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# On a certain generalized condition for starlikeness and convexity

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**Abstract**

In this paper, we applied Salagean differential operator to the integral operator of the form

$$F(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds.$$

and obtained the starlikeness, convexity and convolution properties.

**Mathematics Subject Classification:** 30C45
**Keywords:** Salagean differential operator; integral operator; Starlikeness; Convexity; Convolution

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# 1 Introduction

Let $A$ be the class of all analytic functions $f(z)$ defined in the open unit disk $U = \{z \in C : |z| < 1\}$ and $S$ the subclass of $A$ condisting of univalent functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

While

$$S^* = \left\{ f \in S : \mathrm{Re}\Big(\frac{zf'(z)}{f(z)} > 0,\ z \in U\Big) \right\} \tag{2}$$

$$S^c = \left\{ f \in S : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U \right\} \tag{3}$$

are called respectively univalent starlike and convex functions with respect to the origin.

**Lemma 1.**[1] Let $M$ and $N$ be analytic in $U$ with $M(0) = N(0) = 0$. If $N(z)$ maps onto a many sheeted region which is starlike with respect to the origin and $\mathrm{Re}\{\frac{M'(z)}{N'(z)}\} >$ in $U$, then $\mathrm{Re}\{\frac{M(z)}{N(z)}\} >$ in $U$.

Seenivasagan [4] obtained a sufficient condition for the univalence of the integral operator of the form:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

Where

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{4}$$

While Makinde and Opoola [2] obtained a condition for the starlikeness of the integral operator of the form:

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds,\ \alpha \in C$$

The differential operator $D^n (n \in N_0)$ was introduced by Salagean [5]. Where $D_n f(z)$ is defined by

$$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \text{ with } D^0 f(z) = f(z) \tag{5}$$

Makinde [3] investigated some properties for $\Gamma_{\alpha}^{n}(\zeta_1, \zeta_2; \gamma)$ of $D^n f_i(z)$ in the integral operator of the form:

$$F_{\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{D^n f_i(s)}{s} \right)^{1/\alpha} ds, \ \alpha \in C \quad |\alpha| \leq 1$$

Where she gave the Salagean Differential operator for the function $f_i(z)$ in (4) to be of the form:

$$D^n f_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i z^n \tag{6}$$

Let $D^n f_{i1}(z) = z + \sum_{n=2}^{\infty} n^k a_{n1}^i z^n$ an $D^n f_{i2}(z) = z + \sum_{n=2}^{\infty} n^k a_{n2}^i z^n$ we define the convolution of $D^n f_{i1}(z)$ and $D^n f_{i2}(z)$ by

$$D^n f_{i1}(z) * D^n f_{i2}(z) = (D^n f_{i1} * D^n f_{i2})(z) = z + \sum_{n=2}^{\infty} n^k a_{n1}^i a_{n2}^i z^n$$

In the present paper, we investigated certain conditions for starlikeness and convexity of the integral operator of the form:

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{D^n f_i(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

and its convolution properties.

## 2 Main Results

**Theorem 2.1.** *Let $F_{D\alpha}(z)$ be the function in $U$ defined by*

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{D^n f_i(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

*If $D^n f_i(s)$ is starlike, then $F_{D\alpha}(z)$ is convex.*

*Proof.* Let

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{D^n f_i(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

Then

$$\frac{zF'_{D\alpha}(z)}{F_{D\alpha}(z)} = \frac{\prod_{i=1}^{k}\left(\frac{D^n f_i(s)}{s}\right)^{\frac{1}{\alpha}}}{\int_0^z \prod_{i=1}^{k}\left(\frac{D^n f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds}$$

$$= \frac{z\left(\frac{D^n f_1(z)}{z}\right)^{\frac{1}{\alpha}}\left(\frac{D^n f_2(z)}{z}\right)^{\frac{1}{\alpha}}\cdots\left(\frac{D^n f_k(z)}{z}\right)^{\frac{1}{\alpha}}}{\int_0^z \prod_{i=1}^{k}\left(\frac{D^n f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds}$$

Now, let

$$M(z) = zF'_{D\alpha}(z) = z\left(\frac{D^n f_1(z)}{z}\right)^{\frac{1}{\alpha}}\left(\frac{D^n f_2(z)}{z}\right)^{\frac{1}{\alpha}}\cdots\left(\frac{D^n f_k(z)}{z}\right)^{\frac{1}{\alpha}}$$

and

$$N(z) = F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k}\left(\frac{D^n f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds$$

Then

$$\frac{M'(z)}{N'(z)} = 1 + \frac{zF''_{D\alpha}(z)}{F'_{D\alpha}(z)} = 1 + \frac{\sum_{i=1}^{k}\frac{1}{\alpha}\left(\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1\right)}{\prod_{i=1}^{k}\left(\frac{D^n f_i(z)}{z}\right)^{\frac{1}{\alpha}}}$$

But

$$\left|\frac{M'(z)}{N'(z)} - 1\right| = \frac{|\sum_{i=1}^{k}\frac{1}{\alpha}\left(\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1\right)|}{|\prod_{i=1}^{k}\left(\frac{D^n f_i(z)}{z}\right)^{\frac{1}{\alpha}}|}$$

$$\leq \frac{\sum_{i=1}^{k}|\frac{1}{\alpha}||\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1|}{|\prod_{i=1}^{k}\left(\frac{D^n f_i(z)}{z}\right)^{\frac{1}{\alpha}}|}$$

$D^n f_i(z)$ is starlike by hypothesis implies that

$$|\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1| < 1$$

Thus

$$|\frac{M'(z)}{N'(z)} - 1| < 1$$

Hence, Re

$$\{\frac{M'(z)}{N'(z)}\} > 0$$

which shows that $F_{D\alpha}(z)$ is convex and so by Lemma 1

$$\text{Re } \{\frac{M(z)}{N(z)}\} > 0$$

which implies that $F_{D\alpha}(z)$ is starlike.
This concludes the proof of Theorem 1. □

**Theorem 2.2.** *Let $F_{D\alpha}(z)$ be the function in $U$ defined by*

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{D^n f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

*Then $F_{D\alpha}(z)$ is starlike If $D^n f_i(s)$ is starlike.*

*Proof.* From Theorem 1

$$\left|\frac{M'(z)}{N'(z)} - 1\right| < 1$$

This implies that

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > 0$$

and by Lemma 1 we have

$$Re\left\{\frac{M(z)}{N(z)}\right\} > 0$$

Which implies that

$$Re\left\{\frac{zF'_{D\alpha}(z)}{F_{D\alpha}(z)}\right\} > 0$$

Thus $F_{D\alpha}(z)$ is starlike. □

**Theorem 2.3.** *Let*

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{(D^n f_{i1} * D^n f_{i2})(s)}{s}\right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

*Then $F_{D\alpha}(z)$ is convex if $(D^n f_{i1} * D^n f_{i2})(z)$ is starlike.*

*Proof.* Following the procedure of the proof of the Theorem 1, we obtain the result. □

**Theorem 2.4.** *Let $F_{D\alpha}(z)$ be the function in $U$ defined by*

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{(D^n f_{i1} * D^n f_{i2})(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

*Then $F_{D\alpha}(z)$ is starlike If $(D^n f_{i1} * D^n f_{i2})(z)$ is starlike.*

*Proof.* Following the procedure of the proof of the Theorem 2, we obtain the result. □

**Theorem 2.5.** *Let $F_{D\alpha}(z)$ be the function in $U$ defined by*

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{D^n f_i(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C, \quad |\alpha > 1$$

*Then $F_{D\alpha}(z)$ is convex if*

$$\sum_{i=1}^{k} \sum n^k (n-1)|a_n^i| < 1$$

*Proof.* From Theorem 1, we have

$$\left| \frac{M'(z)}{N'(z)} - 1 \right| \le \frac{\sum_{i=1}^{k} |\frac{1}{\alpha}|| \frac{zD^{n+1} f_i(z)}{D^n f_i(z)} - 1|}{| \prod_{i=1}^{k} \left( \frac{D^n f_i(z)}{z} \right)^{\frac{1}{\alpha}} |}$$

Using (6) we obtain

$$\left| \frac{M'(z)}{N'(z)} - 1 \right| \le \frac{1}{\alpha} \frac{\sum_{i=1}^{k} \sum_{n=2}^{\infty} n^k (n-1)|a_n^i|}{1 + \sum_{i=1}^{k} \sum_{n=2}^{\infty} n^k |a_n^i| \left| \prod_{i=1}^{k} \left( \frac{D^n f_i(z)}{z} \right)^{\frac{1}{\alpha}} \right|} \qquad z \to 1^-$$

and by hypothesis, we have

$$\left| \frac{M'(z)}{N'(z)} - 1 \right| < 1$$

Thus $F_{D\alpha}(z)$ is convex

**Corollary** Let $F_{D\alpha}(z)$ be the function in $U$ defined by

$$F_{D\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{(D^n f_{i1} * D^n f_{i2})(s)}{s} \right)^{\frac{1}{\alpha}} ds, \quad \alpha \in C$$

Then $F_{D\alpha}(z)$ is convex if

$$\sum_{i=1}^{k} \sum n^k (n-1)|a_{n1}^i||a_{n2}^i| < 1.$$

□

## References


[1] S.A. Halim, On a class of analytic functions involving Salagean differential operator, *Tamkang Journal of Mathematics*, **23**(1), (Spring, 1992).

[2] D.O. Makinde and T.O. Opoola, On sufficient condition for sarlikeness, *General Mathematics*, **8**(3), (2010), 35-39.

[3] D.O. Makinde, Application of Salagean differntial operator to certain subclasses of harmonic univalent functions, *International Journal of Scientific and Research publications*, **3**(5), (2013).

[4] N. Seenivasagan, Sufficient conditions for univalence, *Applied Mathemaitics E-notes*, **8**, (2008), 30-35.

[5] E. Yasar and S. Yalcm, Convolution properties of Salagean-type of harmonic univalent functions, *Stud. Univ. Babes-Bolyai Math.*, **57**(2), (2012), 283-289.