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Positive solutions of singular (*k*,*n*-*k*) conjugate eigenvalue problem

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Abstract

Positive solution of singular nonlinear (*k*,*n*-*k*) conjugate eigenvalue problem is studied by employing the positive property of the Green's function, the fixed point theorem of concave function and Krasnoselskii fixed point theorem in cone.

Mathematics Subject Classification: 34B18

Keywords: Eigenvalue; Positive solution; Fixed point theorem

1 Introduction

This paper deals with the following singular (*k*,*n*-*k*) conjugate eigenvalue problem

$$(-1)^{n-k} y^{(n)}(x) = \lambda h(x) f(y) \qquad 0 < x < 1, \tag{1.1}$$

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$$y^{(i)}(0)=0, y^{(j)}(1)=0 \qquad 0\le i\le k-1, 0\le j\le n-k-1, \tag{1.2}$$

Where $1\le k\le n-1$ is a positive number and $\lambda>0$ is a parameter.

If the conjugate eigenvalue problem (1.1), (1.2) has a positive solution $y(x)$ for a particular λ, then λ is called an eigenvalue and $y(x)$ a corresponding eigenfunction of (1.1), (1.2). Let Λ be the set of eigenvalue of the problem (1.1), (1.2), i.e.

$$\Lambda=\{\lambda>0; \text{(1.1), (1.2) has a positive solution}\}.$$

In recent years, the conjugate eigenvalue problem (1.1), (1.2) has been studied extensively, for special case of $\lambda=1$, the existence results of positive solution of the problem (1.1), (1.2) has been established in [1-6], and as for twin positive solutions, several studies to the problem (1.1), (1.2) can be found in [7-9]. For the case of $\lambda>0$, eigenvalue intervals characterizations of the problem (1.1), (1.2) has been discussed in [10] by using Krasnoselskii fixed point theorem if $f(y)$ is superlinear or sublinear. In this paper, by employing property of Green function, the fixed point theorem of concave function and Krasnoselskii fixed point theorem in cone, we give eigenvalue characterizations under different hypothesis condition, and we may allow that $f(x)$ is singular at $x=0,1$. By using different method from [10] we establish not only existence of positive solution but also multiplicity of positive solutions of the problem (1.1), (1.2).

Our assumptions throughout are:

(H_1) $h(x)$ is a nonnegative measurable function defined in (0,1) and do not vanish identically on any subinterval in (0,1) and

$$0<\int_0^1 s^{n-k}(1-s)^k h(s)\mathrm{d}s<\int_0^1 s^{n-k-1}(1-s)^{k-1}h(s)\mathrm{d}s<+\infty;$$

(H_2) $f:[0,+\infty)\to[0,+\infty)$ is a nondecreasing continuous function and $f(y)>0$ for $y>0$;

(H_3) $f_0=\lim_{y\to 0}\dfrac{f(y)}{y}=0$, $f_\infty=\lim_{y\to\infty}\dfrac{f(y)}{y}=0$.

By a positive solution $y(x)$ of the problem (1.1), (1.2), we means that $y(x)$ satisfies

(a) $y(x) \in C^{k-1}[0,1) \cap C^{n-k-1}(0,1] \cap C^{n-1}(0,1)$, $f(y) > 0$ in $(0,1)$ and (1.2) holds;

(b) $y^{(n-1)}(x)$ is locally absolutely continuous in (0,1) and

$$(-1)^{n-k} y^{(n)}(x) = \lambda h(x) f(y(x)) \quad \text{a.e. in } (0,1).$$

The main results of this paper are as follows.

Theorem 1 Assume that (H_1), (H_2) and (H_3) hold. Then there exist two positive numbers λ_*, λ^* with $0 < \lambda_* \le \lambda^* < +\infty$ such that

(i). (1.1),(1.2) has no positive solution for $\lambda \in (0, \lambda_*)$;

(ii). (1.1),(1.2) has at least one positive solution for $\lambda \in (\lambda_*, \lambda^*]$;

(iii). (1.1),(1.2) has at least two positive solutions for $\lambda \in (\lambda^*, +\infty)$;

(iv). (1.1),(1.2) has nonnegative solution for $\lambda = \lambda_*$.

2 Preliminary Notes

In this section, we provide some properties of the Green's function for the problem (1.1),(1.2) which are needed later, and state the fixed point theorems required. As shown in [6], the problem (1.1),(1.2) is equivalent to the integral equation

$$y(x) = \lambda \int_0^1 G(x,s) h(s) f(y(s)) \mathrm{d}s, \tag{2.1}$$

where

$$G(x,s)=\begin{cases}\dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^{s(1-x)}t^{n-k-1}(t+x-s)^{k-1}\mathrm{d}t, & 0\le s\le x\le 1,\\ \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^{x(1-s)}t^{k-1}(t+s-x)^{n-k-1}\mathrm{d}t, & 0\le x\le s\le 1.\end{cases} \tag{2.2}$$

Moreover, the following results have been offered by Kong and Wang [6].

Lemma 2.1 For any $x,s\in[0,1]$, we have

$$\alpha(x)g(s)\le G(x,s)\le\beta(x)g(s), \tag{2.3}$$

$$\left|\frac{\partial G(x,s)}{\partial x}\right|\le\begin{cases}\dfrac{ng(s)}{1-s}, 0\le s\le x\le 1,\\ \dfrac{ng(s)}{s}, 0\le x\le s\le 1.\end{cases} \tag{2.4}$$

where

$$\alpha(x)=\frac{x^k(1-x)^{n-k}}{n-1},\ \beta(x)=\frac{x^{k-1}(1-x)^{n-k-1}}{\min\{k,n-k\}},\ g(x)=\frac{s^{n-k}(1-s)^k}{(k-1)!(n-k-1)!}.$$

Let K be a cone in Banach space E. We say that a map Ψ is a nonnegative continuous concave function on K, if it satisfied: $\Psi:K\to[0,+\infty)$ is continuous and

$$\Psi(\alpha x+(1-\alpha)y)\ge\alpha\Psi(x)+(1-\alpha)\Psi(y)$$

for all $x,y\in K$ and $0\le\alpha\le 1$.

Theorem 2 [12] Le K be a cone in Banach space E. For given $R>0$, define $K_R=\{u\in K;\|u\|<R\}$. Assume that $T:\bar{K}_R\to\bar{K}_R$ is a completely continuous operator and Ψ is a nonnegative continuous concave function on K such that $\Psi(y)\le\|y\|$ for all $y\in\bar{K}_R$. Suppose that there exist $0<a<b\le R$ such that

(A) $\{y\in K(\Psi,a,b);\Psi(y)>a\}\ne\phi$, and $\Psi(Ty)>a$ for all $y\in K(\Psi,a,b)$ where

$$K(\Psi,a,b)=\{y\in K;\Psi(y)\ge a;\|y\|\le b\};$$

(B) $\|Ty\|<r$ for all $y\in\bar{K}_r$;

(C) $\Psi(Ty) > a$ for all $y \in K(\Psi, a, R)$ with $\|Ty\| > b$,

then T has at least three fixed points y_1, y_2 and y_3 in $\bar{K}_R$ satisfying $y_1 \in K_r$, $y_2 \in \{y \in K(\Psi, a, R); \Psi(y) > a\}$ and $y_3 \in \bar{K}_R \setminus (\overline{K(\Psi, a, R)} \cup \bar{K}_r)$.

Theorem 3 [13] Let E be a Banach space, and $K \subseteq E$ a cone in E **.** Assume Ω_1, Ω_2 are open subset of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let $\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(I) $\|\Phi u\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$; or

(II) $\|\Phi u\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$,

then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main Results

Let $\alpha = \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \alpha(x)$, $\beta = \max_{x \in [0,1]} \beta(x)$ and $\gamma = \dfrac{\alpha}{\beta}$. Define the cone in Banach space $C[0,1]$ given as

$$P = \{y(x) \in \mathrm{C}[0,1]; y(x) \ge 0\},$$

$$K = \{y(x) \in \mathrm{C}[0,1]; \min_{x \in \left[\frac{1}{4}, \frac{3}{4}\right]} y(x) \ge \gamma \|y\|\}.$$

We define the operator $T : P \to P$ by

$$(Ty)(x) := \lambda \int_0^1 G(x,s) h(s) f(y(s)) \mathrm{d}s. \tag{3.1}$$

Lemma 3.1 Suppose that (H_1)-(H_3) hold. Then $T : P \to P$ is a completely continuous mapping and $T(K) \subset K$. Moreover, for $y \in K$ we have

$$(Ty)(x) \in C^{k-1}[0,1) \cap C^{n-k-1}(0,1] \cap C^{n-1}(0,1),$$

$$(-1)^{n-k}(Ty)^{(n)}(x)=\lambda h(x)f(y(x)) \quad \text{a.e. } x\in(0,1),$$

$$(Ty)^{(i)}(0)=0,(Ty)^{(j)}(1)=0,\ 0\le i\le k-1,\ 0\le j\le n-k-1.$$

Proof We only prove $T(K)\subset K$. The proof of the remainder of Lemma 3.1 can be found in [6].

For $y\in K$, by employing (2.3) we have

$$\begin{aligned}\min_{x\in[\frac{1}{4},\frac{3}{4}]}(Ty)(x)&\ge\lambda\min_{x\in[\frac{1}{4},\frac{3}{4}]}\alpha(s)\int_0^1 g(s)h(s)f(y(s))\mathrm{d}s\\&\ge\frac{\lambda\alpha}{\beta}\max_{x\in[0,1]}\int_0^1 G(x,s)h(s)f(y(s))\mathrm{d}s\\&=\gamma\|Ty\|,\end{aligned}$$

this implies $T(K)\subset K$. The proof is complete. □

It follows from the lemma 3.1 that we know that $T(K)\subset K$ and fixed point in K of T is a solution of the problem (1.1),(1.2) and vice versa.

Lemma 3.2 Suppose that (H_1), (H_2) and (H_3) hold. Then there exists $0<\lambda^*<+\infty$ such that the problem (1.1), (1.2) has at least two positive solutions for $\lambda\in(\lambda^*,+\infty)$.

Proof It follows from $\lim\limits_{y\to+\infty}\frac{f(y)}{y}=0$ that there exists $R_1>0$ such that $f(y)\le\varepsilon y$ for all $y\ge R_1$, where ε satisfies $\varepsilon\lambda\beta\int_0^1 g(s)h(s)\mathrm{d}s<1$. Let $M=\max\limits_{0\le y\le R_1}f(y)$, then $f(y)\le M+\varepsilon y$ for all $y\ge 0$. By (H_3) we get $\lim\limits_{y\to 0}\frac{y}{f(y)}=\lim\limits_{y\to+\infty}\frac{y}{f(y)}=+\infty$, thus, there exists $0<b<+\infty$ such that $\frac{b}{f(b)}=\min\limits_{y\ge 0}\frac{y}{f(y)}$, and hence there exists $0<\lambda^*<+\infty$ such that $\lambda^*=[\frac{ab}{f(b)}\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)\mathrm{d}s]^{-1}$. Clearly, for all $y\ge 0$ we have

$$f(y) \geq \frac{b}{f(b)} y . \tag{3.2}$$

We shall now show that the conditions of Theorem 2 are satisfied. Choose $R = \max\{b+1, 2M\lambda\beta(1-\varepsilon\lambda\beta\int_0^1 g(s)h(s)\mathrm{d}s)^{-1}\int_0^1 g(s)h(s)\mathrm{d}s\}$. For $y \in \overline{K}_R$ we have

$$\begin{aligned}(Ty)(x) &\leq \lambda \max_{x\in[0,1]}\int_0^1 G(x,s)h(s) f(y(s))\mathrm{d}s \\ &\leq \lambda \max_{x\in[0,1]} \beta(x)\int_0^1 g(s)h(s)(M+\varepsilon y(s))\mathrm{d}s \\ &\leq \lambda\beta(M+\varepsilon\|y\|)\int_0^1 g(s)h(s)\mathrm{d}s \\ &= \lambda\beta(M+\varepsilon R)\int_0^1 g(s)h(s)\mathrm{d}s < R ,\end{aligned}$$

this shows $T(\overline{K}_R) \subset K_R \subset \overline{K}_R$.

Let $\Psi : K \to [0,+\infty)$ be defined by

$$\Psi(y) = \min_{x\in[\frac{1}{4},\frac{3}{4}]} y(x) ,$$

clearly, Ψ is a nonnegative continuous concave function on K such that $\Psi(y) \leq \|y\|$ for all $y \in K$.

It is noted that

$$y(x) = \frac{1}{2}(\gamma b + b) \in \{ y \in K(\Psi, \gamma b, b); \Psi(y) \geq \gamma b\} \neq \phi ,$$

let $y \in K(\Psi, \gamma b, b)$, then $\min_{x\in[\frac{1}{4},\frac{3}{4}]} y(x) = \Psi(y) \geq \gamma b$ and $y \leq \|y\| \leq b$. Using this together with (3.2), for $\lambda > \lambda^*$ we get

$$\begin{aligned}\Psi(Ty)(x) &= \min_{x\in[\frac{1}{4},\frac{3}{4}]} (Ty)(x) \\ &\geq \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \alpha(x)\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s) f(y(s))\mathrm{d}s \\ &\geq \frac{\lambda\alpha b}{f(b)}\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)y(s)\mathrm{d}s\end{aligned}$$

$$\geq \frac{\lambda\alpha\gamma b^2}{f(b)}\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)\mathrm{d}s$$

$$= \frac{\lambda\gamma b}{\lambda^*} > \gamma b.$$

Hence, condition (A) of Theorem 2 is satisfied.

By $\lim\limits_{y\to 0}\frac{f(y)}{y} = 0$, there exists $0 < r < \gamma b$ such that $f(y) < \varepsilon y$ for all $y \leq r$, where ε satisfies $\varepsilon\lambda\beta\int_0^1 g(s)h(s)\mathrm{d}s < 1$. For $y \in \bar{K}_r$ we have

$$\|Ty\| \leq \lambda\beta\int_0^1 g(s)h(s)f(y(s))\mathrm{d}s \leq \varepsilon\lambda\beta\|y\|\int_0^1 g(s)h(s)\mathrm{d}s < r,$$

this implies that condition (B) of Theorem 2 is satisfied.

Finally, for $y \in K(\Psi, \gamma b, b)$ with $\|Ty\| > b$, we obtain

$$\Psi(Ty)(x) = \min_{x\in[\frac{1}{4},\frac{3}{4}]} (Ty)(x)$$

$$\geq \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \alpha(x)\int_0^1 g(s)h(s)f(y(s))\mathrm{d}s$$

$$\geq \frac{\alpha\lambda}{\beta}\max_{x\in[0,1]}\int_0^1 G(x,s)h(s)f(y(s))\mathrm{d}s$$

$$= \gamma\|Ty\| > \gamma b.$$

Therefore, the condition (C) of Theorem 2 is also satisfied. Consequently, an application of Theorem 2 shows that the problem (1.1), (1.2) has at least three solutions $y_1, y_2, y_3 \in \bar{K}_R$.

Further,

$y_1 \in K_r$, $y_2 \in \{y \in K(\Psi, \gamma b, R); \Psi(y) > \gamma b\}$ and $y_3 \in \bar{K}_R \setminus (\overline{K(\Psi, \gamma b, R)} \cup \bar{K}_r)$. This shows that $y_2(x)$ and $y_3(x)$ are two positive solution of the problem (1.1), (1.2), and $y_1(x)$ is a nonnegative solution of (1.1), (1.2). □

Lemma 3.3 Suppose that (H_1), (H_2) and (H_3) hold. If λ is sufficiently small, then $\lambda \notin \Lambda$.

Proof If $\lambda \in \Lambda$, then the problem (1.1),(1.2) has a positive solution $y_\lambda(x) \in K$ and it satisfies (2.1). We note that (H_3) implies the existence of a constant $\eta > 0$ such that $f(y) \le \eta y$ for all $y \ge 0$. By employing (2.1) we have

$$\|y_\lambda\| \le \lambda\beta\int_0^1 g(s)h(s)f(y_\lambda(s))\mathrm{d}s$$
$$\le \lambda\beta\eta\|y_\lambda\|\int_0^1 g(s)h(s)\mathrm{d}s,$$

this means $\lambda\beta\eta\int_0^1 g(s)h(s)\mathrm{d}s \ge 1$, which contradicts with λ sufficiently small. □

Lemma 3.4 Suppose that (H_1), (H_2) and (H_3) hold. Then there exists a $\lambda_0 > 0$ such that $[\lambda_0, +\infty) \subset \Lambda$.

Proof Let $K_1 = \{y \in K, \|y\| < 1\}$, choose $\lambda_0 = (\alpha f(\gamma))(\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)\mathrm{d}s)^{-1}$, for $y \in K \cap \partial K_1$ and $\lambda \in [\lambda_0, +\infty)$, then $\min_{x\in\left[\frac{1}{4},\frac{3}{4}\right]} y(x) \ge \gamma\|y\| = \gamma$, by using (H_1) and (H_2) we have

$$\|Ty\| \ge \lambda_0 \min_{x\in\left[\frac{1}{4},\frac{3}{4}\right]} \alpha(x)\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)f(y(s))\mathrm{d}s$$
$$\ge \lambda_0\alpha f(\gamma)\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)\mathrm{d}s$$
$$= 1 = \|y\|,$$

i.e. $\|Ty\| \ge \|y\|$ for $y \in K \cap \partial K_1$.

It follows from (H_3) that there exist $R_0 > 1$ such that $f(y) \le \varepsilon y$ for all $y \ge R_0$, where ε satisfies $\varepsilon\lambda^*\beta\int_0^1 g(s)h(s)\mathrm{d}s < 1$. Let

$$R = R_0 + \lambda^*\beta \max_{0\le y\le R_0} f(y)\int_0^1 g(s)h(s)\mathrm{d}s,$$

and $K_R = \left\{ y \in K, \|y\| < R \right\}$,then for $y \in K \cap \partial K_R$, we have

$$\begin{aligned}
\|Ty\| &\le \lambda^* \beta \int_0^1 g(s)h(s) f(y(s))\mathrm{d}s \\
&\le \lambda^* \beta [\int_{0 \le y(s) \le R_0} g(s)h(s) f(y(s))\mathrm{d}s + \int_{R_0 \le y(s) \le R} g(s)h(s) f(y(s))\mathrm{d}s] \\
&\le \lambda^* \beta (\max_{0 \le y \le R_0} f(y) + \varepsilon R_0) \int_0^1 g(s)h(s)\mathrm{d}s \\
&< \lambda^* \beta \max_{0 \le y \le R_0} f(y) \int_0^1 g(s)h(s)\mathrm{d}s + R_0 \\
&= R = \|y\|,
\end{aligned}$$

i.e. $\|Ty\| \le \|y\|$ for $y \in K \cap \partial K_R$. □

In view of the Theorem 3, we know that T has a fixed point $y(x)$ in $K \cap (\bar{K}_R \setminus K_1)$. That is to say, the integral equation (2.1) has at least one positive solution $y(x)$, and hence $y(x)$ is a positive solution of (*k*,*n*-*k*) conjugate eigenvalue problem (1.1),(1.2).

Lemma 3.5 Suppose that (H_1), (H_2) and (H_3) hold. Then the problem (1.1),(1.2) has a nonnegative solution for $\lambda = \lambda_*$.

Proof Without loss of generality, let $\{\lambda_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence, $\lim_{n\to\infty} \lambda_n = \lambda_*$, and $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ be corresponding positive solution sequence where $\lambda_n \in \Lambda$. We claim that $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ is uniformly bounded. If it is not true, then $\lim_{n\to\infty} \|y_{\lambda_n}\| = +\infty$. It follows from $\lim_{y\to+\infty} \frac{f(y)}{y} = 0$; that there exists $M > 0$ such that $f(y) \le M + \varepsilon y$ for all $y \ge 0$, where ε satisfies $\varepsilon \lambda \beta \int_0^1 g(s)h(s)\mathrm{d}s < 1$. So we get

$$\|y_{\lambda_n}\| \le \lambda_n \beta \int_0^1 g(s)h(s) f(y_{\lambda_n}(s))\mathrm{d}s$$

$$\leq \lambda_n \beta (M + \varepsilon \| y_{\lambda_n} \|) \int_0^1 g(s)h(s)\mathrm{d}s. \tag{3.3}$$

Let $n \to \infty$ in (3.3) to yield

$$\lim_{n\to\infty} \| y_{\lambda_n} \| \leq \frac{\lambda\beta\int_0^1 g(s)h(s)\mathrm{d}s}{1-\varepsilon\lambda\beta\int_0^1 g(s)h(s)\mathrm{d}s} < +\infty,$$

which is a contradiction. Thus, there exists a number L with $0 < L < +\infty$ such that $\| y_{\lambda_n} \| \leq L$ for all n.

It follows from (H_1) and (2.4) that we have

$$\| y'_{\lambda_n} \| \leq \lambda_n \int_0^1 \frac{\partial G(x,s)}{\partial x} h(s) f(y_{\lambda_n}(s))\mathrm{d}s$$

$$\leq \frac{\lambda^* n f(L)}{(k-1)!(n-k-1)!}\left(\int_0^x s^{n-k}(1-s)^{k-1}h(s)\mathrm{d}s + \int_x^1 s^{n-k-1}(1-s)^k h(s)\mathrm{d}s\right)$$

$$\leq \frac{2n\lambda^* f(L)}{(k-1)!(n-k-1)!}\int_0^1 s^{n-k-1}(1-s)^{k-1}h(s)\mathrm{d}s := Q,$$

this shown that $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ is equicontinuous. Ascoli-Arzela theorem claims that $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ has a uniformly convergent subsequence, denoted again by $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$, and $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ converges to $y^*(x)$ uniformly on [0,1]. Inserting $y_{\lambda_n}(x)$ into (2.1) and letting $n \to \infty$, using the Lebesgue dominated convergence theorem, we obtain

$$y^*(x) = \lambda^* \int_0^1 G(x,s)h(s)f(y^*(s))\mathrm{d}s \geq 0,$$

thus, $y^*(x)$ is a nonnegative solution of (1.1),(1.2). □

Remark It is possible that $y^*(x) = 0$.

Let $\lambda_* = \inf \Lambda$, then from Lemma 3.3 and lemma 3.4 we know $\lambda_* > 0$ and $0 < \lambda_* \leq \lambda^* < +\infty$. So (*k*,*n*-*k*) conjugate eigenvalue problem

(1.1),(1.2) has no positive solution for $\lambda \in (0, \lambda_*)$, has at least one positive solution for $\lambda \in (\lambda_*, \lambda^*]$, has at least two positive solutions for $\lambda \in (\lambda^*, +\infty)$, and has a nonnegative solution for $\lambda = \lambda_*$.

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