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Robust Parameter Estimation Methods: L_Moments and TL-Moments of Probability Distributions

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Abstract

This paper deals with the application of such robust methods of point parameter estimation, as the methods of L-moments and TL-moments on economic data. The advantages of these highly robust parametric estimation methods are aware when applied to small data sets, especially in the field of hydrology, meteorology and climatology, in particular considering extreme precipitation. The main aim of this contribution is to use these methods on large datasets, and comparison the accuracy of these two methods of parametric estimation with the accuracy of the method of maximum likelihood, especially in terms of efficiency of parametric estimation. The study is divided into a theoretical part, in which mathematical and

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statistical aspects are described, and an analytical part, during which the results of the use of three robust parametric estimation methods are presented. Total 168 income distributions of the years from 1992 to 2007 in the Czech Republic (distribution of net annual income per capita in CZK) were analyzed. There are a total income distribution for all households of the Czech Republic together and further the income distributions broken down by gender, job classification, classification of economic activities, age and educational attainment. Three-parametric lognormal curves represent the basic theoretical probability distribution.

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1 Introduction

L-moments form the basis for a general theory, which includes the summarization and description of the theoretical probability distributions, summarization and description of obtained sample data sets, parameter estimation of theoretical probability distributions and hypothesis testing of parameter values for the theoretical probability distributions. The theory of L-moments includes such established methods such as the use of order statistics and Gini middle difference and this leads to some promising innovations in the area of measuring skewness and kurtosis of the distribution and it provides the relatively new methods of parameter estimation for individual distribution. L-moments can be defined for any random variable whose expected value exists. The main advantage of L-moments over conventional moments is that the L-moments can be estimated by linear functions of sample values and they are more resistant to the influence of sample variability. L-moments are more robust than conventional moments to the existence of outliers in the data and they allow better conclusions obtained on the basis of the small samples of basic probability distribution. L-moments sometimes bring even more efficient parameter estimations of parametric distribution than the estimations acquired using maximum likelihood method, particularly for small samples.

L-moments have certain theoretical advantages over conventional moments resting in the ability to characterize a wider range of distribution and they are more resistant to the -compared with conventional moments, L-moments are less prone to estimation bias and approximation by asymptotic normal distribution is more accurate in finite samples.

Let X be a random variable having a distribution with distribution function $F(x)$ and quantile function $x(F)$, and let $X_1, X_2, \ldots, X_n$ is a random sample of sample size n from this distribution. Then $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ are the order statistics of random sample of sample size n, which comes from the distribution of random variable X.

L-moments are analogous to conventional moments. They can be estimated based on linear combinations of sample order statistics, i.e. L-statistics. L-moments are an alternative system describing the shape of the probability distribution.

2 Methods

2.1 L-Moments of Probability Distribution

The problem of L-moments is discussed for example in [1] or [2]. Let X be a continuous random variable that has a distribution with distribution function $F(x)$ and the quantile function $x(F)$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of random sample of sample size n, which comes from the distribution of random variable X. L-moment of the r-th order of random variable X is defined

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r-j:r}), \quad r = 1, 2, \dots. \tag{1}$$

The expected value of the r-th order statistic of random sample of sample size n has the form

$$E(X_{r:n}) = \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-1} \cdot [1 - F(x)]^{n-r} \mathrm{d}\, F(x). \tag{2}$$

If we substitute equation (2) into equation (1), we obtain after adjustments

$$\lambda_r = \int_0^1 x(F) \cdot P^*_{r-1}[F(x)]\, \mathrm{d}\, F(x), \quad r = 1, 2, \dots, \tag{3}$$

where

$$P^*_r[F(x)] = \sum_{j=0}^{r} p^*_{r,j} \cdot [F(x)]^j \quad \text{a} \quad p^*_{r,j} = (-1)^{r-j} \cdot \binom{r}{j} \cdot \binom{r+j}{j}, \tag{4}$$

and $P^*_r[F(x)]$ is the r-th shifted Legendre polynomial. We also obtain substituting expression (2) into expression (1)

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{r!}{(r-j-1)! \cdot j!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-j-1} \cdot [1 - F(x)]^j \,\mathrm{d}\, F(x),$$

$$r = 1, 2, \dots. \tag{5}$$

The letter „L“ in title „L-moments“ stresses that the r-th L-moment λ_r is a linear function of the expected value of a certain linear combination of order statistics. Own estimation of the r-th L-moment λ_r based on the obtained data sample is then a linear combination of order data values, i.e. L-statistics. The first four L-moments of the probability distribution are now defined

$$\lambda_1 = E(X_{1:1}) = \int_0^1 x(F)\, \mathrm{d}\, F(x), \tag{6}$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x(F) \cdot [2F(x) - 1] \,\mathrm{d}\, F(x), \tag{7}$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F) \cdot \{6[F(x)]^2 - 6F(x) + 1\} \,\mathrm{d}\, F(x), \tag{8}$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$
$$= \int_0^1 x(F) \cdot \{20[F(x)]^3 - 30[F(x)]^2 + 12[F(x)] - 1\} \,\mathrm{d}\, F(x) \tag{9}$$

The probability distribution can be specified by its L-moments, even if some of its conventional moments do not exist, but the opposite is not true. It can be proved that the first L-moment λ_1 is the location characteristic, the second L-moment λ_2 is the variability characteristic. It is often desirable to standardize higher L-moments λ_r, $r \geq 3$, so that they are independent on specific units of the random variable X. The ratio of L-moments of the *r*-th order of random variable *X* is defined

$$\tau_r = \frac{\lambda_r}{\lambda_2}, \quad r = 3, 4, \ldots. \tag{10}$$

We can also define a function of L-moments, which is analogous to the classical coefficient of variation, i.e. the so called L-coefficient of variation

$$\tau = \frac{\lambda_2}{\lambda_1}. \tag{11}$$

The ratio of L-moments τ_3 is the skewness characteristic and the ratio of L-moments τ_4 is the kurtosis characteristic of the corresponding probability distribution. Main properties of the probability distribution are very well summarized by the following four characteristics: L-location λ_1, L-variability λ_2, L-skewness τ_3 and L-kurtosis τ_4. L-moments λ_1 and λ_2, L-coefficient of variation τ and ratios of L-moments τ_3 and τ_4 are the most useful characteristics for summarization of probability distribution. Their main properties are: existence (if the expected value of the distribution exists, then all its L-moments exist) and uniqueness (if the expected value of the distribution exists, then L-moments define the only one distribution, i.e. no two distributions have the same L-moments.

Using equations (6)−(9) and equation (10) we obtain the expressions for L-moments, respectively for the ratios of L-moments for the case of chosen probability distributions, see Table 1.

Table 1: Formulas for the distribution function or quantile function, and for L-moments and ratios of L-moments of chosen probability distributions

Distribution	Distribution function $F(x)$ or quantile function $x(F)$	L-moments and ratios of L-moments
Uniform	$x(F) = \alpha + (\beta - \alpha) \cdot F(x)$	$\lambda_1 = \frac{\alpha + \beta}{2}$ $\lambda_2 = \frac{\beta - \alpha}{6}$ $\tau_3 = 0$ $\tau_4 = 0$
Exponential	$x(F) = \xi - \alpha \cdot \ln[1 - F(x)]$	$\lambda_1 = \xi + \alpha$ $\lambda_2 = \frac{\alpha}{2}$ $\tau_3 = \frac{1}{3}$ $\tau_4 = \frac{1}{6}$
Gumbel	$x(F) = \xi - \alpha \cdot \ln[-\ln F(x)]$	$\lambda_1 = \xi + e \cdot \alpha$ $\lambda_2 = \alpha \cdot \ln 2$ $\tau_3 = 0{,}1699$ $\tau_4 = 0{,}1504$

Table 1: Continuation

Logistic	$x(F) = \xi + \alpha \cdot \ln \frac{F(x)}{1 - F(x)}$	$\lambda_1 = \xi$ $\lambda_2 = \alpha$ $\tau_3 = 0$ $\tau_4 = \frac{1}{6}$
Normal	$F(x) = \Phi\left[\frac{x(F) - \mu}{\sigma}\right]$	$\lambda_1 = \mu$ $\lambda_2 = \pi^{-1} \cdot \sigma$ $\tau_3 = 0$ $\tau_4 = 30 \cdot \pi^{-1} \cdot (\tan\sqrt{2})^{-1} - 9 = 0{,}1226$
General Pareto	$x(F) = \xi + \alpha \cdot \frac{1 - [1 - F(x)]^k}{k}$	$\lambda_1 = \xi + \frac{\alpha}{1 + k}$ $\lambda_2 = \frac{\alpha}{(1 + k) \cdot (2 + k)}$ $\tau_3 = \frac{1 - k}{3 + k}$ $\tau_4 = \frac{(1 - k) \cdot (2 - k)}{(3 + k) \cdot (4 + k)}$
General extermal	$x(F) = \xi + \alpha \cdot \frac{1 - [-\ln F(x)]^k}{k}$	$\lambda_1 = \xi + \alpha \cdot \frac{1 - \Gamma(1 + k)}{k}$ $\lambda_2 = \alpha \cdot \frac{(1 - 2^{-k}) \cdot \Gamma(1 + k)}{k}$ $\tau_3 = \frac{2 \cdot (1 - 3^{-k})}{1 - 2^{-k}} - 3$ $\tau_4 = \frac{1 - 6{,}2^{-k} + 10{,}3^{-k} - 5{,}4^{-k}}{1 - 2^{-k}}$

Table 1: Continuation

General logistic	$x(F) = \xi + \alpha \cdot \dfrac{1 - \left[\dfrac{1-F(x)}{F(x)}\right]^k}{k}$	$\lambda_1 = \xi + \alpha \cdot \dfrac{1 - \Gamma(1+k)\cdot\Gamma(1-k)}{k}$ $\lambda_2 = \alpha \cdot \Gamma(1+k)\cdot\Gamma(1-k)$ $\tau_3 = -k$ $\tau_4 = \dfrac{1+5k^2}{6}$
Lognormal	$F(x) = \Phi\left\{\dfrac{\ln[x(F)-\xi]-\mu}{\sigma}\right\}$	$\lambda_1 = \xi + \exp\left(\mu + \dfrac{\sigma^2}{2}\right)$ $\lambda_2 = \exp\left(\mu + \dfrac{\sigma^2}{2}\right)\cdot \operatorname{erf}\left(\dfrac{\sigma}{2}\right)$ $\tau_3 = 6\pi^{-\frac{1}{2}} \cdot \dfrac{\int_0^{\frac{\sigma}{2}} \operatorname{erf}\left(\frac{x}{\sqrt{3}}\right)\cdot\exp(-x^2)\,\mathrm{d}x}{\operatorname{erf}\left(\frac{\sigma}{2}\right)}$
Gamma	$F(x) = \dfrac{\beta^{-\alpha}}{\Gamma(\alpha)} \cdot \int_0^{x(F)} t^{\alpha-1}\cdot\exp\left(-\dfrac{t}{\beta}\right)\mathrm{d}t$	$\lambda_1 = \alpha\cdot\beta$ $\lambda_2 = \pi^{-\frac{1}{2}}\cdot\beta\cdot\dfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}$ $\tau_3 = 6I_{\frac{1}{3}}(\alpha, 2\alpha) - 3$ [1)]

Source:[3]; own research

2.2 Sample L-Moments

L-moments are usually estimated by random sample obtained from an

[1)] $I_x(p, q)$ is incomplete beta function unknown distribution. Since the r-th L-moment λ_r is a function of the expected values of order statistics of a random sample of sample size r, it is natural to estimate it using the so-called U-statistic, i.e. the corresponding function of sample order statistics (averaged over all subsets of sample size r, which may be formed from the obtained random sample of sample size n).

Let $x_1, x_2, \dots, x_n$ is the sample and $x_{1:n} \le x_{2:n} \le \dots \le x_{n:n}$ is order sample. Then the r-th sample L-moment can be written as

$$l_r = \binom{n}{r}^{-1} \sum_{1 \le i_1 < i_2 < \dots < i_r \le n} \sum \dots \sum \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot x_{i_{r-j}:n}, \quad r = 1, 2, \dots, n. \tag{12}$$

Hence the first four sample L-moments have the form

$$l_1 = \frac{1}{n} \cdot \sum_i x_i, \tag{13}$$

$$l_2 = \frac{1}{2} \cdot \binom{n}{2}^{-1} \cdot \sum_{i > j} \sum (x_{i:n} - x_{j:n}), \tag{14}$$

$$l_3 = \frac{1}{3} \cdot \binom{n}{3}^{-1} \cdot \sum_{i > j > k} \sum \sum (x_{i:n} - 2x_{j:n} + x_{k:n}), \tag{15}$$

$$l_4 = \frac{1}{4} \cdot \binom{n}{4}^{-1} \cdot \sum_{i > j > k > l} \sum \sum \sum (x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{l:n}) \tag{16}$$

U-statistics are widely used especially in nonparametric statistics. Their positive properties are: absence of bias, asymptotic normality and some slight resistance due to the influence of outliers.

When calculating the r-th sample L-moment it is not necessary to repeat the process over all sub-sets of sample size r, but this statistic can be expressed directly as a linear combination of order statistics of a random sample of sample size n.

If we thing the estimation of $E(X_{r:r})$ obtained using U-statistics, this estimation can be written as $r \cdot b_{r-1}$, where

$$b_r = \frac{1}{n} \cdot \binom{n-1}{r}^{-1} \cdot \sum_{j=r+1}^{n} \binom{j-1}{r} \cdot x_{j:n} \,, \tag{17}$$

namely

$$b_0 = \frac{1}{n} \cdot \sum_{j=1}^{n} x_{j:n} \,, \tag{18}$$

$$b_1 = \frac{1}{n} \cdot \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} \cdot x_{j:n} \,, \tag{19}$$

$$b_2 = \frac{1}{n} \cdot \sum_{j=3}^{n} \frac{(j-1)\cdot(j-2)}{(n-1)\cdot(n-2)} \cdot x_{j:n} \,, \tag{20}$$

therefore generally

$$b_r = \frac{1}{n} \cdot \sum_{j=r+1}^{n} \frac{(j-1)\cdot(j-2)\cdot\ldots\cdot(j-r)}{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r)} \cdot x_{j:n} \,. \tag{21}$$

The first sample L-moments can be therefore written as

$$l_1 = b_0, \tag{22}$$

$$l_2 = 2b_1 - b_0, \tag{23}$$

$$l_3 = 6b_2 - 6b_1 + b_0, \tag{24}$$

$$l_4 = 20\,b_3 - 30\,b_2 + 12\,b_1 - b_0. \tag{25}$$

We can therefore write generally

$$l_{r+1} = \sum_{k=0}^{r} p^{*}_{r,k} \cdot b_k, \quad r = 0, 1, \ldots, n-1, \tag{26}$$

where

$$p^{*}_{r,k} = (-1)^{r-k} \cdot \binom{r}{k} \cdot \binom{r+k}{k} = \frac{(-1)^{r-k} \cdot (r+k)!}{(k!)^{2} \cdot (r-k)!}. \tag{27}$$

Use of sample L-moments is similar to the use of sample conventional

L-moments. Sample L-moments summarize the basic properties of the sample distribution, which are the location (level), variability, skewness and kurtosis. Thus, sample L-moments estimate the corresponding properties of the probability distribution from which the sample comes, and they can be used in estimating the parameters of the relevant probability distribution. L-moments are often preferred over conventional moments within such applications, since, as the linear functions of sample values, sample L-moments are less sensitive to sample variability than conventional moments, or to measurement errors in the case of extreme observations. L-moments therefore lead to more accurate and robust estimations of parameters or characteristics of the basic probability distribution.

Sample L-moments have been used previously in the statistics, although not as part of a unified theory. The first sample L-moment l_1 is a sample L-location (sample average), the second sample L-moment l_2 is a sample L-variability. Natural estimation of the ratio of L-moments (10) is the sample ratio of L-moments

$$t_r = \frac{l_r}{l_2}, \quad r = 3, 4, \tag{28}$$

Hence t_3 is a sample L-skewness and t_4 is a sample L-kurtosis. Sample ratios of L-moments t_3 and t_4 may be used as characteristics of skewness and kurtosis of the sample data set.

Gini middle difference relates to the sample L-moments and it has the form

$$G = \binom{n}{2}^{-1} \cdot \sum_{i>j}\sum (x_{i:n} - x_{j:n}), \tag{29}$$

and Gini coefficient, which depends only on a single parameter σ in the case of two-parametric lognormal distribution, but it depends on the values of all three parameters in the case of three-parametric lognormal distribution. Table 2 presents the expressions for parameter estimations of chosen probability distributions obtained using the method of L-moments. For more details see for example [3]-[13] or [14].

Table 2: Formulas for estimations of parameters taken by the method of L-moments of chosen probability distributions

Distribution	Parameter estimation
Exponential	(ξ known) $\hat{\alpha} = l_1$
Gumbel	$\hat{\alpha} = \frac{l_2}{\ln 2}$ $\hat{\xi} = l_1 - e \cdot \hat{\alpha}$
Logistic	$\hat{\alpha} = l_2$ $\hat{\xi} = l_1$
Normal	$\hat{\sigma} = \pi^{\frac{1}{2}} \cdot l_2$ $\hat{\mu} = l_1$
General Pareto	(ξ known) $\hat{k} = \frac{l_1}{l_2} - 2$ $\hat{\alpha} = (1 + \hat{k}) \cdot l_1$
General extermal	$z = \frac{2}{3 + t_3} - \frac{\ln 2}{\ln 3}$ $\hat{k} = 7{,}8590\, z + 2{,}9554\, z^2$ $\hat{\alpha} = \frac{l_2 \cdot \hat{k}}{(1 - 2^{-\hat{k}}) \cdot \Gamma(1 + \hat{k})}$ $\hat{\xi} = l_1 + \hat{\alpha} \cdot \frac{\Gamma(1 + \hat{k}) - 1}{\hat{k}}$

Table 2: Continuation

General logistic	$\hat{k} = -t_3$ $\hat{\alpha} = \dfrac{l_2}{\Gamma(1+\hat{k}) \cdot \Gamma(1-\hat{k})}$ $\hat{\xi} = l_1 + \dfrac{l_2 - \hat{\alpha}}{\hat{k}}$
Lognormal	$z = \sqrt{\dfrac{8}{3}} \cdot \Phi^{-1}\left(\dfrac{1+t_3}{2}\right)$ [2)] $\hat{\sigma} = 0{,}999\,281 z - 0{,}006\,118\, z^3 + 0{,}000\,127\, z^5$ $\hat{\mu} = \ln \dfrac{l_2}{\operatorname{erf}\left(\dfrac{\sigma}{2}\right)} - \dfrac{\hat{\sigma}^2}{2}$ $\hat{\xi} = l_1 - \exp\left(\hat{\mu} + \dfrac{\hat{\sigma}^2}{2}\right)$
Gamma	(ξ known) $t = \dfrac{l_2}{l_1}$ if $0 < t < \dfrac{1}{2}$, then: $z = \pi \cdot t^2$ $\hat{\alpha} \approx \dfrac{1 - 0{,}3080\, z}{z - 0{,}058\,12\, z^2 + 0{,}017\,65\, z^3}$ if $\dfrac{1}{2} \le t < 1$, then: $z = 1 - t$ $\hat{\alpha} \approx \dfrac{0{,}7213\, z - 0{,}5947\, z^2}{1 - 2{,}1817\, z + 1{,}2113\, z^2}$ $\hat{\beta} = \dfrac{l_1}{\hat{\alpha}}$

[2)] $\Phi^{-1}(\cdot)$ is quantile function of standardized normal distribution.

2.3 TL-Moments of Probability Distribution

An alternative robust version of L-moments will be introduced now. This modification of L-moments is called the „trimmed L-moments“ and it is noted TL-moments. In this modification of L-moments the expected values of order statistics of random sample in the definition of L-moments of probability distributions are replaced with expected values of order statistics of larger random sample and the sample size grows in such a way that it corresponds to the total size of the adjustment, as shown below.

TL-moments have certain advantages over conventional L-moments and central moments. TL-moment of probability distribution may exist even if the corresponding L-moment or central moment of this probability distribution does not exist, as it is the case of Cauchy distribution. Sample TL-moments are more resistant to outliers in the data. Method of TL-moments is not intended to replace the existing robust methods, but rather as their supplement, especially in situations where we have outliers in the data.

In this alternative robust modification of L-moments the expected value $E(X_{r-j:r})$ is replaced by the expected value $E(X_{r+t_1-j:r+t_1+t_2})$. Thus, for each r we increase the sample size of random sample from the original r to $r + t_1 + t_2$ and we work only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, \ldots, X_{t_1+r:r+t_1+t_2}$ by trimming the t_1 smallest and the t_2 largest from the conceptual sample. This modification is called the r-th trimmed L-moment (TL-moment) and it is marked $\lambda_r^{(t_1,t_2)}$. Thus, TL-moment of the r-th order of random variable X is defined

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t_1-j:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{30}$$

It is evident from the expressions (30) and (1) that TL-moments reduce to L-moments where $t_1 = t_2 = 0$. Although we can also consider applications where the adjustment values are not equal, i.e. $t_1 \neq t_2$, we focus here only on the symmetric case $t_1 = t_2 = t$. Then the expression (30) can be rewritten

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots . \tag{31}$$

Thus, for example $\lambda_1^{(t)} = E(X_{1+t:1+2t})$ is the expected value of the median of the conceptual random sample of sample size 1 + 2*t*. It is necessary to note here that $\lambda_1^{(t)}$ is equal to zero for distributions that are symmetrical around zero.

The first four TL-moments have the form for $t = 1$

$$\lambda_1^{(1)} = E(X_{2:3}), \tag{32}$$

$$\lambda_2^{(1)} = \frac{1}{2} E(X_{3:4} - X_{2:4}), \tag{33}$$

$$\lambda_3^{(1)} = \frac{1}{3} E(X_{4:5} - 2X_{3:5} + X_{2:5}), \tag{34}$$

$$\lambda_4^{(1)} = \frac{1}{4} E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{35}$$

Measurements of location, variability, skewness and kurtosis of the probability distribution analogous to conventional L-moments (6)−(9) are based on $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ a $\lambda_4^{(1)}$.

Expected value $E(X_{r:n})$ can be written using the formula (2). Using equation (2) we can re-express the right side of equation (31)

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{(r+2t)!}{(r+t-j-1)! \cdot (t+j)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r+t-j-1} \cdot [1-F(x)]^t \tag{36}$$

$r = 1, 2, \ldots .$

It is necessary to notify here that $\lambda_r^{(0)} = \lambda_r$ represents normal the *r*-th L-moment with no adjustment.

Expressions (32)−(35) for the first four TL-moments ($t = 1$) may be written in an alternative manner

$$\lambda_1^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \, \mathrm{d} F(x), \tag{37}$$

$$\lambda_2^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot [2F(x) - 1] \, \mathrm{d} F(x), \tag{38}$$

$$\lambda_3^{(1)} = \frac{20}{3} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{5[F(x)]^2 - 5F(x) + 1\} \, \mathrm{d} F(x), \tag{39}$$

$$\lambda_4^{(1)} = \frac{15}{2} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{14[F(x)]^3 - 21[F(x)]^2 + 9[F(x)] - 1] \, \mathrm{d} F(x). \tag{40}$$

Distribution may be identified by its TL-moments, although some of its L-moments and conventional moments do not exit. For example $\lambda_1^{(1)}$ (the expected value of median of conceptual random sample of sample size three) exists for Cauchy distribution, although the first L-moment λ_1 does not exit.

TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ can be defined analogously as L-skewness τ_3 and L-kurtosis τ_4

$$\tau_3^{(t)} = \frac{\lambda_3^{(t)}}{\lambda_2^{(t)}}, \tag{41}$$

$$\tau_4^{(t)} = \frac{\lambda_4^{(t)}}{\lambda_2^{(t)}}. \tag{42}$$

2.4 Sample TL-Moments

Let $x_1, x_2, \ldots, x_n$ is the sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ is order sample. The expression

$$\hat{E}(X_{j+1:j+l+1}) = \frac{1}{\binom{n}{j+l+1}} \cdot \sum_{i=1}^{n} \binom{i-1}{j} \cdot \binom{n-i}{l} \cdot x_{i:n} \tag{43}$$

is considered to be an unbiased estimation of the expected value of the (j + 1)-th order statistic $X_{j+1:j+l+1}$ in the conceptual random sample of sample size ($j + l$ + 1). Now we assume that in the definition of the TL-moment $\lambda_r^{(t)}$ in (31) we replace the expression $E(X_{r+t-j:r+2t})$ by its unbiased estimation

$$\hat{E}(X_{r+t-j:r+2t}) = \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \tag{44}$$

which is obtained by assigning $j \rightarrow r + t - j - 1$ a $l \rightarrow t + j$ in (43). Now we get the r-th sample TL-moment

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \hat{E}(X_{r+t-j:r+2t}), \quad r = 1, 2, ..., n-2t, \tag{45}$$

i.e.

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \tag{46}$$

$$r = 1, 2, ..., n-2t,$$

which is an unbiased estimation of the r-th TL-moment $\lambda_r^{(t)}$. Note that for each $j = 0, 1, \ldots, r - 1$ the values $x_{i:n}$ in (46) are not equal to zero only for $r + t - j \leq i \leq n - t - j$ relative to the combination numbers. Simple adjustment of equation (46) provides an alternative linear form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] \cdot x_{i:n}. \tag{47}$$

For example, we obtain for the first sample TL-moment for r = 1

$$l_1^{(t)} = \sum_{i=t+1}^{n-t} w_{i:n}^{(t)} \cdot x_{i:n}, \tag{48}$$

where the weights are given by

$$w_{i:n}^{(t)} = \frac{\binom{i-1}{t} \cdot \binom{n-i}{t}}{\binom{n}{2t+1}}. \tag{49}$$

The above results can be used to estimate TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ by simple ratios

$$t_3^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}, \tag{50}$$

$$t_4^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}. \tag{51}$$

We can choose $t = n\alpha$ representing the amount of adjustment from each end of the sample, where α is a certain ratio, where $0 \le \alpha < 0{,}5$.

Table 3 contains the expressions for TL-moments and ratios of TL-moments and expressions for parameter estimations of chosen probability distributions obtained using the method of TL-moments ($t = 1$), more see for example in [15].

Table 3: Formulas for TL-moments and ratios of TL-moments and formulas for estimations of parameters taken by the method of TL-moments of chosen probability distributions ($t = 1$)

Distribution	TL-moments and ratios of TL-moments	Parameter estimation
Normal	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}297\,\sigma$	$\hat{\mu} = l_1^{(1)}$

	$\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}062$	$\hat{\sigma} = \dfrac{l_2^{(1)}}{0{,}297}$
Logistic	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}500\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}083$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = 2\,l_2^{(1)}$
Cauchy	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}698\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}343$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = \dfrac{l_2^{(1)}}{0{,}698}$
Exponential	$\lambda_1^{(1)} = \dfrac{5\,\alpha}{6}$ $\lambda_2^{(1)} = \dfrac{\alpha}{4}$ $\tau_3^{(1)} = \dfrac{2}{9}$ $\tau_4^{(1)} = \dfrac{1}{12}$	$\hat{\alpha} = \dfrac{6\,l_1^{(1)}}{5}$

Source:[15]; own research

2.5 Maximum Likelihood Method

Let the random sample of sample size n comes from three-parametric lognormal distribution with probability density function

$$f(x;\,\mu,\,\sigma^2,\,\theta) = \frac{1}{\sigma\cdot(x-\theta)\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{[\ln(x-\theta)-\mu]^2}{2\sigma^2}\right], \qquad x > \theta,$$
$$= 0, \qquad \text{else}, \tag{52}$$

where $-\infty < \mu < \infty$, $\sigma^2 > 0$, $-\infty < \theta < \infty$ are parameters. Three-parametric lognormal distribution is described in detail for example in [8], [9], [11] or [12].

The likelihood function then has the form

$$L(\mathbf{x};\mu,\sigma^2,\theta) = \prod_{i=1}^{n} f(x_i;\mu,\sigma^2,\theta)$$
$$= \frac{1}{(\sigma^2)^{n/2}\cdot(2\pi)^{n/2}\cdot\prod_{i=1}^{n}(x_i-\theta)}\cdot\exp\left\{\sum_{i=1}^{n} -\frac{[\ln(x_i-\theta)-\mu]^2}{2\sigma^2}\right\}. \tag{53}$$

We determine the natural logarithm of the likelihood function

$$\ln L(\mathbf{x};\mu,\sigma^2,\theta)$$
$$= \sum_{i=1}^{n} -\frac{[\ln(x_i-\theta)-\mu]^2}{2\sigma^2} - \frac{n}{2}\cdot\ln\sigma^2 - \frac{n}{2}\cdot\ln(2\pi) - \sum_{i=1}^{n}\ln(x_i-\theta). \tag{54}$$

We put the first partial derivations of the logarithm of the likelihood function according to μ and σ^2 in the equality to zero. We obtain a system of likelihood equations

$$\frac{\partial \ln L(\mathbf{x};\mu,\sigma^2,\theta)}{\partial\mu} = \frac{\sum_{i=1}^{n}[\ln(x_i-\theta)-\mu]}{\sigma^2} = 0, \tag{55}$$

$$\frac{\partial \ln L(\mathbf{x};\mu,\sigma^2,\theta)}{\partial\sigma^2} = \frac{\sum_{i=1}^{n}[\ln(x_i-\theta)-\mu]^2}{2\sigma^4} - \frac{n}{2\sigma^2} = 0. \tag{56}$$

After adjustment we obtain maximum likelihood estimations of the parameters μ and σ^2 for the parameter θ

$$\hat{\mu}(\theta) = \frac{\sum_{i=1}^{n} \ln(x_i - \theta)}{n}, \tag{57}$$

$$\hat{\sigma}^2(\theta) = \frac{\sum_{i=1}^{n} [\ln(x_i - \theta) - \hat{\mu}(\theta)]^2}{n}. \tag{58}$$

If the value of the parameter θ is known, we get maximum likelihood estimations of the remaining two parameters of three-parametric lognormal distribution using equations (57) and (58). However, if the value of the parameter θ is unknown, the problem is more complicated. It can be proved that if the parameter θ closes to $\min\{X_1, X_2, \ldots, X_n\}$, then the maximum likelihood approaches to infinity. The maximum likelihood method is also often combined with Cohen method, where we put the smallest sample value to be equal to the $100 \cdot (n+1)^{-1}$-percentage quantile

$$x_{\min}^{V} = \hat{\theta} + \exp(\hat{\mu} + \hat{\sigma} \cdot u_{(n+1)^{-1}}). \tag{59}$$

Equation (59) is then combined with a system of equations (57) and (58).

For solving of maximum likelihood equations (57) and (58) it is also possible to use $\hat{\theta}$ satisfying the equation

$$\sum_{i=1}^{n} (x_i - \hat{\theta}) + \frac{\sum_{i=1}^{n} \frac{z_i^{\backslash}}{(x_i - \hat{\theta})}}{\hat{\sigma}(\hat{\theta})} = 0, \tag{60}$$

where

$$z_i^{\backslash} = \frac{\ln(x_i - \hat{\theta}) - \hat{\mu}(\hat{\theta})}{\hat{\sigma}(\hat{\theta})}, \tag{61}$$

where $\hat{\mu}(\hat{\theta})$ and $\hat{\sigma}(\hat{\theta})$ satisfy equations (57) and (58) with the parameter θ replaced by $\hat{\theta}$. We may also obtain the limits of variances

$$n \cdot D(\hat{\theta}) = \frac{\sigma^2 \cdot \exp(2\mu)}{\omega \cdot \left[\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1 \right]}, \tag{62}$$

$$n \cdot D(\hat{\mu}) = \frac{\sigma^2 \cdot [\omega \cdot (1 + \sigma^2) - 2\sigma^2]}{\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1}, \tag{63}$$

$$n \cdot D(\hat{\sigma}) = \frac{\sigma^2 \cdot [\omega \cdot (1 + \sigma^2) - 1]}{\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1}. \tag{64}$$

3 Results and Discussion

In the past L-moments were mainly used in hydrology, climatology and meteorology in the research of extreme precipitation, see for example [14]. There are mainly small data sets in these cases. This study presents an application of L-moments and TL-moments on large sets of economic data, Table 4 presents the sample sizes of obtained sample sets of households.

The researched variable is the net annual household income per capita (in CZK) within the Czech Republic (nominal income). The data obtained come from a statistical survey Microcensus – years 1992, 1996, 2002, and statistical survey EU-SILC (The European Union Statistics on Income and Living Conditions) – the period 2004-2007, from the Czech Statistical Office. Total 168 income distributions were analyzed this way, both for all households of the Czech Republic together and also broken down by gender, country (Bohemia and Moravia, see Figure 1), social groups, municipality size, age and the highest educational attainment, while households are classified into different subsets according to the head of household, which is man in the vast majority of households. Sharply smaller sample sizes for women than for men in Table 4 correspond to this fact. Head of household is always a man in two-parent families

Table 4: Sample sizes of income distributions broken down by relatively homogeneous categories

	Set	Year						
		1992	1996	2002	2004	2005	2006	2007
Gender	Men	12,785	21,590	5,870	3,203	5,456	7,151	8,322
	Women	3,448	6,558	2,103	1,148	2,027	2,524	2,972
Country	Czech Republic	16,233	28,148	7,973	4,351	7,483	9,675	11,294
	Bohemia	9,923	22,684	5,520	2,775	4,692	6,086	7,074
	Moravia	6,310	5,464	2,453	1,576	2,791	3,589	4,220
Social group	Lower employee	4,953	4,963	1,912	1,068	1,880	2,385	2,811
	Self-employed	932	1,097	740	391	649	802	924
	Higher employee	3,975	4,248	2,170	1,080	1,768	2,279	2,627
	Pensioner with s EA	685	594	278	178	287	418	493
	Pensioner without EA	4,822	4,998	2,533	1,425	2,577	3,423	4,063
	Unemployed	189	135	172	131	222	258	251
Munic. size	0–999 inhabitants	2,458	3,069	999	727	1,164	1,607	1,947
	1,000–9,999 inhabitants	4,516	4,471	2,300	1,233	2,297	3,034	3,511
	10,000–99,999 inhabitants	5,574	5,755	2,401	1,508	2,655	3,347	3,947
	100,000 and more inhabit.	3,685	2,853	2,273	883	1,367	1,687	1,889
Age	To 29 years	1,680	2,809	817	413	627	649	827
	From 30 to 39 years	3,035	4,718	1,398	716	1,247	1,620	1,655
	From 40 to 49 years	3,829	6,348	1,446	738	1,249	1,609	1,863
	From 50 to 59 years	2,621	5,216	1,642	919	1,581	2,051	2,391
	From 60 years	5,068	9,057	2,670	1,565	2,779	3,746	4,558
Education	Primary	9,302	15,891	3,480	553	940	1,183	1,385
	Secondary	4,646	3,172	2,493	3,186	5,460	7,168	8,371
	Complete secondary	1,951	6,356	1,129	118	282	266	319
	Tertial	334	2,729	871	494	801	1,058	1,219

Source: Own research

of the type the husband and wife or two partners, regardless of the economic activity. In single-parent families of the type only one parent with children and in non-family households, where persons are not related by marriage or by union partner, nor parent-child relationship, the first decisive criterion for determining the head of household is the economic activity and the second aspect is the amount of money income of individual household members. This criterion also applies in the case of more complex types of households, such as the case of joint management of more than two-parent families.

Source: www.zemepis.com

Figure 1: Map of the Czech republic (Bohemia and Moravia)

The parameters of three-parametric lognormal curves were estimated simultaneously using three robust methods of parametric estimation, namely the method of TL-moments, the method of L-moments and the maximum likelihood method and accuracy of these methods were compared with each other using the familiar test criterion

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n\,\pi_i)^2}{n\,\pi_i}, \qquad (65)$$

where n_i are the observed frequencies in individual income intervals, π_i are the theoretical probabilities of belonging of statistical unit to the i-th interval, $n \cdot \pi_i$ are the theoretical frequencies in individual income intervals, , $i = 1, 2, ..., k$, n is the total sample size of the corresponding statistical set and k is the number of intervals.

However, the question of the appropriateness of the model curve for income distribution is not quite common mathematical and statistical issue in which we test the null hypothesis

H_0: The sample comes from the assumed theoretical distribution

against the alternative hypothesis

H_1: non H_0,

because in goodness of fit tests in the case of income distribution we meet frequently with the fact that we work with large sample sizes and therefore the tests would almost always lead to the rejection of the null hypothesis. This results not only from the fact that with such large sample sizes the power of the test is so high at the chosen significance level that the test uncovers all the slightest deviations of the actual income distribution and a model, but it also results from the principle of construction of the test.

But practically we are not interested in such small deviations, so only gross agreement of the model with reality is sufficient and we so called “borrow” the model (curve). Test criterion χ^2 can be used in that direction only tentatively. When evaluating the suitability of the model we proceed to a large extent subjective and we rely on experience and logical analysis.

Table 5: Parameter estimations of three-parametric lognormal curves obtained using three various robust methods of point parameter estimation and the value of χ^2 criterion

Year	Method of TL-moments			Method of L-moments			Maximum likelihood method		
	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
1992	9.722	0.521	14,881	9.696	0.700	14,491	10.384	0.390	-325
1996	10.334	0.573	25,981	10.343	0.545	25,362	10.995	0.424	52.231
2002	10.818	0.675	40,183	10.819	0.773	37,685	11.438	0.459	73.545
2004	10.961	0.552	39,899	11.028	0.675	33,738	11.503	0.665	7.675
2005	11.006	0.521	40,956	11.040	0.677	36,606	11.542	0.446	-8.826
2006	11.074	0.508	44,941	11.112	0.440	40,327	11.623	0.435	-42.331
2007	11.156	0.472	48,529	11.163	0.654	45,634	11.703	0.421	-171.292

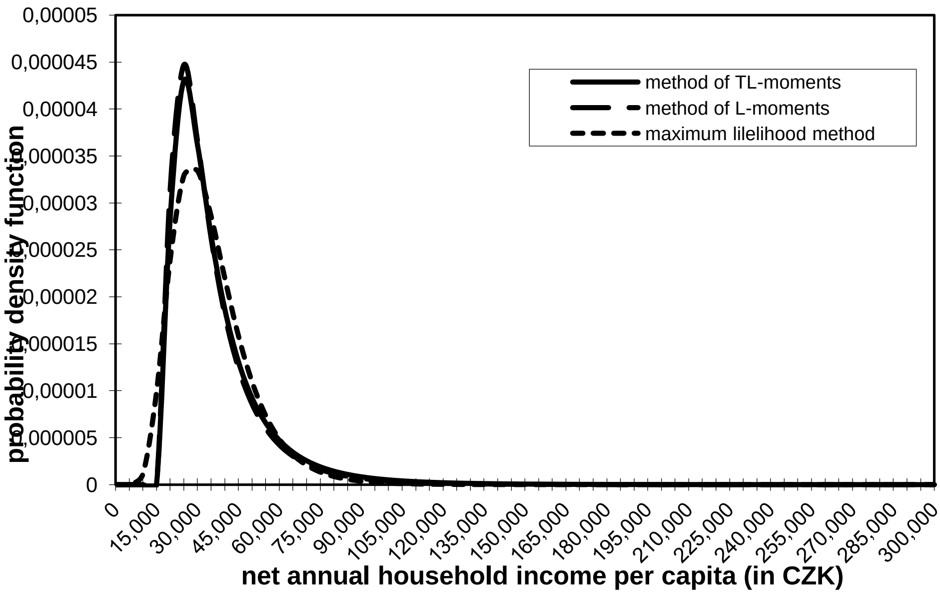
Year	Criterion χ^2	Criterion χ^2	Criterion χ^2
1992	739.512	811.007	1,227.325
1996	1,503.878	1,742.631	2,197.251
2002	998.325	1,535.557	1,060.891
2004	494.441	866.279	524.478
2005	731.225	899.245	995.855
2006	831.667	959.902	1,067.789
2007	1,050.105	1,220.478	1,199.035

Source: Own research

Method of TL-moments provided the most accurate results in almost all cases, with the negligible exceptions. Method of L-moments results as the second in more than half of the cases, although the differences between the method of L-moments and maximum likelihood method are not distinctive enough to turn in the number of cases where the method of L-moments came out better than maximum likelihood method. Table 5 is a typical representative of the results for all 168 income distributions. This table provides the results for the total household sets in the Czech Republic. It contains the estimated values of the parameters of three-parametric lognormal distribution, which were obtained simultaneously using the method of TL-moments, method of L-moments and maximum likelihood method, and the value of test criterion (65). This is evident from the values of the criterion that the method of L-moments brought more accurate results than maximum likelihood method in four of seven cases. The most accurate results were obtained using the method of TL-moments in all seven cases.

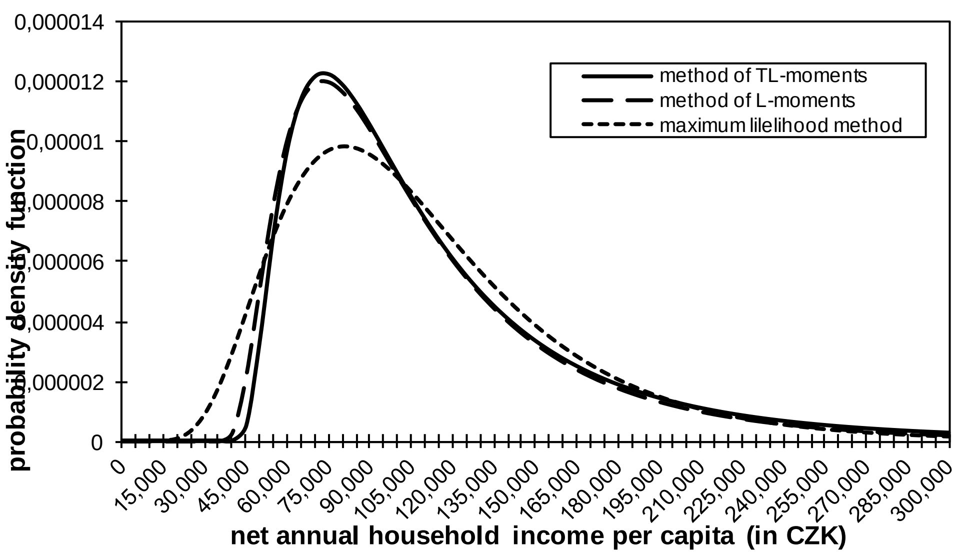
The estimation of the value of the parameter θ (beginning of the distribution, theoretical minimum) obtained using the maximum likelihood method is negative in 1992 and 2005–2007. This means that three-parametric lognormal curve gets into negative values initially its course in terms of income. Since at first the curve has very tight contact with the horizontal axis, it does not matter good agreement of model with real distribution.

Figures 2–4 allow the comparison of these methods in terms of model probability density functions in choosing years (1992, 2004 and 2007) for the total set of households throughout the Czech Republic together. It should be noted at this point that other scale is on the vertical axis in Figure 2 than in Figures 3 and 4 for better legibility, because income distribution just after the transformation of the Czech economy from a centrally planned to a marked economy still showed different behaviour (lower level and variability, higher skewness and kurtosis) than the income distribution closer to the present. It is clear from these three figures that the methods of TL-moments and L-moments bring the very similar results, while the probability density function with the parameters estimated by maximum likelihood method is very different from model probability density functions constructed using the method of TL-moments and the method of L-moments.



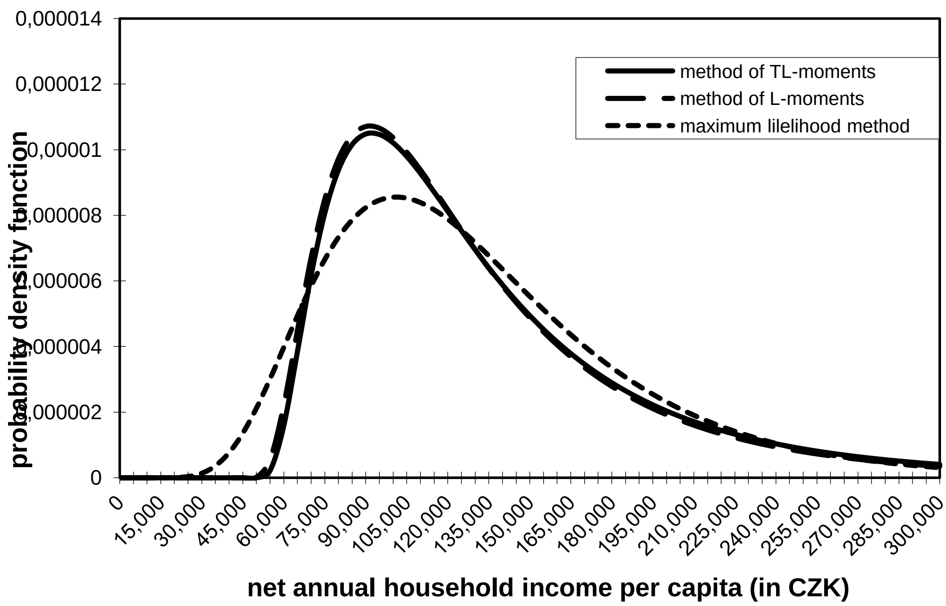
Source: Own research

Figure 2: Model probability density functions of three-parametric lognormal curves in 1992 with parameters estimated using three various robust methods of point parameter estimation



Source: Own research

Figure 3: Model probability density functions of three-parametric lognormal curves in 2004 with parameters estimated using three various robust methods of point parameter estimation

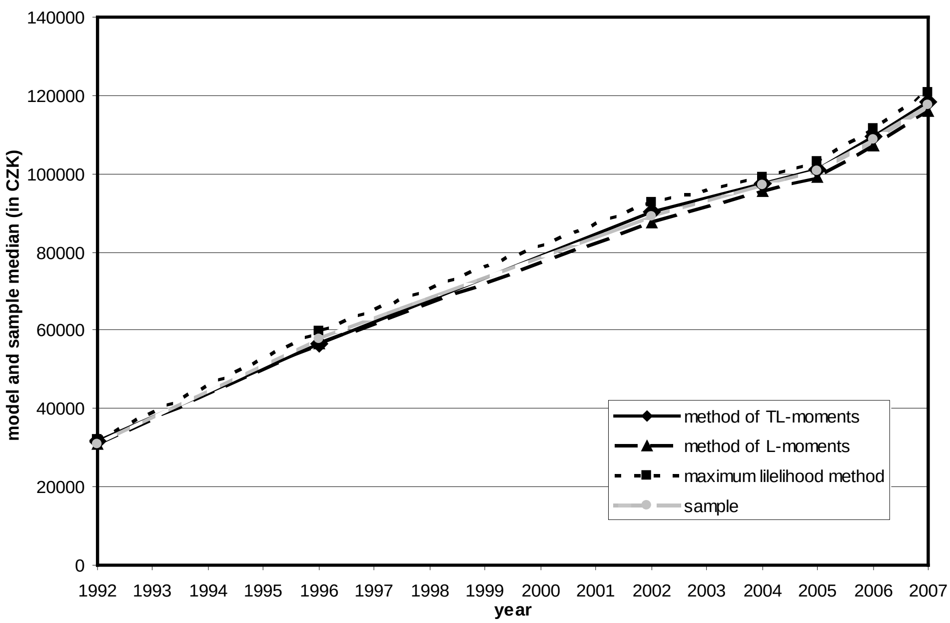


Source: Own research

Figure 4: Model probability density functions of three-parametric lognormal curves in 2007 with parameters estimated using three various robust methods of point parameter estimation

Figure 5 also provides some comparison of the accuracy of these three methods of point parameter estimation. It represents the development of the sample median and the theoretical medians of lognormal distribution with parameters estimated using the method of TL-moments, method of L-moments and maximum likelihood method in the researched period again for the total set of households of the Czech Republic. It is also clear from this figure that the curve representing the course of the theoretical medians of lognormal distribution with parameters estimated by methods of TL-moments and L-moments are more tightly to the curve showing the course of sample median compared with the curve representing the development of theoretical median of lognormal distribution with parameters estimated by maximum likelihood method.

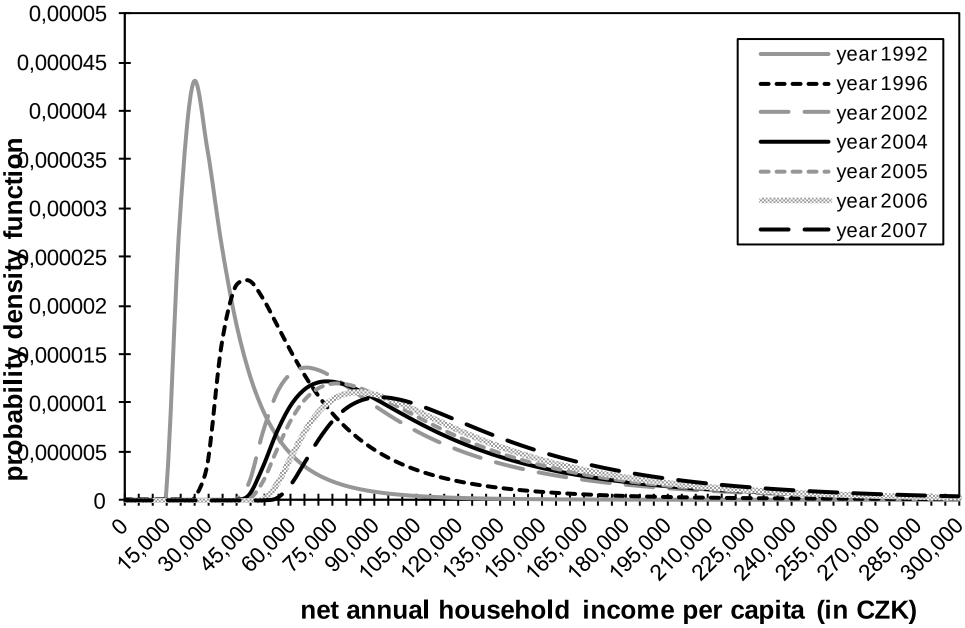
Figures 6–8 show the development of the model probability density functions of three-parametric lognormal distribution again with parameters estimated using three researched methods of parameter estimation in the analysed period for total set of households of the Czech Republic. Also, in view of these figures income distribution in 1992 shows a strong difference from the income distributions in next years. Also here, we can observe a certain similarity of the results taken using the methods of TL-moments and L-moments and considerable divergence of the results obtained using these two methods of point parameter estimation from the results obtained using the maximum likelihood method.



Source: Own research

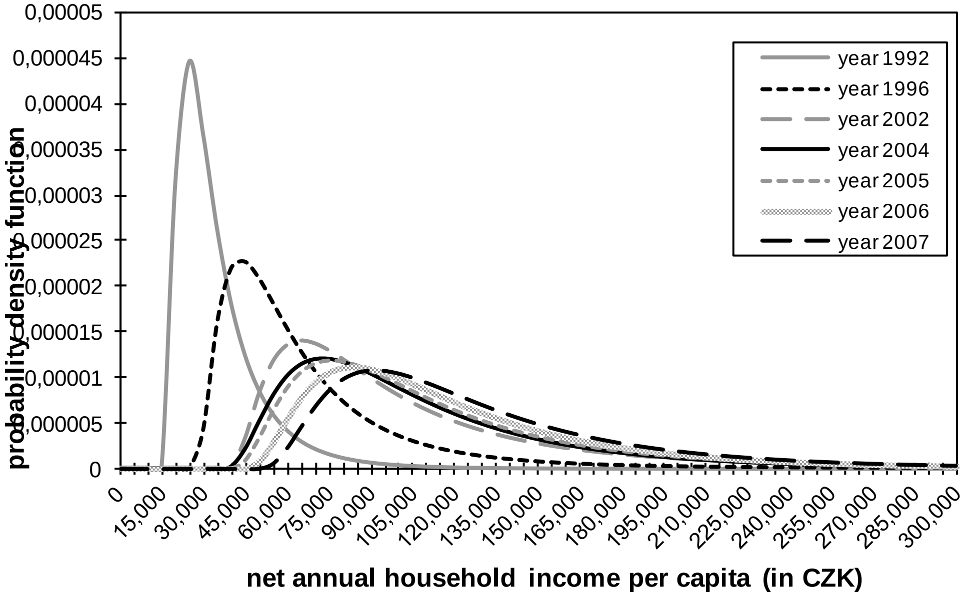
Figure 5: Development of model and sample median of net annual household income per capita (in CZK)

Figures 9–11 then represent the model relative frequencies (in %) of employees by the band of net annual household income per capita in 2007 obtained using three-parametric lognormal curves with parameters estimated by the method of TL-moments, method of L-moments and maximum likelihood method. These figures also allow some comparison of the accuracy of the researched methods of point parameter estimation compared with Figure 12, where are the really observed relative frequencies in individual income intervals obtained from a sample.



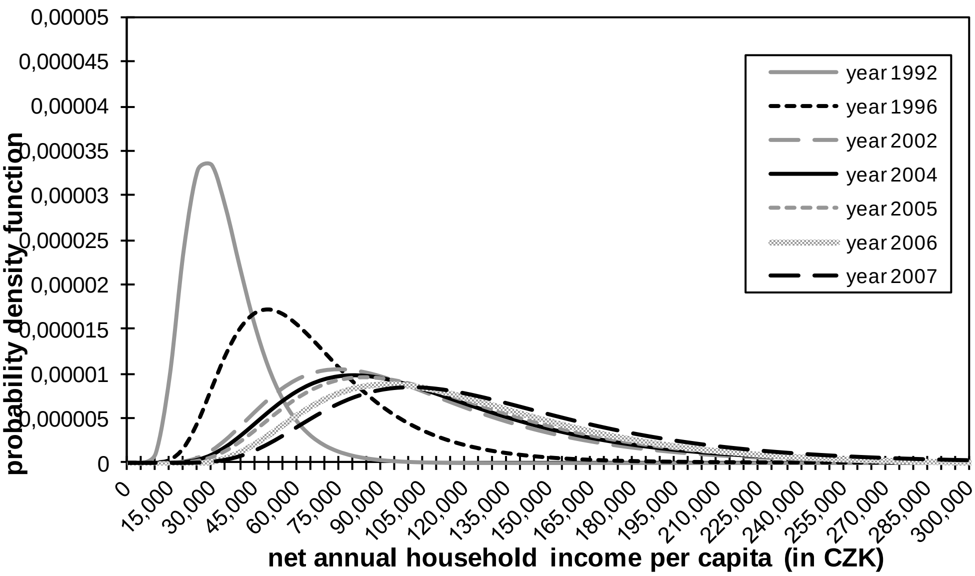
Source: Own research

Figure 6: Development of probability density function of three-parameter lognormal curves with parameters estimated using the method of TL-moments



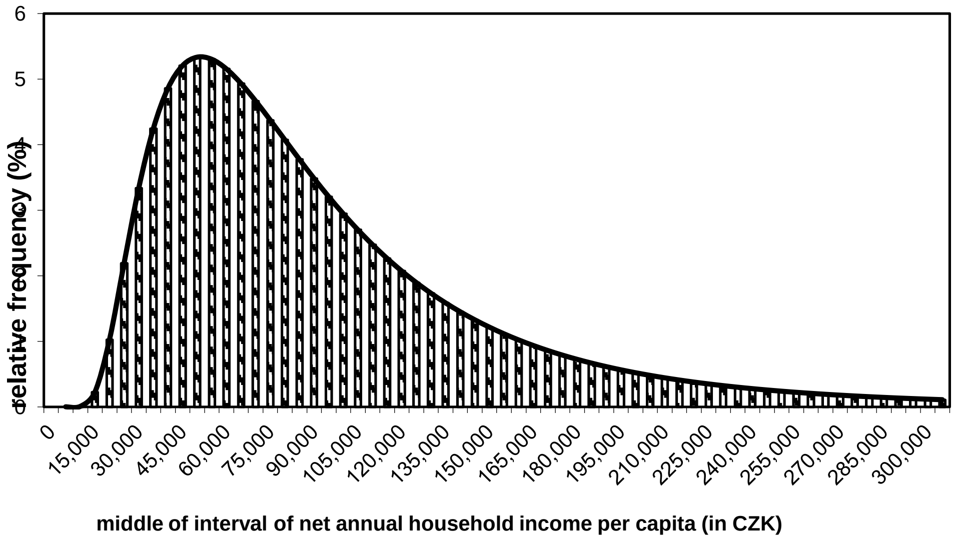
Source: Own research

Figure 7: Development of probability density function of three-parameter lognormal curves with parameters estimated using the method of L-moments



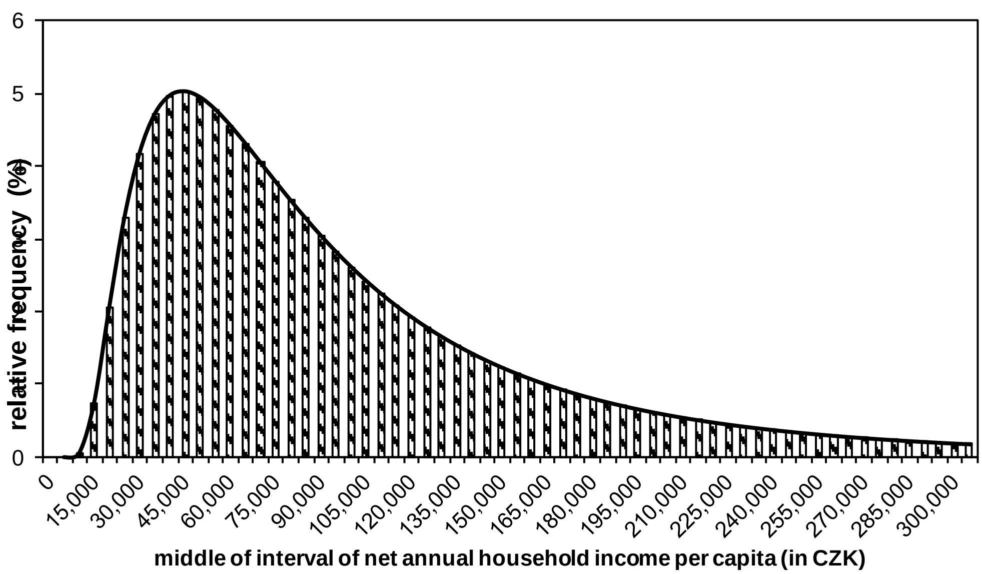
Source: Own research

Figure 8: Development of probability density function of three-parameter lognormal curves with parameters estimated using the maximum likelihood method



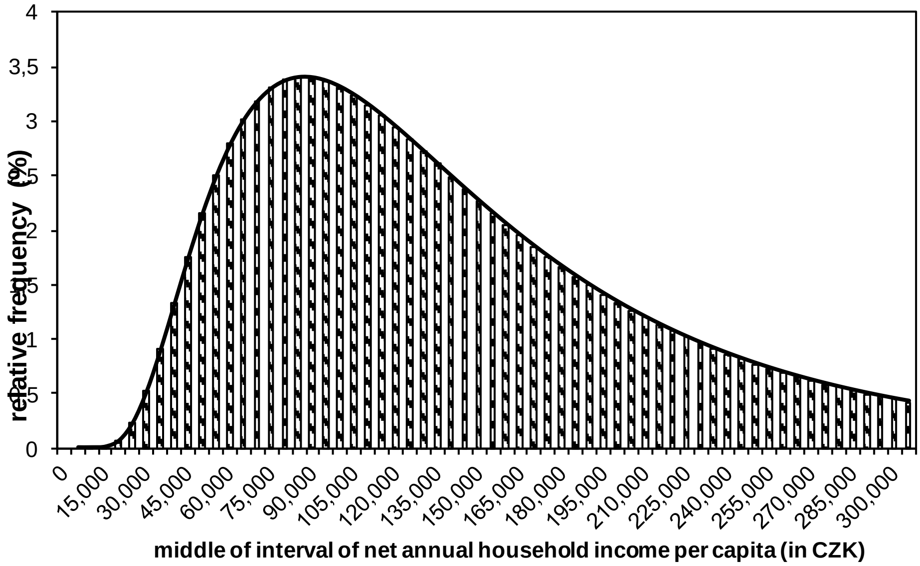
Source: Own research

Figure 9: Model ratios of employees by the band of net annual household income per capita with parameters of three-parametric lognormal curves estimated by the method of TL-moments in 2007



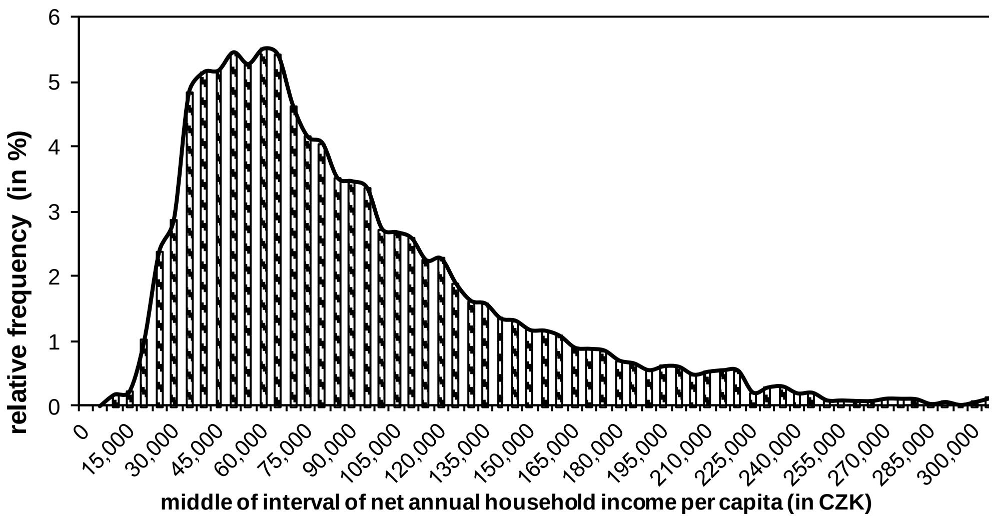
Source: Own research

Figure 10: Model ratios of employees by the band of net annual household income per capita with parameters of three-parametric lognormal curves estimated by the method of L-moments in 2007



Source: Own research

Figure: 11: Model ratios of employees by the band of net annual household income per capita with parameters of three-parametric lognormal curves estimated by the maximum likelihood method in 2007



Source: Own research

Figure 12: Sample ratios of employees by the band of net annual householdincome per capita in 2007

4 Conclusion

Relatively new class of moment characteristics of probability distributions were here introduced. There are the characteristics of location (level), variability, skewness and kurtosis of probability distributions constructed using L-moments and TL-moments that are robust extension of L-moments. Own L-moments have been introduced as a robust alternative to classical moments of probability distributions. However, L-moments and their estimations lack some robust features that belong to TL-moments.

Sample TL-moments are linear combinations of sample order statistics, which assign zero weight to a predetermined number of sample outliers. Sample TL-moments are unbiased estimations of the corresponding TL-moments of probability distributions. Some theoretical and practical aspects of TL-moments are still the subject of research or they remain for future research. Efficiency of TL-statistics depends on the choice of α, for example, $l_1^{(0)}, l_1^{(1)}, l_1^{(2)}$ have the smallest variance (the highest efficiency) among other estimations for random samples from normal, logistic and double exponential distribution.

The above methods can be also used for modeling the wage distribution or other analysis of economic data (among other methods, see for example [16] or [17]).

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