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Positive solutions of singular (*k*,*n*-*k*) conjugate boundary value problem

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Abstract

Positive solution of singular nonlinear $(k, n-k)$ conjugate boundary value problem is studied by employing a priori estimates, the cone theorem and the fixed point index.

Keywords: Positive solution; Multiplicity; Fixed point index

1 Introduction

In this paper, we are concerned with positive solutions for singular (*k*,*n*-*k*) conjugate boundary value problem

$$(-1)^{n-k} y^{(n)}(x) = \lambda h(x) f(y), \qquad 0 < x < 1, \tag{1.1}$$

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$$y^{(i)}(0)=0,\, y^{(j)}(1)=0, 0\le i\le k-1, 0\le j\le n-k-1 \tag{1.2}$$

where $1\le k\le n-1$ is a positive number and $\lambda>0$ is a parameter. Our assumptions throughout are:

(H_1) $h(x)$ is a nonnegative measurable function defined in $(0,1)$ and do not vanish identically on any subinterval in $(0,1)$ and

$$0<\int_0^1 s^{n-k}(1-s)^k h(s)ds<\int_0^1 s^{n-k-1}(1-s)^{k-1}h(s)ds<+\infty;$$

(H_2) $f:[0,+\infty)\to[0,+\infty)$ is a nondecreasing continuous function

and $f(y)>0$, for $y>0$;

(H_3) $f(0)>0$, $\lim\limits_{y\to+\infty}\dfrac{f(y)}{y}=+\infty$;

By a positive solution $y(x)$ of the problem $(1.1),(1.2)$, we means that $y(x)$ satisfies

(i) $y(x)\in C^{k-1}[0,1)\cap C^{n-k-1}(0,1]\cap C^{n-1}(0,1)$, $f(y)>0$ >0 in $(0,1)$ and (1.2) holds;

(ii) $y^{(n-1)}(x)$ is locally absolutely continuous in $(0,1)$ and

$$(-1)^{n-k}y^{(n)}(x)=\lambda h(x)f(y(x)) \quad \text{a.e. in } (0,1).$$

If for a particular λ the conjugate boundary value problem $(1.1),(1.2)$ has a positive solution $y(x)$, then λ is called an eigenvalue and $y(x)$ a corresponding eigenfunction of $(1.1),(1.2)$. We let Λ be the set of eigenvalue of the problem $(1.1),(1.2)$, i.e.

$$\Lambda=\{\lambda>0;\ (1),\ (2) \text{ has a positive solution }\}.$$

The conjugate boundary value problem $(1.1),(1.2)$ has been studied extensively, for detail,see, for instance, $[1-11]$ and references therein. For special case of $\lambda=1$, the existence results of positive solution of $(1.1),(1.2)$ has been established in $[1-4]$ .As for twin positive solutions, several studies to the problem $(1.1),(1.2)$ can be found in $[5-7]$. For the case of $\lambda>0$, eigenvalue intervals characterizations of theproblem

The equations in $(1.1),(1.2)$ has been discussed in $[8-11]$. In this paper, we only use fixed point index and fixed point theorem in cone which allow us to establish not only existence of positive solution but also multiplicity of positive solutions of the problem $(1.1),(1.2)$ under weaker conditions. In fact, we may allow that h possesses strong singularity at $x=0,1$, for example,

$$h(x)=x^{n-k-\alpha+1}(1-x)^{-k-\beta+1}$$

with $0<\alpha,\ \ \beta<1$ satisfies (H_1). Our results generalized and extend some known theorems and improve the work of some author of the above references $[8-11]$.

2 Preliminary Notes

To obtain positive solutions for the problem $(1.1),(1.2)$, we state some properties of Green's function for $(1.1),(1.2)$.

As shown in $[4]$, the problem $(1.1),(1.2)$ is equivalent to the integral equation

$$y(x)=\lambda\int_0^1 G(x,s)h(s)f(y(s))ds \tag{2.1}$$

where

$$G(x,s)=\begin{cases}\dfrac{1}{(k-1)!(n-k-1)!}\int_0^{s(1-x)} t^{n-k-1}(t+x-s)^{k-1}dt, & 0\le s\le x\le 1,\\ \dfrac{1}{(k-1)!(n-k-1)!}\int_0^{x(1-s)} t^{k-1}(t+s-x)^{n-k-1}dt, & 0\le x\le s\le 1.\end{cases} \tag{2.2}$$

Moreover, the following results have been recently offered by Kong and Wang[4].

Lemma 1 For any $x,s\in[0,1]$, we have

$$\alpha(x)g(s)\le G(x,s)\le \beta(x)g(s), \tag{2.3}$$

$$\left|\frac{\partial G(x,s)}{\partial x}\right|\le\begin{cases}\dfrac{ng(s)}{1-s}, 0\le s\le x\le 1,\\ \dfrac{ng(s)}{s}, 0\le x\le s\le 1.\end{cases} \tag{2.4}$$

where

$$\alpha(x)=\frac{x^k(1-x)^{n-k}}{n-1},\ \beta(x)=\frac{x^{k-1}(1-x)^{n-k-1}}{\min\{k,n-k\}},\ g(x)=\frac{s^{n-k}(1-s)^k}{(k-1)!(n-k-1)!}.$$

Let $\alpha=\min_{x\in[\frac{1}{4},\frac{3}{4}]}\alpha(x)$, $\beta=\max_{x\in[0,1]}\beta(x)$ and $\gamma=\dfrac{\alpha}{\beta}$. Define the cone in Banach space $C[0,1]$ give by

$$P:=\{y(x)\in \mathrm{C}[0,1];\ y(x)\ge 0\},$$

$$K:=\{y(x)\in \mathrm{C}[0,1];\ \min_{x\in\left[\frac{1}{4},\frac{3}{4}\right]} y(x)\ge\gamma\|y\|\}.$$

We define the operator $T:P\to P$ by

$$(Ty)(x):=\lambda\int_0^1 G(x,s)h(s)f(y(s))ds. \tag{2.5}$$

Lemma 2 Suppose that $(H_1)-(H_3)$ hold. Then $T:P\to P$ is a completely continous mapping and $T(P) \subset K$ .Moveover, for $y \in K$ we have

$$(Ty)(x) \in C^{k-1}[0,1) \cap C^{n-k-1}(0,1] \cap C^{n-1}(0,1), \tag{2.6}$$

$$(-1)^{n-k}(Ty)^{(n)}(x) = \lambda h(x) f(y(x)) \quad \text{a.e. } x \in (0,1), \tag{2.7}$$

$$(Ty)^{(i)}(0) = 0, (Ty)^{(j)}(1) = 0,\ 0 \le i \le k-1,\ 0 \le j \le n-k-1. \tag{2.8}$$

Proof. We only prove $T(P) \subset K$. The proof of the remainder of Lemma 2 can be found in [4] .

For $y \in P$,by employing (2.3) and (2.5) we have

$$\begin{aligned}\min_{x\in[\frac{1}{4},\frac{3}{4}]}(Ty)(x) &\ge \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \alpha(s)\int_0^1 g(s)h(s)f(y(s))ds \\ &\ge \frac{\lambda\alpha}{\beta}\max_{x\in[0,1]}\int_0^1 G(x,s)h(s)f(y(s))ds \\ &= \gamma\|Ty\|,\end{aligned}$$

this implies $T(P) \subset K$.

From the Lemma 1, we know that $T(K) \subset K$ and fixed point of T in K is a solution of (1.1),(1.2) and vice versa. □

3 Main Results

The main results of this paper are as follows.

Theorem 1 Assume that $(H_1)-(H_3)$ hold. Then there exists a number λ^* with $0 < \lambda^* < +\infty$ such that

(i) $(1.1),(1.2)$ has two positive solution for $\lambda \in (0,\lambda^*)$;

(ii) $(1.1),(1.2)$ has a positive solution for $\lambda = \lambda^*$;

(iii) $(1.1),(1.2)$ has no positive solution for $\lambda \in (\lambda^*,+\infty)$.

The following theorem will be used in our proof.

Theorem 2 [13] Let E be a Banach space, and $K \subseteq E$ a cone in E. For $\rho > 0$, define $K_\rho = \{u \in K; \|u\| \le \rho\}$. Assume that $\Phi : K_\rho \to K$ is a compact map such that $\Phi u \ne u$ for $u \in \partial K_\rho = \{u \in K; \|u\| = \rho\}$.

(a) If $\|u\| \le \|\Phi u\|$ for $u \in \partial K_\rho$, then $i(\Phi, K_\rho, K) = 0$;

(b) If $\|u\| \ge \|\Phi u\|$ for $u \in \partial K_\rho$, then $i(\Phi, K_\rho, K) = 1$.

Lemma 3 Suppose that $(H_1)-(H_3)$ hold. If λ is sufficiently large, then $\lambda \notin \Lambda$.

Proof. if $\lambda \in \Lambda$, then the problem $(1.1),(1.2)$ has a positive solution $y_\lambda(x)$, and Lemma 2 means that $y_\lambda(x) \in K$ is a fixed point of T. It follows from $\lim\limits_{y\to+\infty} \frac{f(y)}{y} = +\infty$ that there exists a $M > 0$ such that $f(y) \ge y$ for all $y \ge M$.

If $\|y_\lambda\| \ge \frac{M}{\gamma}$, since $\min\limits_{x\in[\frac{1}{4},\frac{3}{4}]} y_\lambda(x) \ge \gamma \|y_\lambda\| \ge M$, we get

$$\|y_\lambda\| = \lambda \max_{x\in[0,1]} \int_0^1 G(x,s)h(s)f(y_\lambda(s))ds$$

$$\ge \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(x,s)h(s)f(y_\lambda(s))ds$$

$$\ge \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \alpha(x) \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)y_\lambda(s)ds$$

$$\geq \lambda\alpha\gamma\|y_\lambda\|\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)ds$$

Thus, $\lambda \leq (\alpha\gamma\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)h(s)ds)^{-1}$,which contradicts with λ sufficiently large.

If $\|y_\lambda\| < \dfrac{M}{\gamma}$, then

$$\frac{M}{\gamma} > \|y_\lambda\| \geq \lambda \min_{x\in[\frac{1}{4},\frac{3}{4}]} \alpha(x)\int_0^1 g(s)h(s)f(y_\lambda(s))ds$$

$$\geq \lambda\alpha f(0)\int_0^1 g(s)h(s)ds,$$

i.e. $\lambda \leq M(\alpha\gamma f(0)\int_0^1 g(s)h(s)ds)^{-1}$,which is also contradiction. □

Lemma 4 Suppose that $(H_1)-(H_3)$ hold. Then there exists a $\lambda_0 > 0$ such that $(0,\lambda_0] \subset \Lambda$.

Proof. Let $K_1 = \{y \in K; \|y\| < 1\}$, choose $\lambda_0 = (2\beta f(1)\int_0^1 g(s)h(s)ds)^{-1}$. For $y \in \partial K_1 = \{y \in K; \|y\| = 1\}$ and $\lambda \in (0,\lambda_0]$, using $(H_1),(H_2)$ and (2.3) we have

$$(Ty)(x) \leq \lambda \max_{x\in[0,1]} \int_0^1 G(x,s)h(s)f(y(s))ds$$

$$\leq \lambda_0\beta f(1)\int_0^1 g(s)h(s)ds < 1 = \|y\|,$$

i.e. $\|Ty\| < \|y\|$ for $y \in \partial K_1$. Thus, Theorem 2 implies $i(T,K_1,K) = 1$. Hence, T has a fixed point $y_\lambda(x)$ in K_1 and it satisfies

$$y_\lambda(x) = \lambda\int_0^1 G(x,s)h(s)f(y_\lambda(s))ds$$

$$\geq \lambda f(0)\alpha(x)\int_0^1 g(s)h(s)ds,$$

this shows that $y_\lambda(x)$ is a positive solution of problem (1.1), (1.2). □

Lemma 5 Suppose that $(H_1)-(H_3)$ hold. Let $\lambda^*=\sup\Lambda$, then $\lambda^*\in\Lambda$.

Proof. Without loss of generality, let $\{\lambda_n\}_{n=1}^{\infty}$ be a monotone increasing sequence and $\lim_{n\to\infty}\lambda_n=\lambda^*$, where $\lambda_n\in\Lambda$. We claims that the corresponding positive solution sequence $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ is uniformly bounded. In fact, from $\lim_{y\to+\infty}\frac{f(y)}{y}=+\infty$,there exists a $M>0$ such that $f(y)\ge Ny$ for all $y\ge M$, where N is chosen so that $N\lambda_n\alpha\gamma\int_{\frac{1}{4}}^{\frac{3}{4}}h(s)g(s)ds>1$, If $\left\|y_{\lambda_n}\right\|\ge\frac{M}{\gamma}$, then $\min_{x\in[\frac{1}{4},\frac{3}{4}]}y_{\lambda_n}\ge\gamma\left\|y_{\lambda_n}\right\|\ge M$, hence we obtain

$$\begin{aligned}
\|y_{\lambda_n}\| &\ge \lambda_n\min_{x\in[\frac{1}{4},\frac{3}{4}]}\int_0^1 G(x,s)h(s)f(y_{\lambda_n}(s))ds\\
&\ge N\lambda_n\alpha\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)y_{\lambda_n}(s)ds\\
&\ge N\lambda_n\alpha\gamma\|y_{\lambda_n}\|\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)ds\\
&>\|y_{\lambda_n}\|,
\end{aligned}$$

which is a contradiction. Thus, there exists a number L with $0<L<+\infty$ such that $\|y_{\lambda_n}\|\le L$ for all n.

In addition, using (H_1) and (2.4), we have

$$\|y'_{\lambda_n}\|\le\lambda_n\int_0^1\frac{\partial G(x,s)}{\partial x}h(s)f(y_{\lambda_n}(s))ds$$

$$\le\frac{\lambda^* nf(L)}{(k-1)!(n-k-1)!}\left(\int_0^x s^{n-k}(1-s)^{k-1}h(s)ds+\int_x^1 s^{n-k-1}(1-s)^k h(s)ds\right)$$

$$\leq \frac{2n\lambda^* f(L)}{(k-1)!(n-k-1)!}\int_0^1 s^{n-k-1}(1-s)^{k-1}h(s)ds =: Q,$$

this shown that $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ is equicontinous. Ascoli-Arzela theorem claims that $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ has a uniformly convergent subsequence, denoted again by $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$, and $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ converges to $y^*(x)$ uniformly on $[0,1]$. Inserting $y_{\lambda_n}(x)$ into (2.1) and letting $n \to \infty$, using the Lebesgue dominated convergence theorem, we obtain

$$y^*(x) = \lambda^* \int_0^1 G(x,s)h(s)f(y^*(s))ds.$$

Therefore, $y^*(x)$ is a positive solution of (1.1),(1.2) and $\lambda^* \in \Lambda$. □

Lemma 6 Suppose that $(H_1)-(H_3)$ hold.Then the problem (1.1),(1.2) has two positive solution for $\lambda \in (0,\lambda^*)$.

Proof. As shown in [9], it follows from (H_2) and uniform continous of $f(y)$ that there exists a $\delta > 0$ such that

$$(Ty)(x) = \lambda\int_0^1 G(x,s)h(s)f(y(s))ds < y^*(x) + \delta \tag{3.1}$$

Let $\Omega = \{y(x) \in C[0,1]; -\delta < y(x) < y^*(x) + \delta\}$, then Ω is a bounded open subset in $C[0,1]$ and $K \cap \Omega$ is a bounded open subset in K and $y^*(x) \in K \cap \Omega$. It is clear that

$$K \cap \overline{\Omega} = \{y(x) \in K; 0 \leq y(x) \leq y^*(x) + \delta\}.$$

Consider the homotopy

$$H(t,y) = (1-t)Ty + ty^*$$

it is obvious that

$$H:[0,1]\times(K\cap\overline{\Omega})\to K$$

is completely continous. For $(t,y)\in[0,1]\times(K\cap\overline{\Omega})$, form (3.1) we have

$$H(t,y)=(1-t)Ty+ty^*<(1-t)y^*+ty^*<y^*+\delta\,.$$

Hence $H(t,y)\in K\cap\Omega$. Thus, we get $H(t,y)\neq y$ for $(t,y)\in[0,1]\times(K\cap\partial\Omega)$.

By the homotopy invariance and normality of the fixed point index we obtain

$$i(T,K\cap\Omega,K)=i(y^*,K\cap\Omega,K)=1 \tag{3.2}$$

this shows that T has a fixed point $y_\lambda^{(1)}(x)$ in $K\cap\Omega$ and $y_\lambda^{(1)}(x)$ is a positive solution of (1.1), (1.2).

From $\lim\limits_{y\to+\infty}\dfrac{f(y)}{y}=+\infty$, there exists a $R_1>0$ such that $f(y)\geq\eta y$ for all $y\geq R_1$, where η is chosen so that $\eta\gamma\alpha\lambda\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)ds>1$. Let $R=\max\{\dfrac{R_1}{\gamma},\|y^*\|+\delta+1\}$, then for $y\in\partial K_R$, since $\min\limits_{x\in[\frac{1}{4},\frac{3}{4}]}y(x)\geq\gamma\|y\|=\gamma R\geq R_1$, we have

$$\|Ty\|=\lambda\max_{x\in[0,1]}\int_0^1 G(x,s)h(s)f(y(s))ds$$

$$\geq\lambda\min_{x\in[\frac{1}{4},\frac{3}{4}]}\alpha(x)\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)\eta y(s)ds$$

$$\geq\eta\lambda\alpha\gamma\|y\|\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)h(s)ds>\|y\|,$$

thus, Theorem 2 implies

$$i(T,K_R,K)=0 \tag{3.3}$$

Consequently, the additivity of the fixed point index and (3.2), (3.3) together implies

$$i(T, K_R \setminus (K \cap \overline{\Omega}), K) = -1,$$

therefore, T has another fixed point $y_\lambda^{(2)}(x)$ in $K_R \setminus (K \cap \overline{\Omega})$, and $y_\lambda^{(2)}(x)$ is also a positive solution of $(1.1), (1.2)$.

Up to now, the proof of theorem is complete. □

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