On higher order ultra–hyperbolic kernel related to the spectrum

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Abstract

In this paper, the solutions of the equation $-(-1)^k u_k = f(x)$, $k \geq 1$ in $\mathcal{R}^n$ where $f \in L^2(\mathcal{R}^n)$ and $\frac{\partial v_k}{\partial t} = \Box v_k$ in $\mathcal{R}^n \times (0, \infty)$, $v_k(x, 0) = f_k(x)$, $k \geq 1$ are considered, the operator $\Box$ is named the ultra–hyperbolic operator defined by $\Box = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)$, $p + q = n$ is the dimension of Euclidean space $\mathcal{R}^n$. We define the ultra–hyperbolic kernels $E_k$ of higher order, then we get recurrence relations between $u_k$ and $E_k$, we obtain also an estimation of $u_k$ and $E_k$ related to the spectrum, then we show that $u_k$ and $E_k$ are bounded. A relation between $v_k$ and $u_k$ under certain conditions on $f_k$ is obtained.

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1 Introduction

The solutions of the equation
\[-(\Delta - 1)^k u_k(x) = f(x), \ k \geq 1 \ \text{in } \mathbb{R}^n, \ f \in L^2(\mathbb{R}^n)\] (1.1)
where \(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\) were investigated in [1], Bessel potential of higher order were defined and recurrence relations between these solutions and these Bessel potentials are obtained.

Now, the purpose of this work is to study the solutions of

\[-(\Box - 1)^k u_k = f(x), k \geq 1 \ \text{in } \mathbb{R}^n, f \in L^2(\mathbb{R}^n)\] (1.2)

and we obtain an estimation of the solutions \(u_k\) such equation which is related to the spectrum and also of the ultra-hyperbolic kernels \(E_k\). There are a lot of problems use the ultra-hyperbolic operator, see [3], [4] and [5]. Then under certain conditions on \(f_k\) we obtain a relation between these solutions \(u_k\) and the solutions \(v_k\) of

\[\frac{\partial v_k}{\partial t} = \Box v_k \ \text{in } \mathbb{R}^n \times (0, \infty)\] (1.3)

\[v_k(x, 0) = f_k(x), k \geq 1\]

2 Preliminary Notes

Definition 2.1 Four transform take the form

\[\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} f(x) dx\] (2.1)

and its inverse

\[f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} \hat{f}(\xi) d\xi\] (2.2)

See [2], [6] and [7].

Definition 2.2 let \(\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n\) denote by \(\Gamma_+ = \{\xi \in \mathbb{R}^n: \xi_1^2 + \cdots +\)
\( \xi^2_p - \xi^2_{p+1} - \cdots - \xi^2_{p+q} > 0 \) and \( \xi_1 > 0 \) \{ the set of an interior of the forward cone, and \( \bar{\Gamma}, \) denotes the closure of \( \Gamma \).

**Definition 2.3** Let \( \Omega \subset \bar{\Gamma} \) is the spectrum of \( E_k \) and

\[
E_k(x) = \frac{1}{(2\pi)^{n/2}} \int_\Omega \int_0^\infty \exp\left[ -t(1 + \|\xi\|^2)^k + i(\xi, x) \right] d\xi \, dt
\]

where \( E_k \) is called ultra hyperbolic kernel (of order \( k \)).

**Definition 2.4** Bipolar coordinates

Let \( \xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \),

and

\( \xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q} \),

where

\( \sum_{i=1}^{p} \omega_i^2 = 1 \) and \( \sum_{j=p+1}^{p+q} \omega_j^2 = 1 \) where \( d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q \), and \( d\Omega_p, d\Omega_q \) are the elements of surface area of the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively, and we suppose \( 0 \leq r \leq R \) and \( 0 \leq s \leq L \) where \( R \) and \( L \) are constants.

### 3 Main Results

#### 3.1 The solution \( u_1 \)

\[
-(\Box - l)u_1(x) = f(x) \text{ in } \mathbb{R}^n
\]

The Fourier transform of (3.1) is

\[
\hat{u}_1 = \frac{\hat{f}}{1 + \|\xi\|^2}
\]

where

\[
|\|\xi\||^2 = \sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 > 0
\]

and its inverse Fourier transform is

\[
u_1 = \frac{r^p E_1}{(2\pi)^{n/2}}
\]

where
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\[ \overline{E_1} = \frac{1}{1 + ||\xi||^2} \] (3.4)

and

\[ E_1(x) = \frac{1}{(2\pi)^n/2} \int_0^\infty \int_0^\infty \exp[-t(1 + ||\xi||^2) + i(\xi, x)] d\xi dt \] (3.5)

where \( E_1 \) is called ultra hyperbolic kernel (of order one), \( \Omega \subset \Gamma_+ \) is the spectrum of \( E_1 \), and the estimation of \( E_1 \) is

\[ |E_1(x)| \leq \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q \int_0^R \int_0^L \int_0^\infty \exp[-t + t(s^2 - r^2)] r^{p-1}s^{q-1} \, dr \, ds \, dt \]

\[ = \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q M_1 \] (3.6)

where

\[ M_1 = \int_0^\infty \int_0^R \int_0^L \exp[-t + t(s^2 - r^2)] r^{p-1}s^{q-1} \, dr \, ds \, dt, \quad \Omega_p = \frac{2\pi^p}{\Gamma(\frac{p}{2})} \text{ and } \Omega_q = \frac{2\pi^q}{\Gamma(\frac{q}{2})} \] (3.7)

Thus for any fixed \( t > 0 \), \( E_1 \) is bounded.

Also,

\[ u_1(x) = \frac{1}{(2\pi)^{n/2}} \int_\Omega E_1(x - y) f(y) dy \] (3.8)

\[ |u_1(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_\Omega |E_1 f(y)| dy \leq \frac{1}{(2\pi)^{n/2}} |E_1| N \]

\[ \leq \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q M_1 N \leq \frac{1}{(2\pi)^n} \Omega_p \Omega_q M_1 N \] (3.9)

Where \( M_1, \Omega_p, \Omega_q \) are defined in (3.7) and

\[ \int_\Omega |f(y)| dy = N \] (3.10)

i.e. \( u_1 \) is bounded.

### 3.2 The solution \( u_2 \)

At first we consider now the PDE
Which is equivalent to the system
\[
-(\Box - 1) u_2 = f \quad \text{in} \quad \mathcal{R}^n
\] (3.11)

Applying Fourier transform, we get
\[
-(1 + |\xi|^2) \hat{u}_2 = \hat{u}_1
\]
\[
\hat{u}_2 = -\frac{i f}{(1 + |\xi|^2)^2}, \quad u_2(x) = -\frac{f * E_2}{(2\pi)^2}, \quad E_2 = \frac{E_1 * E_1}{(2\pi)^2},
\]
\[
E_2(x) = \frac{E_1 * E_1}{(2\pi)^2} = \frac{1}{(2\pi)^2} \int_{\Omega} E_1(y)E_1(x - y)dy
\] (3.13)

Where \( E_1 \) and \( u_1 \) are given in (3.5) and (3.8) respectively. So we have for estimation of \( E_2(x) \)
\[
E_2(x) = \frac{E_1 * E_1}{(2\pi)^2} \leq \frac{1}{(2\pi)^2} \int_{\Omega} |E_1(y)| |E_1(x - y)| dy
\]
\[
\leq \frac{1}{(2\pi)^2} \Omega_p^3 \Omega_q^3 M_1^2 \int_0^R \int_0^L r^{p-1}s^{q-1} dr ds = \frac{1}{(2\pi)^2} \Omega_p^3 \Omega_q^3 M_1^2 S
\]

Where \( M_1, \Omega_p, \Omega_q \) are defined in (3.7) and \( S = \frac{r^p s^q}{p^q} \). Thus for any fixed \( t > 0, E_2 \) is bounded.

Or
\[
E_2(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \int_0^\infty e^{-t(1 + |\xi|^2)^2} e^{i(x \cdot \xi)} d\xi dt
\]
\[ |E_2(x)| \leq \frac{1}{(2\pi)^2} \Omega_p \Omega_q M_2 \]

Where
\[
M_2 = \int_0^R \int_0^L \int_0^\infty \exp[-t(1 + r^2 - s^2)^2] r^{p-1}s^{q-1} \, dr \, ds \, dt
\tag{3.14}
\]

And for the estimation of \( u_2 \) we have
\[
u_2(x) = -\frac{1}{(2\pi)^2} \int_\Omega f(y) E_2(x - y) \, dy
\]
\[|u_2| \leq \frac{1}{(2\pi)^2} \int_\Omega |f(y) E_2(x - y)| \, dy \]
\[\leq \frac{1}{(2\pi)^{2n}} \Omega_p^3 \Omega_q^3 M_1^2 N S \]

or
\[|u_2| \leq \frac{1}{(2\pi)^n} \Omega_p \Omega_q M_2 N \tag{3.15}\]

Where \( M_1, M_2, \Omega_p, \Omega_q \) and \( N \) are defined in (3.7), (3.14) and (3.10)

### 3.3. Recurrence relations

**Theorem 3.3.1**

For all \( k \geq 1 \) the unique solution of the equation \( -(\Box - 1)^k u_k = f(x) \)

is given via
\[
\tilde{u}_k = \frac{(-1)^{k-1} \xi^k}{(1 + ||\xi||^2)^k}
\]
\[u_k = (-1)^{k-1} \frac{\xi^k E_k}{(2\pi)^{n/2}}, \tag{3.16}\]

where
\[
\tilde{E}_k = \frac{1}{(1 + ||\xi||^2)^k} \quad \text{and} \quad ||\xi||^2 \quad \text{is defined in (3.2)}
\]

**Proof.**

The formula
\[
\tilde{u}_k = \frac{(-1)^{k-1} \xi^k}{(1 + ||\xi||^2)^k},
\]

was shown for \( k = 1, 2 \). Let (3.16) be true for some \( k \). In order to prove that
\[
\widehat{u}_{k+1} = \frac{(-1)^k \hat{f}}{(1 + ||\xi||^2)^{k+1}}
\]

holds for the solution to the equation

\[-(\Box - 1)^{k+1} u_{k+1} = f
\]

This equation is written as equivalent system

\[-(\Box - 1)v_{k+1} = f
\]

\[(\Box - 1)^{k} u_{k+1} = v_{k+1}
\]

Using (3.16), we get

\[
\widehat{u}_{k+1} = \frac{(-1)^{k-1}(-\widehat{v}_{k+1})}{(1 + ||\xi||^2)^k} = \frac{(-1)^k (\widehat{v}_{k+1})}{(1 + ||\xi||^2)^k}
\]

But since

\[
\widehat{v}_{k+1} = \frac{\hat{f}}{(1 + ||\xi||^2)^{k+1}}
\]

Then, \[\widehat{u}_{k+1} = \frac{(-1)^k \hat{f}}{(1 + ||\xi||^2)^{k+1}}\]

So we get our result.

Also we note that

\[u_k = (-1)^{k-1} \frac{f * E_k}{(2\pi)^{n/2}}, \text{ since } \widehat{u}_k = \frac{(-1)^{k-1} \hat{f}}{(1 + ||\xi||^2)^k} = \hat{f} \widehat{E}_k \quad \text{where } \widehat{E}_k = \frac{1}{(1 + ||\xi||^2)^k},
\]

and from properties of Fourier transform \((u * v)^* = (2\pi)^{n/2} \hat{u} \hat{v}\).

**Theorem 3.3.2**

For any \(k \geq 1\) the equation \(-(\Box - 1)^{k} u_k = f\) is uniquely solvable by

\[(i) u_k = -\frac{u_{k-1} + E_k}{(2\pi)^{n/2}} \text{ for } k \geq 2 \text{ and } u_k = \frac{u_{k-2} + E_k}{(2\pi)^{n/2}} \text{ for } k \geq 3
\]

More over, if \(2 \geq k\)

\[(ii) f * E_k = (-1)^{r-1} u_r * E_{k-r} \quad \text{for } 1 \leq r \leq k - 1
\]

\[(iii) E_k = \frac{E_{k-r} * E_r}{(2\pi)^{n/2}}
\]

**Proof.**

Since \[\widehat{u}_k = \frac{(-1)^{k-1} \hat{f}}{(1 + ||\xi||^2)^k}\] Can be written in the form
\[
\hat{u}_k = \frac{(-1)^{k-2} \hat{f}}{(1 + \|\xi\|^2)^{k-1}} \cdot \frac{(-1)}{(1 + \|\xi\|^2)},
\]
we get for all \( k \geq 2 \)

\[
u_k = -\frac{u_{k-1} \cdot E_1}{(2\pi)^n}.
\]

Also since

\[
\hat{u}_k = \frac{(-1)^{k-3} \hat{f}}{(1 + \|\xi\|^2)^{k-2}} \cdot \frac{(-1)^2}{(1 + \|\xi\|^2)^2},
\]
We get for all \( k \geq 3 \)

\[
u_k = \frac{u_{k-2} \cdot E_2}{(2\pi)^n}.
\]

Thus (i) is proven. Similarly, since

\[
\hat{u}_k = \frac{(-1)^{k-1+r-r} \hat{f}}{(1 + \|\xi\|^2)^{k-r+r}} = (-1)^{r-1} \frac{\hat{f}}{(1 + \|\xi\|^2)^r} \cdot \frac{(-1)^{k-r}}{(1 + \|\xi\|^2)^{k-r}}
\]

So

\[
u_k = (-1)^{k-r} u_r \cdot E_{k-r} \frac{E_k}{(2\pi)^n}, \quad \text{and} \quad \nu_k = (-1)^{k-1} \frac{f \cdot E_k}{(2\pi)^n}
\]

Thus we get

\[f \ast E_k = (-1)^{r-1} u_r \ast E_{k-r}\]

where \( 1 \leq r \leq k - 1 \), and thus (ii) is proven.

From properties of Fourier transform \((u \ast v) = (2\pi)^{n/2} \hat{u} \hat{v} \Rightarrow \hat{u} \hat{v} = \frac{(u \ast v)^\wedge}{(2\pi)^{n/2}}\),

we obtain

\[
E_k = \frac{1}{(1 + \|\xi\|^2)^{k}} \ast \frac{1}{(1 + \|\xi\|^2)^{k-r}} \cdot \frac{1}{(1 + \|\xi\|^2)^r} = E_{k-1} \cdot E_r
\]

\[
E_k = \frac{E_{k-r} \ast E_r}{(2\pi)^{n/2}}, \quad \text{and then} \quad E_k = \frac{E_{k-1} \ast E_1}{(2\pi)^{n/2}}
\]

Thus (iii) is proven.

**3.4. Estimation of \( E_k(x) \)**

It holds
\begin{equation}
|E_k| \leq \left(\frac{1}{(2\pi)^{2}}\right)^{2k-1} \Omega_p^{2k-1} \Omega_q^{2k-1} M_1^k S^{k-1} \tag{3.21}
\end{equation}

or

\begin{equation}
|E_k| \leq \left(\frac{1}{(2\pi)^{2}}\right)^{k} \Omega_p \Omega_q M_k , \quad k \geq 1 \tag{3.22}
\end{equation}

where \(\Omega_p, \Omega_q\) are defined in (3.7), \(S = \frac{r^p s^q}{p q}, \int_{\mathbb{R}^n} |f(x)| \, dx = N\) and

\begin{equation}
M_k = \int_0^\infty \int_0^R \int_0^L \exp[-t(1 + r^2 - s^2)^k] r^{p-1}s^{q-1} \, dr \, ds \, dt , \quad k \geq 1 \tag{3.23}
\end{equation}

We have,

\begin{equation*}
|E_1| \leq \left(\frac{1}{(2\pi)^{2}}\right)^{k} \Omega_p \Omega_q M_1 , \quad |E_2| \leq \left(\frac{1}{(2\pi)^{2}}\right)^{3} \Omega_p^3 \Omega_q^3 M_1^2 S
\end{equation*}

Let (3.21) is true for some \(k\). We prove now that

\begin{equation*}
|E_{k+1}| \leq \left[\left(\frac{1}{n}\right)^{2k+1} \Omega_p^{2k+1} \Omega_q^{2k+1} M_1^{k+1} S^k \right]
\end{equation*}

\textbf{Proof.}

\begin{equation*}
E_{k+1} = \frac{E_k \ast E_1}{(2\pi)^{n/2}} = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} E_k(y) E_1(x-y) \, dy
\end{equation*}

Then

\begin{equation*}
|E_{k+1}| \leq \left[\left(\frac{1}{n}\right)^{2k+1} \Omega_p^{2k+1} \Omega_q^{2k+1} M_1^{k+1} S^k \right]
\end{equation*}

Thus we get the result (3.21). Also from (3.16) we get

\begin{equation*}
E_k(x) = \frac{1}{(2\pi)^{n}} \int_{\Omega} \int_0^\infty \int_0^\infty \exp[-t\left(1 + ||\xi||^2\right)^k] + i(\xi, x)] \, d\xi \, dt
\end{equation*}

So

\begin{equation*}
|E_k| \leq \left(\frac{1}{(2\pi)^{2}}\right)^{k} \Omega_p \Omega_q \int_0^\infty \int_0^R \int_0^L \exp[-t(1 + r^2 - s^2)^k] r^{p-1}s^{q-1} \, dr \, ds \, dt
\end{equation*}

\begin{equation*}
\leq \left(\frac{1}{(2\pi)^{2}}\right)^{k} \Omega_p \Omega_q M_k
\end{equation*}

Thus for any fixed \(t > 0\), \(E_k\) is bounded

Now we consider the problem

\begin{equation*}
-(\Box - I)^k u_k = f
\end{equation*}

or equivalent
On higher order ultra-hyperbolic kernel related to the spectrum

\((\Box - I)u_k = u_{k-1}, k \geq 1, u_0 = -f \) \hspace{1cm} (3.24)

Then

\[ u_k = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} E_1(x-y)u_{k-1}(y)dy, u_0 = -f \]

\[ |u_1| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\Omega_p\Omega_q M_1 N \right], |u_2| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\Omega_p^2\Omega_q^2 M_1^2 N \right] S \]

3.5. The estimation of \( u_k(x) \)

We now prove that

\[ |u_k| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k} \Omega_p^{2k-1}\Omega_q^{2k-1} M_1^k N S^{k-1}, k \geq 1 \] \hspace{1cm} (3.25)

or

\[ |u_k| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \Omega_p\Omega_q M_k N \]

where \( \Omega_p, \Omega_q, M_k \) are defined in (3.7), (3.23), \( S = \frac{r^p s^q}{p^q} \) and \( \int_{\mathbb{R}^n} |f(x)| dx = N \)

**Proof.**

Let (3.25) is true for some \( k \), we prove for \( k + 1 \)

\[ u_{k+1} = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} E_1(x-y)u_k(y)dy \]

\[ |u_{k+1}| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} |E_1||u_k(y)|\Omega_p\Omega_q S \]

\[ |u_{k+1}| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k+1} \Omega_p^{2k+1}\Omega_q^{2k+1} M_1^{k+1} N S^k \]

Then

\[ |u_k| \leq \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\right]^{2k-1} \Omega_p^{2k-1}\Omega_q^{2k-1} M_1^k N S^{k-1} \]

is true for all \( k \geq 1 \). Or

\[-(\Box - I)^k u_k = f \]

Which equivalent \( (\Box - I)u_k = u_{k-1}, k \geq 1, u_0 = -f \) ;
\[ u_k(x) = (-1)^{k-1} \frac{f \ast E_k}{(2\pi)^{n/2}} \]

\[ u_k(x) = \frac{(-1)^{k-1}}{(2\pi)^{n/2}} \int_{\Omega} E_k(x - y)f(y)dy \]  (3.26)

\[ \widetilde{E}_k = \frac{1}{(1 + ||\xi||^2)^k} \]

\[ E_k(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \int_0^\infty \exp[-t(1 + ||\xi||^2)^k + i(\xi, x)]d\xi dt \]

\[ |E_k(x)| \leq \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q M_k \]  (3.27)

From (3.26) and (3.27)

\[ |u_k(x)| \leq \frac{1}{(2\pi)^n} \Omega_p \Omega_q M_k \int_{\mathbb{R}^n} f(y)dy = \frac{1}{(2\pi)^n} \Omega_p \Omega_q M_k \cdot N \]  (3.28)

### 3.6. The relation between \( v_k \) and \( u_k \) under certain conditions on \( f_k \)

Consider now the ultra-hyperbolic equation (3.24) for the solution \( u_k \) and

\[ \frac{\partial v_k}{\partial t} = \Box v_k \text{ in } \mathbb{R}^n \times (0, \infty), \quad v_k(x, 0) = f_k(x) \]  (3.29)

Where

\[ f_k = -u_{k-1}, \quad k \geq 1, u_0 = -f(x); \quad v_k^\#(x, s) \text{ is the Laplace transform with respect to time } t, \]

i.e.

\[ v_k^\#(x, s) = \int_0^\infty e^{-st} v_k(x, t)dt \quad (s > 0) \]

**Theorem 3.6.1**

\[ u_k(x) = v_k^\#(x, s) \]

Where \( u_k \) and \( v_k \) are the solutions of (3.24) and (3.29) respectively.

**Proof.**

We perform Laplace transform w.r.t. time \( t \) for (3.29), we get

\[ \Box v_k^\#(x, s) = \int_0^\infty e^{-st} \Box v_k(x, t)dt = \int_0^\infty e^{-st}(v_k)_t(x, t)dt \]
\[ = s v_k^\#(x, s) - v_k(x, 0) \]

When
\[ s = 1 , u_k(x) = v_k^\#(x, s) , \]
we get
\[ -(\square - I)u_k = f_k(x) \]

### 3.6.1 Estimation of \( v_k(x, t) , k \geq 1 \)

\[ |v_k(x, t)| \leq \frac{1}{(2 \pi)^{n/2}} 2^{k-1} \Omega_p^{2k-1} \Omega_q^{2k-1} M_1^{k-1} M(t)N S^{k-1} \]

Or
\[ |v_k(x, t)| \leq \frac{1}{(2 \pi)^{n/2}} 3 \Omega_p^{3} \Omega_q^{3} M_{k-1}M(t)NS , \]

where \( \Omega_p , \Omega_q , M_k , N \) are defined in (3.7), (3.23) and
\[ M(t) = \int_0^R \int_0^L e^{(s^2-r^2)} r^{p-1}s^{q-1}drds , t > 0 , k \geq 1 \]

**Proof.**

Since
\[ u_k = (-1)^{k-1} \frac{u_{k-1} * E_1}{n} , u_0 = -f \]

then using (3.3),(3.5)and (3.8) we get
\[ u_k = \frac{(-1)^{k-1}}{n (2 \pi)^{n/2}} \int_\Omega \int_\Omega \int_0^\infty e^{-t(1+||\xi||^2) + i(\xi,x-y)} u_{k-1}(y)dyd\xi dt \]

where \( \Omega \) is the spectrum of \( u_k(x) \). Since \( u_k = v_k^\# , \)
then
\[ v_k(x, t) = \frac{(-1)^{k-1}}{n (2 \pi)^{n/2}} \int_\Omega \int_\Omega e^{-t||\xi||^2 + i(\xi,x-y)} u_{k-1}(y)dyd\xi \]

Then by changing to bipolar coordinates
\[ |v_k(x, t)| \leq \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q \int_0^R \int_0^L e^{(s^2-r^2)} r^{p-1}s^{q-1}drds \]
\[ \times |u_{k-1}(y)| \Omega_p \Omega_q \int_0^R \int_0^L r^{p-1}s^{q-1}drds \]
\[ \leq \frac{1}{(2\pi)^{n/2}} \Omega_p \Omega_q M(t) |u_{k-1}(y)| \Omega_p \Omega_q S \]
\[
\leq \frac{1}{(2m)^{n/2}} \Omega_{p}^{2} \Omega_{q}^{2} M(t) S|u_{k-1}(y)|
\]
So using (3.25) we obtain the result.

4 Conclusion

We find the solutions of the equation (1.1). We define the ultra –hyperbolic kernels \( E_k \) of higher order, then we get recurrence relations between \( u_k \) and \( E_k \), we obtain also an estimation of \( u_k \) and \( E_k \) related to the spectrum, then we show that \( u_k \) and \( E_k \) are bounded. A relation between \( u_k \) and \( v_k \) of (3.29) under certain conditions on \( f_k \) is obtained.

References


