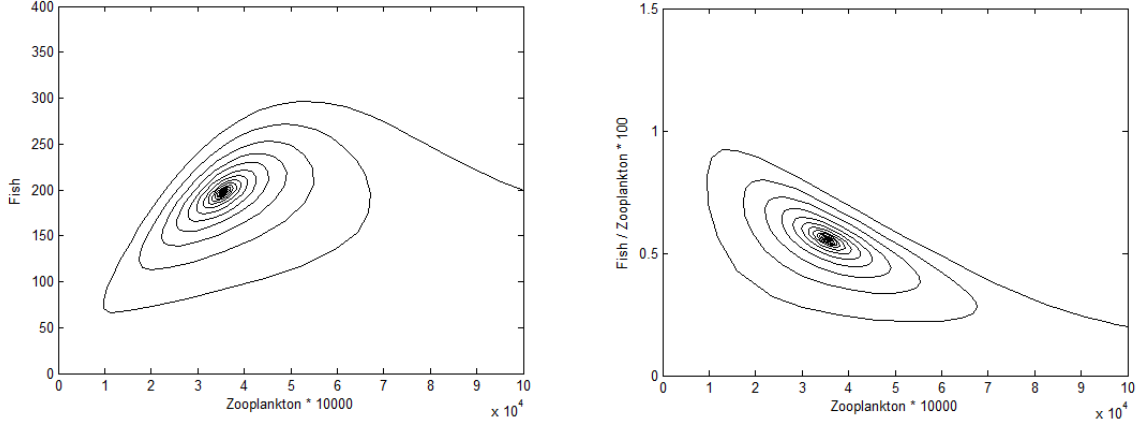


Figure 11: Zoooplankton-Phytoplankton(B-R)

used are given by the following list, [3, 6, 8, 9].

Parameters	B_0	R_0	μ_r	μ_b	P_0	q	K_r	K_b	μ_ζ	β
Value used	50000	2000	180	180	100	1	20000	100000	0,02	1,5

4.2.1 Numerical simulations on the Phytoplankton-Zooplankton-Fish system evolve in no exploited area

First, we look at the Phytoplankton-Zooplankton-Fish system which evolve in no exploited area and so we take the fishing effort $E = 0$.

The Figure 12 explain the stability of the different populations. As a consequence global we have the existence with large time stabilization of this dynamic.

4.2.2 Numerical simulations of the Phytoplankton-Zooplankton-Fish system evolve in exploited area

Second, we continue ours numerical simulations supposing that the Phytoplankton-Zooplankton-Fish system evolve in exploited area with a fishing effort $E = 0,5$.

The Figure 13 explain the stability of the Zooplankton and phytoplankton, and a imbalance of the fish populations. As a global consequence we observe the existence with large time of the fish population and his a imbalance. In this case, we speak about to the normally exploited of area. So, if a fish area is exploited with a fishing effort $E = 0,5$, there is not risk for the fish population.

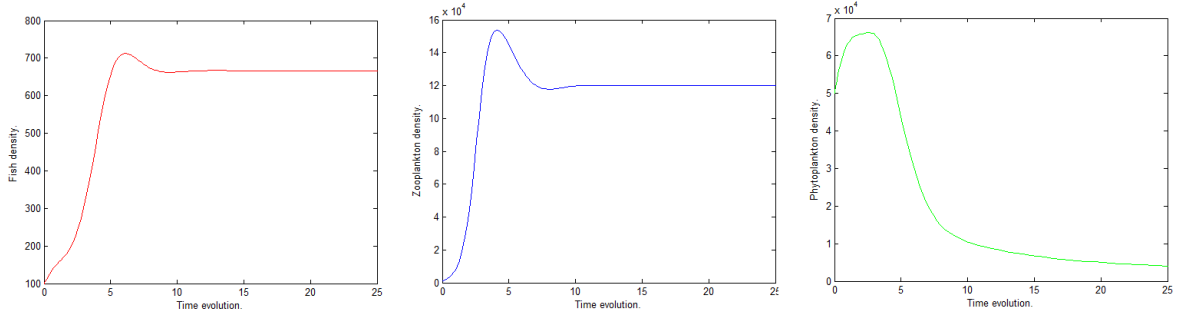


Figure 12: Evolution of three species: Phytoplankton-Zooplankton-Fish system evolve in no exploited area

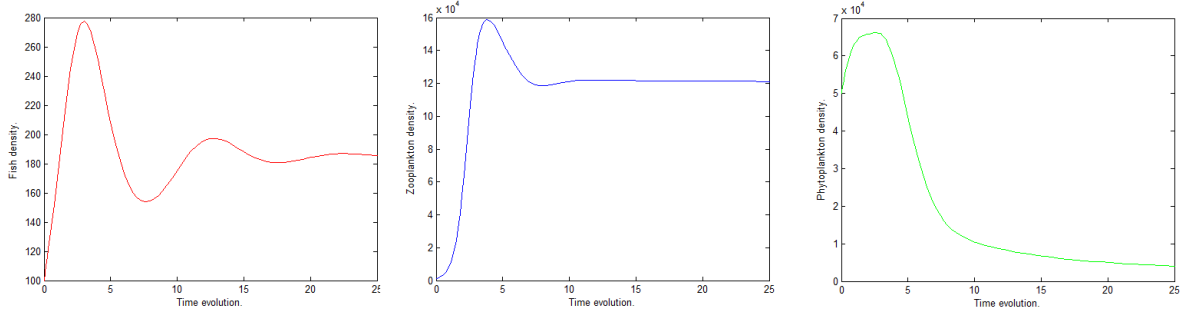


Figure 13: Evolution of the Phytoplankton-Zooplankton-Fish system evolve in normally exploited area

4.2.3 Numerical simulations on the Phytoplankton-Zooplankton-Fish system evolve in severely exploited area

Finally, we consider that the system Phytoplankton-Zooplankton-Fish evolve in exploited area with a fishing effort $E = 2, 3$.

The Figure 14 show the stability of the Zooplankton and phytoplankton species, and endangered fish species. So we have the extinction with large time of the fish species. In this case, we speak about the severely exploitation of the area. We can say that, if a fish area is exploited with a fishing effort $E = 2, 3$, there is a risk for the fish population and so it's necessary to develop political management for this area, otherwise fish species risk to disappear.

5 Conclusion

In this paper, we are interested in the study of fish population dynamic under a

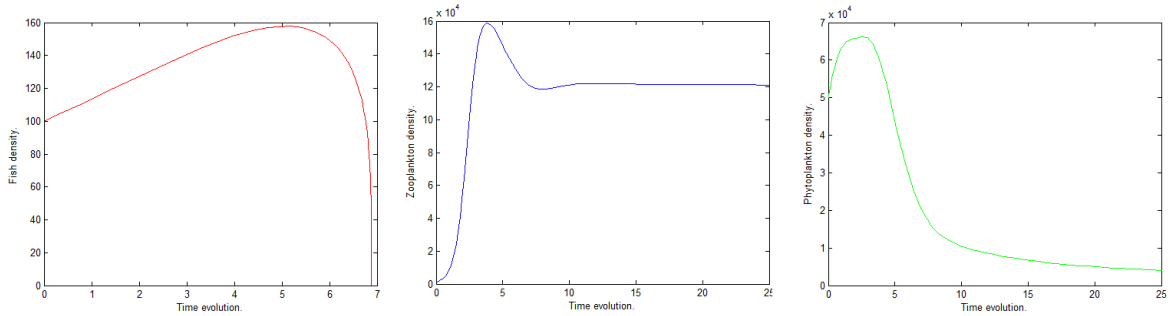


Figure 14: Evolution of the Phytoplankton-Zooplankton-Fish system evolve in severely exploited area

diet of a plankton (phytoplakton and zooplakton). A mathematic model associate of this dynamic based ODE singular system, the denominator of one the reaction terms being conclable,[9]. The mathematical analysis permits to isolate extinction condition based on a growth-rate, in finished or persisting time. We use numerical experiments to point out the main effects of the fishing on the dynamic. It is important to note that, for the fish dynamic, we could obtain stable coexistence. Equally, computational simulations, show that if the fishing effort is $E = 0,5$ area is exploited normally, while if $E > 0,5$ area is overexploited. In the future, we shall extend this work by taking into account two classes of the fish population: Larva and adult. This approach should give additional numerical results and should help us to determine which will be the more efficient strategy to protect the fish species.

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