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On higher order ultra –hyperbolic kernel related to the spectrum

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Abstract

In this paper, the solutions of the equation $-(-I)^k u_k = f(x)$, $k \geq 1$ in $\mathcal{R}^n$ where $f \in L^2(\mathcal{R}^n)$ and $\frac{\partial v_k}{\partial t} = \Box v_k$ in $\mathcal{R}^n X\,(0,\infty)$, $v_k(x,0) = f_k(x)$, $k \geq 1$ are considered, the operator $\Box$ is named the ultra –hyperbolic operator defined by $\Box = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)$, $p + q = n$ is the dimension of Euclidean space $\mathcal{R}^n$ .We define the ultra –hyperbolic kernels E_k of higher order, then we get recurrence relations between u_k and E_k, we obtain also an estimation of u_k and E_k related to the spectrum, then we show that u_k and E_k are bounded. A relation between v_k and u_k under certain conditions on f_k is obtained.

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1 Introduction

The solutions of the equation

$$-(\Delta - I)^k u_k(x) = f(x) \ , k \geq 1 \ \text{in } \mathcal{R}^n, f \in L^2(\mathcal{R}^n) \tag{1.1}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ were investigated in [1], Bessel potential of higher order were defined and recurrence relations between these solutions and these Bessel potentials are obtained.

Now, the purpose of this work is to study the solutions of

$$-(\Box - I)^k u_k = f(x), k \geq 1 \text{ in } \mathcal{R}^n, f \epsilon L^2(\mathcal{R}^n) \tag{1.2}$$

and we obtain an estimation of the solutions u_k such equation which is related to the spectrum and also of the ultra-hyperbolic kernels E_k .There are a lot of problems use the ultra –hyperbolic operator, see [3], [4] and [5].Then under certain conditions on $f_k$we obtain a relation between these solutions u_k and the solutions v_k of

$$\frac{\partial v_k}{\partial t} = \Box v_k \ \text{in } \mathcal{R}^n X (0, \infty) \tag{1.3}$$

$$v_k(x, 0) = f_k(x) \ , k \geq 1$$

2 Preliminary Notes

Definition 2.1 Fourier transform take the form

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{R}^n} e^{-i(\xi,x)} f(x) dx \tag{2.1}$$

and its inverse

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{R}^n} e^{i(\xi,x)} \hat{f}(\xi) d\xi \tag{2.2}$$

See [2], [6] and [7].

Definition 2.2 let $\xi = (\xi_1, \xi_2, \dots \dots, \xi_n) \in \mathcal{R}^n$denote by $\Gamma_+ = \{\xi \in \mathcal{R}^n : \xi_1^2 + \cdots +$

$\xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 > 0$ and $\xi_1 > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma}$,denotes the closure of Γ.

Definition 2.3 let $\Omega \subset \overline{\Gamma_+}$ is the spectrum of E_k and

$$E_k(x) = \frac{1}{(2\pi)^{n/2}} \int_\Omega \int_0^\infty \exp[-t(1+||\xi||^2)^k + i(\xi, x)] d\xi \, dt \tag{2.3}$$

where E_k is called ultra hyper-bolic kernel (of order k).

Definition 2.4 Bipoolar coordinates

Let $\xi_1 = r\omega_1, \xi_2 = r\omega_2, \dots\dots\dots, \xi_p = r\omega_p$,

and

$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \dots\dots\dots, \xi_{p+q} = s\omega_{p+q}$,

where

$\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$ where $d\xi = r^{p-1}s^{q-1} dr ds d\Omega_p d\Omega_q$, and $d\Omega_p$, $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathcal{R}^p$, $\mathcal{R}^q$ respectively, and we suppose $0 \le r \le R$ and $0 \le s \le L$ where R and L are constants.

3 Main Results

3.1 The solution $\mathbf{u_1}$

$$-(\square - I)u_1(x) = f(x) \text{ in } \mathcal{R}^n \tag{3.1}$$

The Fourier transform of (3.1) is

$$\widehat{u_1} = \frac{\hat{f}}{1+||\xi||^2} \tag{3.2}$$

where

$$||\xi||^2 = \sum_{i=1}^p \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 > 0$$

and its inverse Fourier transform is

$$u_1 = \frac{f * E_1}{(2\pi)^{n/2}} \tag{3.3}$$

where

$$\widehat{E_1}=\frac{1}{1+\|\xi\|^2} \tag{3.4}$$

and

$$E_1(x) = \frac{1}{(2\pi)^{n/2}} \int_\Omega \int_0^\infty \exp[-t(1+\|\xi\|^2) + i(\xi, x)]d\xi\, dt \tag{3.5}$$

where E_1 is called ultra hyperbolic kernel (of order one) , $\Omega \subset \overline{\Gamma_+}$ is the spectrum of E_1 , and the estimation of E_1 is

$$|E_1(x)| \le \frac{1}{(2\pi)^{n/2}} \Omega_p\Omega_q \int_0^R \int_0^L \int_0^\infty \exp[-t + t(s^2 - r^2)]\, r^{p-1}s^{q-1}\, drdsdt$$

$$= \frac{1}{(2\pi)^{n/2}} \Omega_p\, \Omega_q M_1 \tag{3.6}$$

where

$$M_1 = \int_0^\infty \int_0^R \int_0^L \exp[-t + t(s^2 - r^2)]\, r^{p-1}s^{q-1} drdsdt\, ,\ \Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \text{ and } \Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} \tag{3.7}$$

Thus for any fixed $t > o$, E_1 is bounded.

Also,

$$u_1(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_1(x-y).f(y)dy \tag{3.8}$$

$$|u_1(x)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega |E_1.f(y)|dy \le \frac{1}{(2\pi)^{\frac{n}{2}}} |E_1|N$$

$$\le \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{(2\pi)^{\frac{n}{2}}} \Omega_p\, \Omega_q M_1 N \le \frac{1}{(2\pi)^n} \Omega_p\, \Omega_q M_1 N \tag{3.9}$$

Where M_1 , Ω_p , Ω_q are defined in (3.7) and

$$\int_\Omega |f(y)|\, dy = N \tag{3.10}$$

i.e. $u_1$is bounded.

3.2 The solution u_2.

At first we consider now the PDE

$$- (\square - I)^2 u_2 = f \quad \text{in} \quad \mathcal{R}^n \tag{3.11}$$

Which is equivalent to the system

$$\begin{aligned} -(\square - I)\, u_1 &= f \\ (\square - I) u_2 &= u_1 \end{aligned} \tag{3.12}$$

Applying Fourier transform, we get

$$-(1 + \|\xi\|^2)\, \widehat{u_2} = \widehat{u_1}$$

$$\widehat{u_2} = -\frac{\hat{f}}{(1+\|\xi\|^2)^2}, \quad u_2(x) = -\frac{f * E_2}{(2\pi)^{\frac{n}{2}}} \quad , E_2 = \frac{E_1 * E_1}{(2\pi)^{\frac{n}{2}}} \quad ,$$

$$u_2(x) = -\frac{u_1 * E_1}{(2\pi)^{\frac{n}{2}}} \tag{3.13}$$

Where E_1 and u_1 are given in (3.5) and (3.8) respectively. So we have for estimation of $E_2(x)$

$$E_2(x) = \frac{E_1 * E_1}{(2\pi)^{\frac{n}{2}}} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_1(y) E_1(x-y) dy$$

Where $E_2(x)$ is called ultra –hyperbolic kernel of second order, $\Omega \subset \overline{\Gamma_+}$ is the spectrum of E_2.

Then we can find $|E_2(x)|$ by using bipolar coordinates, where $d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$, and $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathcal{R}^p$ and $\mathcal{R}^q$ respectively. Then using (3.6) and (3.7), we obtain

$$|E_2(x)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega |E_1(y). E_1(x-y)|\, dy$$

$$\le \frac{1}{(2\pi)^{\frac{3n}{2}}} {\Omega_p}^3 {\Omega_q}^3 M_1^2 \int_0^R \int_0^L r^{p-1} s^{q-1} dr ds = \frac{1}{(2\pi)^{\frac{3n}{2}}} {\Omega_p}^3 {\Omega_q}^3 M_1^2 S$$

Where M_1 , Ω_p , Ω_q are defined in (3.7) and $S = \frac{r^p}{p} \frac{s^q}{q}$. Thus for any fixed $t > o$, E_2 is bounded.

Or

$$\widehat{E_2} = \frac{1}{(1+\|\xi\|^2)^2}$$

$$E_2(x) = \frac{1}{(2\pi)^{n/2}} \int_\Omega \int_0^\infty e^{-t(1+\|\xi\|^2)^2} e^{i(\xi,x)}\, d\xi\, dt \quad ,$$

$$|E_2(x)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\Omega_p\Omega_q M_2$$

Where

$$M_2 = \int_0^R \int_0^L \int_0^\infty \exp[-t(1 + r^2 - s^2)^2]\, r^{p-1} s^{q-1} drdsdt \qquad (3.14)$$

And for the estimation of u_2 we have

$$u_2(x) = -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_\Omega f(y)E_2(x-y)\,dy$$

$$|u_2| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_\Omega |f(y)E_2(x-y)|\,dy$$

$$\leq \frac{1}{(2\pi)^{2n}}\Omega_p^3\Omega_q^3 M_1^2 N\, S$$

or

$$|u_2| \leq \frac{1}{(2\pi)^n}\Omega_p\Omega_q M_2 N \qquad (3.15)$$

Where M_1 , M_2 , Ω_p , Ω_q and N are defined in (3.7), (3.14) and (3.10)

3.3. Recurrence relations

Theorem 3.3.1

For all $k \geqq 1$ the unique solution of the equation $-(\square - I)^k u_k = f(x)$ is given via

$$\widehat{u_k} = \frac{(-1)^{k-1}\hat{f}}{(1+||\xi||^2)^k}$$

$$u_k = (-1)^{k-1}\frac{f * E_k}{(2\pi)^{n/2}} \quad , \qquad (3.16)$$

where

$$\widehat{E_k} = \frac{1}{(1+||\xi||^2)^k}$$ and $||\xi||^2$ is defined in (3.2)

Proof.

The formula

$$\widehat{u_k} = \frac{(-1)^{k-1}\hat{f}}{(1+||\xi||^2)^k} \, ,$$

was shown for k = 1 , 2. Let (3.16) be true for some k . In order to prove that

$$\widehat{u_{k+1}} = \frac{(-1)^k \hat{f}}{(1+||\xi||^2)^{k+1}} \tag{3.17}$$

holds for the solution to the equation

$$-(\Box - I)^{k+1} u_{k+1} = f \tag{3.18}$$

This equation is written as equivalent system

$$\begin{aligned} -(\Box - I) v_{k+1} &= f \\ (\Box - I)^k u_{k+1} &= v_{k+1} \end{aligned} \tag{3.19}$$

Using (3.16), we get

$$\widehat{u_{k+1}} = \frac{(-1)^{k-1}(-\widehat{v_{k+1}})}{(1+||\xi||^2)^k} == \frac{(-1)^k(\widehat{v_{k+1}})}{(1+||\xi||^2)^k} \tag{3.20}$$

But since

$\widehat{v_{k+1}} = \frac{\hat{f}}{(1+||\xi||^2)}$. Then, $\widehat{u_{k+1}} = \frac{(-1)^k \hat{f}}{(1+||\xi||^2)^{k+1}}$

So we get our result.

Also we note that

$u_k = (-1)^{k-1} \frac{f * E_k}{(2\pi)^{n/2}}$, since $\widehat{u_k} = \frac{(-1)^{k-1}\hat{f}}{\left(1+||\xi||^2\right)^k} = \hat{f}\,\widehat{E_k}$ where $\widehat{E_k} = \frac{1}{\left(1+||\xi||^2\right)^k}$,

and from properties of Fourier transform $(u * v)^{\wedge} = (2\pi)^{n/2} \hat{u}\, \hat{v}$.

Theorem 3.3.2

For any $k \geqq 1$ the equation $-(\Box - I)^k u_k = f$, is uniquely solvable by

(i) $u_k = -\frac{u_{k-1} * E_1}{(2\pi)^{\frac{n}{2}}}$ for $k \geqq 2$ and $u_k = \frac{u_{k-2} * E_2}{(2\pi)^{\frac{n}{2}}}$ for $k \geqq 3$

More over, if $2 \geqq k$

(ii) $f * E_k = (-1)^{r-1} u_r * E_{k-r}$ for $1 \leqq r \leqq k-1$

(iii) $E_k = \frac{E_{k-r} * E_r}{(2\pi)^{\frac{n}{2}}}$

Proof.

Since $\widehat{u_k} = \frac{(-1)^{k-1}\hat{f}}{(1+||\xi||^2)^k}$ Can be written in the form

$$\widehat{u_k} = \frac{(-1)^{k-2}\hat{f}}{(1+\|\xi\|^2)^{k-1}} \cdot \frac{(-1)}{(1+\|\xi\|^2)} ,$$

we get for all $k \geqq 2$

$$u_k = -\frac{u_{k-1} * E_1}{(2\pi)^{\frac{n}{2}}}$$

Also since

$$\widehat{u_k} = \frac{(-1)^{k-3}\hat{f}}{(1+\|\xi\|^2)^{k-2}} \cdot \frac{(-1)^2}{(1+\|\xi\|^2)^2}$$

We get for all $k \geqq 3$

$$u_k = \frac{u_{k-2} * E_2}{(2\pi)^{\frac{n}{2}}}$$

Thus (i) is proven. Similarly, since

$$\widehat{u_k} = \frac{(-1)^{k-1+r-r}\hat{f}}{(1+\|\xi\|^2)^{k-r+r}} = (-1)^{r-1}\frac{\hat{f}}{(1+\|\xi\|^2)^r} \cdot \frac{(-1)^{k-r}}{(1+\|\xi\|^2)^{k-r}}$$

So

$$u_k = (-1)^{k-r}\frac{u_r * E_{k-r}}{(2\pi)^{\frac{n}{2}}} \text{ , and } u_k = (-1)^{k-1}\frac{f * E_k}{(2\pi)^{\frac{n}{2}}}$$

Thus we get

$$f * E_k = (-1)^{r-1} u_r * E_{k-r}$$

where $1 \leqq r \leqq k-1$, and thus (ii) is proven.

From properties of Fourier transform $(u * v)^{\wedge} = (2\pi)^{n/2}\,\hat{u}\,\hat{v} \Longrightarrow \hat{u}\,\hat{v} = \frac{(u*v)^{\wedge}}{(2\pi)^{n/2}}$,

we obtain

$$\widehat{E_k} = \frac{1}{(1+\|\xi\|^2)^k} = \frac{1}{(1+\|\xi\|^2)^{k-r}} \cdot \frac{1}{(1+\|\xi\|^2)^r} = \widehat{E_{k-1}} \,.\, \widehat{E_r}$$

$$E_k = \frac{E_{k-r} * E_r}{(2\pi)^{n/2}} \quad \text{and then} \quad E_k = \frac{E_{k-1} * E_1}{(2\pi)^{n/2}}$$

Thus (iii) is proven.

3.4. Estimation of $E_k(x)$

It holds

$$|E_k| \leq \left(\frac{1}{(2\pi)^{\frac{n}{2}}}\right)^{2k-1} {\Omega_p}^{2k-1} {\Omega_q}^{2k-1} M_1^k \, S^{k-1} \tag{3.21}$$

or

$$|E_k| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \Omega_p \Omega_q M_k \quad , \ k \geq 1 \tag{3.22}$$

where Ω_p , Ω_q are defined in (3.7), $S = \frac{r^p}{p}\frac{s^q}{q}$, $\int_{\mathcal{R}^n} |f(x)| \, dx = N$ and

$$M_k = \int_0^\infty \int_0^R \int_0^L \exp[-t(1 + r^2 - s^2)^k] \, r^{p-1} s^{q-1} drdsdt \ , \quad k \geq 1 \tag{3.23}$$

We have,

$$|E_1| \leq \frac{1}{(2\pi)^{n/2}} \ \Omega_p \, \Omega_q M_1 \ , |E_2| \leq \frac{1}{(2\pi)^{\frac{3n}{2}}} {\Omega_p}^3 {\Omega_q}^3 M_1^2 \, S$$

Let (3.21) is true for some k. We prove now that

$$|E_{k+1}| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k+1} {\Omega_p}^{2k+1} {\Omega_q}^{2k+1} M_1^{k+1} \, S^k$$

Proof.

$$E_{k+1} = \frac{E_k * E_1}{(2\pi)^{n/2}} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_k(y) E_1(x-y) dy$$

Then

$$|E_{k+1}| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k+1} {\Omega_p}^{2k+1} {\Omega_q}^{2k+1} M_1^{k+1} \, S^k$$

Thus we get the result (3.21). Also from (3.16) we get

$$E_k(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega \int_0^\infty \exp[-t\left(1 + \left||\xi|\right|^2\right)^k + i(\xi, x)] d\xi dt$$

So

$$|E_k| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \Omega_p \Omega_q \int_0^\infty \int_0^R \int_0^L \exp[-t(1 + r^2 - s^2)^k] \, r^{p-1} s^{q-1} drdsdt$$

$$\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \Omega_p \Omega_q M_k$$

Thus for any fixed $t > o$, E_k is bounded

Now we consider the problem

$$-(\square - I)^k u_k = f$$

or equivalent

$$(\square - I)u_k = u_{k-1}\ , k \geq 1\ , u_0 = -f \tag{3.24}$$

Then

$$u_k = -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Omega} E_1\,(x-y)u_{k-1}(y)dy\ , u_0 = -f$$

$$|u_1| \leq [\frac{1}{(2\pi)^{\frac{n}{2}}}]^2\Omega_p\Omega_q M_1.N\ ,\ |u_2| \leq [\frac{1}{(2\pi)^{\frac{n}{2}}}]^4{\Omega_p}^3{\Omega_q}^3{M_1}^2 N\,S$$

3.5. The estimation of $u_k(x)$

We now prove that

$$|u_k| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k}{\Omega_p}^{2k-1}{\Omega_q}^{2k-1}{M_1}^k\,N\,S^{k-1}\ , k \geq 1 \tag{3.25}$$

or

$$|u_k| \leq \frac{1}{(2\pi)^n}\Omega_p\Omega_q M_k N$$

where Ω_p , Ω_q , M_k are defined in (3.7), (3.23), $S = \frac{r^p}{p}\frac{s^q}{q}$ and $\int_{\mathcal{R}^n}|f(x)|\,dx = N$

Proof.

Let (3.25) is true for some k, we prove for $k+1$

$$u_{k+1} = -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Omega} E_1\,(x-y)u_k(y)dy$$

$$|u_{k+1}| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}|E_1||u_k(y)|\Omega_p\Omega_q S$$

$$|u_{k+1}| \leq [\frac{1}{(2\pi)^{\frac{n}{2}}}]^{2k+1}{\Omega_p}^{2k+1}{\Omega_q}^{2k+1}{M_1}^{k+1}N\,S^k$$

Then

$$|u_k| \leq [\frac{1}{(2\pi)^{\frac{n}{2}}}]^{2k-1}{\Omega_p}^{2k-1}{\Omega_q}^{2k-1}{M_1}^k\,N\ S^{k-1}$$

is true for all $k \geq 1$. Or

$$-(\square - I)^k u_k = f$$

Which equivalent $(\square - I)u_k = u_{k-1}\ \ k \geq 1\ , u_0 = -f$;

$$u_k(x) = (-1)^{k-1}\frac{f * E_k}{(2\pi)^{n/2}}$$

$$u_k(x) = \frac{(-1)^{k-1}}{(2\pi)^{n/2}}\int_{\Omega} E_k\,(x-y)f(y)dy \tag{3.26}$$

$$\widehat{E_k} = \frac{1}{(1+||\xi||^2)^k}$$

$$E_k(x) = \frac{1}{(2\pi)^{n/2}}\int_{\Omega}\int_0^{\infty}\exp[-t\,(1+||\xi||^2)^k + i(\xi,x)]d\xi\,dt$$

$$|E_k(x)| \le \frac{1}{(2\pi)^{\frac{n}{2}}}\Omega_p\Omega_q M_k \tag{3.27}$$

From (3.26) and (3.27)

$$|u_k(x)| \le \frac{1}{(2\pi)^n}\Omega_p\Omega_q M_k.\int_{\mathcal{R}^n} f(y)dy = \frac{1}{(2\pi)^n}\Omega_p\Omega_q M_k.N \tag{3.28}$$

3.6. The relation between v_k and u_k under certain conditions on f_k

Consider now the ultra-hyperbolic equation (3.24) for the solution u_k and

$$\frac{\partial v_k}{\partial t} = \Box v_k \text{ in } \mathcal{R}^n X\,(0,\infty)\,,\quad v_k(x,0) = f_k(x) \tag{3.29}$$

Where

$f_k = -u_{k-1},\ k \ge 1$,$u_0 = -f(x)$; $v_k^{\#}(x,s)$ is the Laplace transform with respect to time t ,

i.e.

$$v_k^{\#}(x,s) = \int_0^{\infty} e^{-st}\,v_k(x,t)dt \quad (s>0)$$

Theorem 3.6.1

$$u_k(x) = v_k^{\#}(x,s)$$

Where u_k and v_k are the solutions of (3.24) and (3.29) respectively.

Proof.

We perform Laplace transform w.r.t. time t for (3.29), we get

$$\Box v_k^{\#}(x,s) = \int_0^{\infty} e^{-st}\,\Box v_k(x,t)dt = \int_0^{\infty} e^{-st}\,(v_k)_t(x,t)dt$$

$$= sv_k^{\#}(x,s) - v_k(x,0)$$

when

$s = 1$, $u_k(x) = v_k^{\#}(x,s)$, we get

$$-(\square - I)u_k = f_k(x)$$

3.6.1 Estimation of $\mathbf{v_k(x,t)}$, $\mathbf{k \geq 1}$

$$|v_k(x,t)| \leq [\frac{1}{(2\pi)^{\frac{n}{2}}}]^{2k-1}\Omega_p{}^{2k-1}\Omega_q{}^{2k-1}M_1{}^{k-1}\,M(t)N\;S^{k-1}$$

Or

$$|v_k(x,t) \leq| \frac{1}{(2\pi)^{\frac{3n}{2}}}\Omega_p{}^3\Omega_q{}^3 M_{k-1}M(t)NS \quad ,$$

where Ω_p , Ω_q , M_k, N are defined in (3.7), (3.23) and

$$M(t) = \int_0^R\int_0^L e^{t(s^2-r^2)}\, r^{p-1}s^{q-1}drds \;, t > 0 \;, k \geq 1$$

Proof.

Since

$$u_k = (-1)^{k-1}\frac{u_{k-1} * E_1}{(2\pi)^{\frac{n}{2}}} \,, u_0 = -f$$

then using (3.3),(3.5)and (3.8) we get

$$u_k = \frac{(-1)^{k-1}}{(2\pi)^{\frac{n}{2}}}\int_\Omega \int_\Omega \int_0^\infty e^{-t\left(1+\|\xi\|^2\right)+i(\xi,x-y)}u_{k-1}(y)dyd\xi dt$$

where Ω is the spectrum of $u_k(x)$. Since $u_k = v_k^{\#}$,then

$$v_k(x,t) = \frac{(-1)^{k-1}}{(2\pi)^{\frac{n}{2}}}\int_\Omega \int_\Omega e^{-t\|\xi\|^2+i(\xi,x-y)}\,u_{k-1}(y)dyd\xi$$

Then by changing to bipolar coordinates

$$|v_k(x,t)| \leq \frac{1}{(2\pi)^{n/2}}\Omega_p\Omega_q \int_0^R\int_0^L e^{t(s^2-r^2)}\, r^{p-1}s^{q-1}drds$$

$$\times|u_{k-1}(y)|\Omega_p\Omega_q \int_0^R\int_0^L r^{p-1}s^{q-1}drds$$

$$\leq \frac{1}{(2\pi)^{n/2}}\Omega_p\Omega_q M(t).\,|u_{k-1}(y)|\Omega_p\Omega_q S$$

$$\leq \frac{1}{(2\pi)^{n/2}} \Omega_p{}^2 \Omega_q{}^2 M(t) S |u_{k-1}(y)|$$

So using (3.25) we obtain the result.

4 Conclusion

We find the solutions of the equation (1.1). We define the ultra –hyperbolic kernels E_k of higher order, then we get recurrence relations between u_k and E_k, we obtain also an estimation of u_k and E_k related to the spectrum, then we show that u_k and E_k are bounded. A relation between u_k and v_k of (3.29) under certain conditions on f_k is obtained.

References

[1] A.S. Abdel Rady and H. Beghr, higher order Bessel potentials, *Arch. Math.*, **82**, (2004), 361-370.

[2] B. Osgood, *Fourier Transform and its Applications*, Stanford University.

[3] K. Nolaopon and A. Kananthai, On the Ultra-hyperbolic Heat Kernel Related to the Spectrum, *International Journal of Pure and Applied Mathematics*, **17**(1), (2004), 19-28.

[4] K. Nolaopon and A. Kananthai, On the Generalized Ultra -hyperbolic Heat Kernel Related to the Spectrum, *Science Asia*, **32**, (2006), 21-24.

[5] K. Nolaopon and A. Kananthai, On the Generalized Nonlinear Ultra -hyperbolic Heat Kernel Related to the Spectrum, *Computational and Applied Mathematics*, **28**(2), (2009), 157-166.

[6] L.C. Evans, Partial Differential Equations, *Mathematical Society*,**19**, Providence, RI 1998.

[7] Elias M.Stien and Rami Shakarchi, lectures in Analysis (I), Fourier Analysis, an introduction Title of the paper, Princeton University Press , Princeton 2002.