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# Nonlinear Black-Scholes Equation Through Radial Basis Functions

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**Abstract**

The Black-Scholes equation is a partial differential equation (PDE) characterizing the price evolution of a European call option and put option on a stock. In this work, we use Multi-Quadratic Radial Basis Functions (Multi-Quadratic RBF) for approximating the solution of the Black-Scholes equation and show how it can be applied to a case in which the volatility is not constant but is dependent on the transaction costs (a nonlinear case). That is, volatility satisfies an Ornstein-Uhlenbeck stochastic process. These nonlinear models are also presented in the financial markets characterized by lack of liquidity. After discretizing the problem with the Crank-Nicholson algorithm, we expose a numerical example to validate the developed method.

**Mathematics Subject Classification:** 91G30; 91B25; 91G60
**Keywords:** Radial Basis Functions; Black-Scholes Equation; Crank-Nicholson Algorithm; Lack of Liquidity

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# 1 Introduction

In finance, a financial derivative (or derivative) is a financial product whose value varies depending on the price of another asset; the asset of which depends takes the name of the underlying asset (e.g., stock, commodity, futures and index). In practice, the options are a type of derivative, *i.e.*, a contract which we have the option to buy or sell a certain underlying asset at a certain date and at a price established. Therefore, an option can be a purchase option or a selling option, called call option and put option, respectively. Commonly, an option is valued using the Black-Scholes model, which is a partial differential equation which, when solved, produces a function that allows us to know the price of a derivative, depending on the underlying asset price and time. As the name suggests, this model was proposed by Fischer Black and Myron Scholes [1]. Subsequently, Robert Merton published a paper with a deeper and more consistent development of the mathematical model. As a historical note, it is worth noting that Merton and Scholes received the Nobel Prize in Economics in 1997, by the Royal Swedish Academy of Sciences; F. Black had died in 1995.

The Black-Scholes equation (in standard linear version [1], [5]) has been studied in many articles from various points of view; for example, in [3] this equation was studied using the theory of semigroups of operators; in [4] the same equation is solved using the theory of special functions; and in [2] a study for the valuation of financial derivatives with time-dependent parameters approach was carried out.

In this paper we study the Black-Scholes equation for a nonlinear case, first by an interpolation by Multi-Quadratic RBF and then followed by a Crank-Nicholson type algorithm [12] to find a solution to the nonlinear variant, which has been proposed mainly in the work [7] and [8], in financial markets characterized by lack of liquidity.

Accordingly, we first derive the Black-Scholes model for both the standard case and the nonlinear case (Section 2). Then (Section 3) we present a brief introduction to the scheme of Radial Basis Functions (RBF). After, an approach to nonlinear Black-Scholes model using the radial basis functions as interpolator is developed, then the efficiency of the method is illustrated following a Crank-Nicholson type scheme (Section 4). We conclude with some remarks on Black-Scholes model and the method used for approximating a solution of this equation (Section 5).

## 2 The Black-Scholes: The Nonlinear Case

This model is a partial differential equation whose solution describes the value of an European Option, see [1], [5]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_{t\geq 0})$ be a filtered probability space and let $B_t$ be a brownian motion in $\mathbb{R}$. We will consider the stochastic differential equation (SDE)

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dB(t),$$

with $a$ and $\sigma$ continuous in $(t, x)$ and Lipschitz continuous in $x$; *i.e.,* there is a constant M such that

$$|a(t,x) - a(t,y)| \leq M|x-y|, \quad |\sigma(t,x) - \sigma(t,y)| \leq M|x-y| \quad \forall\, t, x, y.$$

The price processes is given by the geometric brownian motion $S(t)$, $S(0) = x_0$, solution of the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t),$$

with $\mu$ and $\sigma$ constants. It is well know that the solution of this SDE is given by:

$$dS(t) = x_0 \exp\{\sigma(B(t) - B(t_0)) + (r - \frac{1}{2}\sigma^2)(t - t_0)\}$$

Let $0 \leq t < T$ and $h$ be a Borel measurable function, $h(X(T))$ denotes the contingent claim, let $E^{x,t}h(X(T))$ be the expectation of $h(X(T))$, with the initial condition $X(t) = x$.

Now we recall the Feynman–Kac theorem [6]. Let $v(t,x) = E^{x,t}h(X(T))$ be, $0 \leq t < T$, where $dX(t) = a(X(t))dt + \sigma(X(t))dW(t)$. Then

$$v_t(t,x) + a(x)v_x(t,x) + \frac{1}{2}\sigma^2(x)v_{xx}(t,x) = 0, \text{ and } v(T,x) = h(x). \tag{1}$$

Now, if we consider the discounted value

$$u(t,x) = e^{-r(T-t)}E^{x,t}h(X(T)) = e^{-r(T-t)}v(t,x). \tag{2}$$

Then if at time $t$, $S(t) = x$, we proceed in a standard way,

$$\begin{aligned} v(t,x) &= e^{r(T-t)}u(t,x), \\ v_t(t,x) &= -re^{r(T-t)}u(t,x) + e^{r(T-t)}u_t(t,x), \\ v_x(t,x) &= e^{r(T-t)}u_x(t,x), \\ v_{xx}(t,x) &= e^{r(T-t)}u_{xx}(t,x). \end{aligned}$$

The Black-Scholes equation is obtained substituting the above equalities in the equation (1) and multiplying by the factor $e^{-r(T-t)}$:

$$-ru(t,x) + u_t(t,x) + rxu_x(t,x) + \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) = 0, \tag{3}$$

where $0 \leq t < T$, and $x \geq 0$; here the risk-free interest rate $r$ and the volatility $\sigma$ are taken constants.

In financial models, we usually assumed that the markets are competitive and there is no friction of any kind, i.e. there are no transaction costs and there is ample liquidity such that a trader can buy or sell any number of assets without changing its price. In some financial models [7],[8], for very short periods of time, the volatility $\sigma$ can be a function that depends on the transaction cost or the second derivative of the asset value while the interest rate $r$, is a constant. The idea is that not only the asset price but also the volatility, is a stochastic process [9]. If the volatility satisfies the Ornstein-Uhlenbeck process we have the two equations

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW_1(t)$$

$$d\sigma(t) = -\beta\sigma(t)dt + \delta dW_2(t),$$

the processes $W_1(t)$ and $W_2(t)$ are not independent and must be $Corr(W_1(t), W_2(t)) = \rho$, where $|\rho| << 1$. Using the Itô's lema, we can obtain an expression for the stochastic differential of $d\sigma^2(t)$, which happens to be

$$d\sigma^2(t) = [2\delta^2 - \beta\sigma^2(t)]dt + 2\delta\sigma(t)dW_2(t)$$

such that eq.(3) with $\beta = 1$ becomes the non-linear equation:

$$u_t(t,x) + \frac{1}{2}\sigma^2 x^2 \cdot \frac{u_{xx}(t,x)}{(1 - \rho x\lambda(x)u_{xx}(t,x))^2} = 0. \tag{4}$$

If we consider the special case where the price of risk is unit, $\lambda(x) = 1$ and if we further assume that $||\rho x u_{xx}|| < \epsilon$ for some $\epsilon > 0$ small enough (low impact of hedging) and considering the functional approximation $\frac{1}{(1-F)^2} \approx 1+2F+\mathcal{O}(F)^3$ we have that eq.(4) becomes

$$u_t(t,x) + \frac{1}{2}\sigma^2 x^2 \cdot u_{xx}(t,x)(1 + 2\rho x u_{xx}(t,x)) = 0. \tag{5}$$

# 3 Radial Basis Functions

**Definition 3.1.** *A radial basis function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function of the form $\phi(r)$ which depends upon the separation distances $r = ||\vec{x} - \vec{\zeta}||$ of a set of nodes $\Pi \subset \mathbb{R}^d$ also called centres, $\vec{\zeta} = (\zeta_j) \in \Pi$ for all $j = 1, 2, \ldots, N$.*

Due to their spherical symmetry about the points $\zeta_j$ these functions are termed radial. The distances $r = ||\vec{x} - \vec{\zeta}||$ are usually measured using the euclidean norm. A typical radial basis function (RBF) usually has the form

$$\phi(r) = (\epsilon||\vec{x} - \vec{\zeta}||) \tag{6}$$

where $\epsilon$ means the shape parameter of the radial basis function. Some of the most popular used RBFs are showed in Table 1.

The Multiquadratic RBF is also sometimes called Hardy's multiquadratic, is so called due to the seminal work of Hardy written in 1971 [10].

Another key feature of the RBF method is that it is mesh-free, which means that it does not require a grid. It only depends on distances to center points $\zeta_j$ in the approximation. As pairwise distances are very easy to compute in any number of space dimensions, it also works well for high-dimensional problem.

A standard RBF interpolant is a linear combination of RBFs $\phi$ centered at the scattered points $\zeta_j$, $j = 1, 2, \ldots, N$ which has the following form

$$S(\vec{x}, \epsilon) = \sum_{j=1}^{N} \lambda_j \phi(\epsilon||\vec{x} - \vec{\zeta}_j||) = \sum_{j=1}^{N} \lambda_j \phi_j(\vec{x}), \tag{7}$$

the coefficients (unknown) $\lambda_j$ are determined through the interpolation condition $S(\zeta_j, \epsilon) = f(\zeta_j)$ and can be computed as the solution of the following linear system

$$A_\phi \vec{\lambda} = \vec{f}, \tag{8}$$

where the symmetric matrix $A_\phi$ is given by $[a_{ij}] = \phi_j(\zeta_i) = \phi(\epsilon||\vec{x} - \vec{\zeta_j}||)$, and $\vec{\lambda}$ , $\vec{f}$ are column vectors.

| **Function** | **Expression** |
|---|---|
| Gaussian (GA) | $\phi(r) = e^{-(\epsilon r)^2}$ |
| Multiquadric (MQ) | $\phi(r) = \sqrt{(\epsilon r)^2 + c^2}$ |
| Inverse Multiquadric (IMQ) | $\phi(r) = \frac{1}{\sqrt{1+(\epsilon r)^2}}$ |
| Inverse Quadric (IQ) | $\phi(r) = \frac{1}{1+(\epsilon r)^2}$ |
| Sigmoid (S) | $\phi(r) = \frac{1}{1+e^{-(\epsilon r)}}$ |

Table 1: Examples of Radial Basis Functions (RBF).

Furthermore, the implementation of a RBF method is also straightforward. However, there are still some issues left such as stability for the time-dependent problems and computational efficiency [11]. With the advantages mentioned above of achieving spectral accuracy using infinitely smooth basis functions, the geometrical flexibility with arbitrary choice of node locations and the ease of implementation, radial basis function (RBF) approximation is rising as an important method for interpolation, approximation, and solution of partial differential equations (PDE).

# 4 Black-Scholes and Radial Basis Functions

Now we will consider Black-Scholes equation (nonlinear) (5), using more conventional symbology

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 x^2 \cdot \frac{\partial^2 U}{\partial x^2}\left(1 + 2\rho x \frac{\partial^2 U}{\partial x^2}\right) = 0. \tag{9}$$

The initial condition is given by the terminal payoff valuation

$$U(x,T) = \begin{cases} max\{K-x,0\} & \text{for put} \\ max\{x-K,0\} & \text{for call} \end{cases}$$

where $T$ is the time of maturity and $K$ is the strike price of the european-type option.

As mentioned in last section, there are many types of RBF such as Gaussians, Sigmoid, Multiquadratics and Inverse Multiquadratics (among many other). In our example we select Hardys Multi-quadratic (MQ), which can be written in terms of the log stock price $x$ (the independent variable in one dimension) as (with $\epsilon = 1$)

$$\phi(||\vec{x}-\vec{\zeta_j}||) = \sqrt{(x-\zeta_j)^2+c^2} \tag{10}$$

The derivatives of $\phi$ can be easily calculated as

$$\frac{\partial \phi}{\partial x} = \frac{x-\zeta_j}{\sqrt{(x-\zeta_j)^2+c^2}} \tag{11}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\sqrt{(x-\zeta_j)^2+c^2}} - \frac{(x-\zeta_j)^2}{(\sqrt{(x-\zeta_j)^2+c^2})^3} \tag{12}$$

Using these definitions, we can now propose an approximation for the option price $U$, solution of the equation (9)

$$U(x,t) = \sum_{j=1}^{N} \lambda_j(t)\phi(||\vec{x}-\vec{\zeta_j}||) \tag{13}$$

The derivatives of this approximation are

$$\frac{\partial U}{\partial x} = \sum_{j=1}^{N} \lambda_j(t)\frac{\partial \phi(||\vec{x}-\vec{\zeta_j}||)}{\partial x} \tag{14}$$

$$\frac{\partial^2 U}{\partial x^2} = \sum_{j=1}^{N} \lambda_j(t)\frac{\partial^2 \phi(||\vec{x}-\vec{\zeta_j}||)}{\partial x^2} \tag{15}$$

$$\frac{\partial U}{\partial t} = \sum_{j=1}^{N} \frac{\partial \lambda_j(t)}{\partial t}\phi(||\vec{x}-\vec{\zeta_j}||) \tag{16}$$

Now that we have discretized all parts of the equation (9), we will use an iterative method, Crank-Nicholson-type, see reference [12] for more information, we obtain

$$FU^{(t+\Delta t)} = GU^{(t)} \tag{17}$$

where

$$U^{(t)} = U(x,t), \quad U^{(t+\Delta t)} = U(x,t+\Delta t),$$
$$G = 1 - \theta\Delta t\Big(\frac{1}{2}\sigma^2 x^2 \cdot \frac{\partial^2}{\partial x^2}\Big[1 + 2\rho x\frac{\partial^2}{\partial x^2}\Big]\Big),$$
$$F = 1 + (1-\theta)\Delta t\Big(\frac{1}{2}\sigma^2 x^2 \cdot \frac{\partial^2}{\partial x^2}\Big[1 + 2\rho x\frac{\partial^2}{\partial x^2}\Big]\Big)$$

here, $\Delta t$ is the time step size and the parameter $\theta$ running in the range $0 \leq \theta \leq 1$, $\theta$ is the Crank-Nicholson parameter and fixed once chosen. The price $U = U(x,t)$ governed by (9) will be estimated with the RBF as in the equation (13), *i.e.*

$$U^{(t)} = \sum_{j=1}^{N} \lambda_j(t)\phi(||\vec{x} - \vec{\zeta_j}||) \tag{18}$$

where $N$ denote the total number of data points. Substituting (18) into (17), we obtain

$$\sum_{j=1}^{N} F\lambda_j^{(t+\Delta t)}(t)\phi(||\vec{x} - \vec{\zeta_j}||) = \sum_{j=1}^{N} G\lambda_j^{(t)}(t)\phi(||\vec{x} - \vec{\zeta_j}||) \tag{19}$$

The Crank-Nicholson scheme suggests the algorithm of seven steps, illustrated in Table 2.

## 4.1 An Illustrative Example

Our illustrative example considers the data of Black-Scholes model for the case of a European option as illustrated in Table 3. The algorithm was implemented in MATLAB 5.2 by Scientific Word 3.0, considering $N = 125$.

The parameter $c$ in equation (10) allows us to adjust the error committed when RBF approach is used to approximate the solution of a partial differential equation *i.e.*, Black Scholes equation (9) with an algorithm as shown in Table

| Step | Steps of the algorithm |
|---|---|
| step 1 | Discretize from $t = 0$ to $t = T$ with time-step $\Delta t = T/N$ |
| step 2 | Specify $U^{(T)}$ at the expire time $T$ from the boundary condition and calculate $\lambda_j^{(T)}$ at time $T$ |
| step 3 | Make $t = T - \Delta t$ |
| step 4 | Calculate $\lambda_j^{(k)}$ with (19) for every $k$ |
| step 5 | Make $t = T - \Delta t$ |
| step 6 | Repeat step 4 and step 5 if $1 \leq j \leq N$ |
| step 7 | Evaluate $U^{(0)} = \sum_{j=1}^{N} \lambda_j(0)\phi(\|\vec{x} - \vec{\zeta_j}\|)$ |

Table 2: Data of Black-Scholes model for the case of a european option.

| Parameter | Value |
|---|---|
| Maturity time | $T = 6\ months$ |
| Strike price | $K = \$15$ |
| Interest rate | $r = 0.01$ |
| Volatility | $\sigma = 0.1$ |
| Crank-Nicholson parameter | $\theta = 0.6$ |

2. As can be seen in [13], the value taken by the parameter $c$ is inversely proportional to coefficient $\lambda$ appearing in (8) (remember that this case is 1-dimensional, so $\lambda \in \mathbb{R}$).

The results of such approach can be seen in Table 4. Note that for this example we consider $c = 0.999$ and therefore $\lambda \approx 4.004$, which produces an error less than $2.8 \times 10^{-4}$.

# 5 Summary

In this paper, we used Multi-Quadratic Radial Basis Functions for approximating the solution of the Black-Scholes equation, for a situation characterized by a non-constant volatility, but that depends on transaction costs (*i.e.*, a nonlinear case). Such nonlinear models frequently occur in financial markets characterized by a lack of liquidity. First, an approximate solution was ob-

Table 3: Results of the numerical example in the validation of the developed method.

| **Stock** $x$ | $\mathbf{U_{using\ RBF}}$ | $\mathbf{U_{analytical}}$ |
|---|---|---|
| 25 | 1.8682741 | 1.8682866 |
| 30 | 5.0188546 | 5.0188442 |
| 35 | 8.3181237 | 8.3181066 |
| 40 | 11.7350982 | 11.735099 |
| 45 | 15.2483041 | 15.248305 |
| 50 | 18.8420513 | 18.842061 |
| 75 | 37.6833322 | 37.683321 |
| 100 | 57.4912555 | 57.491263 |
| 150 | 98.7814899 | 98.78149 |
| 200 | 141.4388217 | 141.43881 |

tained by means of Multi-Quadratic RBF, then the problem was discretized and the Crank-Nicholson algorithm was used. As a final point, a numerical example to validate the developed method has been given, achieving an approximation with an error less than $2.8 \times 10^{-4}$. As part of our future directions, we intend to discretize the Black-Scholes equation for some particular investment and get solutions through evolutionary algorithms or attempt to address the problem using artificial neural networks based on Radial Basis Functions [14].

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# References

[1] F. Black and M. Scholes, The Pricing Options and Corporate Liabilities, *Journal of Political Economy*, **81**, (1973), 637-659.

[2] C. F. Lo, C.H. Hui, Valuation of financial derivatives with time-dependent parameters: Lie-algebraic approach, *Quantitative Finance*, **1**, (2001), 7378.

[3] O. González-Gaxiola, José A. Santiago, The Black-Scholes Operator as the Generator of a $C_0$-Semigroup and Applications, *Int. Journal of Pure and Applied Math.*, **76**(2), (2012),191-200.

[4] O. Gonzalez-Gaxiola, J. A. Santiago, An $\alpha$-Mellin transform and some of its applications, *Int. Journal of Contemp. Math. Sciences*, **7** (45-48), (2012), 2353-2361.

[5] R.C. Merton, heory of Rational Options Pricing, *Bell Journal of Economic and Management Science*, **4**, (1973), 141-183.

[6] S. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer Finance, 2004.

[7] R. Frey, *Market illiquidity as a source of model risk in dynamic hedging*, ed. by R. Gibson, Risk Publications, pp. 125-136, Risk Publications, London, 2000.

[8] U. Cetin, R. Jarrow, P. Protter, *Liquidity risk and arbitrage pricing theory, Finance Stoch.*, **8**, (2004), 311-341.

[9] L. Steven, A. Heston, Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies*, **6**(2), (1993).

[10] R.L. Hardy, Multiquadric equations of topography and other irregular surfaces, *J. Geophys. Res.*, **76**, (1971), 1905-1915.

[11] J. Park, I. W. Sandberg, Universal Approximation Using Radial-Basis-Function Networks, *Neural Computation*, **3**(2), (1991), 246-257.

[12] J. Crank, P. Nicholson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type; *Proc. Cambridge Philos. Soc.*, **43**, (1947), 5067.

[13] E.J. Kansa, Multiquadratics: A scattered data approximation scheme with applications to computational fluid dynamics. I. Surface approximations and partial derivative estimates, *Comput. Math. Applic.*, 19 (**8/9**), (1990), 127-145.

[14] O. González-Gaxiola, Pedro Pablo González-Pérez, Applying the Fock spaces-based tools to the modeling of RBF neural networks: a quantum RBF neural network approach, *Journal of Applied Mathematics & Bioinformatics*, **13**(3), (2013), 89-102.