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Existence Theorem for Abstract Measure Differential Equations Involving the Distributional Henstock-Kurzweil Integral

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Abstract

In this paper, we study the existence of solutions of the abstract measure differential equations by using the Leray-Schauder's nonlinear alternative and distributional Henstock-Kurzweil integral. The distributional Henstock-Kurzweil integral is very general and it includes the Lebesgue integral and Henstock-Kurzweil integral. The main result of the paper extends some previously known results in the literature.

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1 Introduction

In this paper, we consider the abstract measure differential equation (AMDE)

$$\frac{d\lambda}{d\mu} = f(x, \lambda(\bar{S}_x)), \tag{1.1}$$

where $(X,\, \mathcal{M},\, \mu)$ is a measure space, $\bar{S}_x$ is a certain measureable set for each $x \in X$, and $\frac{d\lambda}{d\mu}$ stands for the distributional derivative of $\lambda \in Ca(X,\, \mathcal{M})$, f is a distribution (generalized function).

It is well known that distributional derivative includes ordinary derivative and approximate derivative. Hence, (1.1) has a universality.

The general AMDE of the form

$$\frac{d\lambda}{d\mu} = f(x, \lambda(\bar{S}_x)) \tag{1.2}$$

with Radon-Nikodym derivative $\frac{d\lambda}{d\mu}$ and $f : S_x \times (-a, a)$ (a is a positive real number) has been studied extensively in [1,2,4,5,6]. R.R. Sharma in [1] proved the existence and uniqueness of solution of the equation (1.2) by the principle of contraction mapping. In [6], the authors considered the existence and uniqueness of solution of the equation (1.2) applying the Leray-Schauder alternative [7] under Carath*á*odory conditions. However, as far as we know, few papers have applied distributional derivatives to study the AMDE. In this paper, by using distributional derivatives, we study the existence of solutions of AMDE (1.1).

We organize this paper as follows. In section 2, we introduce a general integral called distributional Henstock-Kurzweil integral or D_{HK}-integral. In section 3, we will give the statement of the problem and the Leray-Schauder's nonlinear alternative that needed later. In section 4, we apply the Leray-Schauder's nonlinear alternative to prove the existence of solutions of the AMDE (1.1).

2 The Distributional Henstock-Kurzweil Integral

In this section, we present the definition and some basic properties of the distributional Henstock-Kurzweil integral.

Define the space

$$C_c^\infty = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty \text{ and } \phi \text{ has compact support in } \mathbb{R}\},$$

where the *support* of a function ϕ is the closure of the set on which ϕ does not vanish, denoted by $supp(\phi)$. A sequence $\{\phi_n\} \subset C_c^\infty$ converges to $\phi \in C_c^\infty$ if there is a compact set K such that all ϕ_n have supports in K and the sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$ uniformly on K for every $m \in \mathbb{N}$. Denote C_c^∞ endowed with this convergence property by $\mathcal{D}$. Also, ϕ is called $test\ function$ if $\phi \in \mathcal{D}$. The dual space to $\mathcal{D}$ is denoted by $\mathcal{D}'$ and its elements are called distributions. That is, if $f \in \mathcal{D}'$ then $f : \mathcal{D} \to \mathbb{R}$, and we write $\langle f, \phi \rangle \in \mathbb{R}$, for $\phi \in \mathcal{D}$.

For all $f \in \mathcal{D}'$, we define the distributional derivative Df of f to be a distribution satisfying $\langle Df, \phi \rangle = -\langle f, \phi' \rangle$, where ϕ is a test function and ϕ' is the ordinary derivative of ϕ. With this definition, all distributions have derivatives of all orders and each derivative is a distribution.

Let (a, b) be an open interval in $\mathbb{R}$. We define

$$\mathcal{D}((a,b)) = \{\phi : (a,b) \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has compact support in } (a,b)\}.$$

The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

Define $B_C = \{F \in C([a,b]) \mid F(a) = 0\}$ is a Banach space with the uniform norm

$$\|F\|_\infty = \max_{[a,b]} |F|.$$

Now we are able to introduce the definition of the D_{HK}-integral.

A distribution f is distributionally Henstock-Kurzweil integrable or briefly D_{HK}-integrable on $[a, b]$ if f is the distributional derivative of a continuous function $F \in B_C$.

The D_{HK}-integral of f on $[a, b]$ is denoted by $(D_{HK}) \int_a^b f = F(b)$, where F is called the primitive of f and "$(D_{HK}) \int$" denotes the D_{HK}-integral. Notice that if $f \in D_{HK}$ then f has many primitives in $C([a, b])$, but f has exactly one primitive in B_C.

The space of D_{HK}-integrable distributions is defined by

$$D_{HK} = \{f \in \mathcal{D}'((a,b)) \mid f = DF \text{ for some } F \in B_C\}.$$

With this definition, if $f \in D_{HK}$ then, for all $\phi \in \mathcal{D}((a, b))$,

$$\langle f, \phi \rangle = \langle DF, \phi \rangle = -\langle F, \phi' \rangle = -\int_a^b F\phi'.$$

Remark 2.2. Integrals defined in the same way have also been proposed in other papers. For example, Ang [3] defined an integral in the plane and called it the G-integral, and Talvila [10] defined the A_C-integral on the extended real line. In fact, these two integrals are equivalent to the D_{HK}-integral for one-dimensional intervals.

The following result is known as the fundamental theorem of calculus.

Lemma 2.3. [10,Theorem 4].

(a) Let $f \in D_{HK}$, and define $F(t) = (D_{HK}) \int_a^t f$. Then $F \in B_C$ and $DF = f$.

(b) Let $F \in C([a,b])$. Then $(D_{HK}) \int_a^t DF = F(t) - F(a)$ for all $t \in [a,b]$.

Example 2.4. We know that the primitive function F of the HK-integral function f is ACG^* (generalized absolutely continuous; see [11,12]). In [12, Example 6.6], Lee pointed out that if F is a continuous function and pointwise differentiable nearly everywhere on [a,b], then F is ACG^*. Furthermore, if F is a continuous function which is differentiable nowhere on [a,b], then F is not ACG^*. Therefore, if $F \in C([a,b])$ but is differentiable nowhere on [a,b], then DF exists and is D_{HK}-integrable but not HK-integrable. Conversely, if F is ACG^* then it also belongs to $C([a,b])$. Therefore, F' is not only HK-integrable but also D_{HK}-integrable. Here F' denotes the ordinary derivative of F.

This example shows that the D_{HK}-integral includes the HK-integral, and hence the Lebesgue and Riemann integrals.

Some other results about the distributional derivative and the D_{HK}-integral are given below.

For $f \in D_{HK}$ and $F \in B_C$ with $DF = f$, we define the *Alexiewicz* norm by

$$\|f\| = \|F\|_\infty = \max_{[a,b]} |F|.$$

The following result has been proved. □

Lemma 2.5. [10, Theorem 2] With the *Alexiewicz* norm, D_{HK} is a Banach space.

It is known that there is a pointwise ordering in $C([a,b])$ (so it is with B_C), that is, $u \leq v$ in $C([a,b])$ if and only if $u(t) \leq v(t)$ for every $t \in [a,b]$.

We now impose a partial ordering on D_{HK}: for $f, g \in D_{HK}$, we say that $f \succeq g$(*or* $g \preceq f$) if and only if $f - g$ is a positive measure on [a,b]. By this definition, if $f, g \in D_{HK}$ then

$$(D_{HK})\int_a^b f \geq (D_{HK})\int_a^b g, \ where \ \ f \succeq g. \tag{2.2}$$

According to the definition of this ordering, we also have the result.

Lemma 2.6. [3, Corollary 1]. If $f_1, f_2, f_3 \in \mathcal{D}'((a,b)), f_1 \preceq f_2 \preceq f_3$, and if f_1 *and* f_3 are D_{HK}-integrable, then f_2 is also D_{HK}-integrable.

We say that a sequence $\{f_n\} \subset D_{HK}$ converges strongly to $f \in D_{HK}$ if $\|f_n - f\| \to 0 \ as \ n \to \infty$. The following two convergence theorems hold.

Lemma 2.7. [3, Corollary4, Monotone convergence theorem for the D_{HK}-integral]. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in D_{HK} such that $f_0 \preceq f_1 \preceq \cdots \preceq f_n \preceq \cdots$, and that $(D_{HK})\int_a^b f_n \to A$ as $n \to \infty$. Then $f_n \to f$ in D_{HK} and $(D_{HK})\int_a^b f = A$.

Lemma 2.8. [1, Corollary 5, Dominated convergence theorem for the D_{HK}-integral]. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in D_{HK} such that $f_n \to f \ in \ \mathcal{D}'$. Suppose that there exist $f_-, f_+ \in D_{HK}$ satisfying $f_- \preceq f_n \preceq f_+$ for all $n \in \mathbb{N}$. Then $f \in D_{HK}$ and $\lim_{n\to\infty}(D_{HK})\int_a^b f_n = (D_{HK})\int_a^b f$.

3 Statement of the problem

Let X be a linear space on $\mathbb{R}$. For each $x \in X$, defined

$$S_x = \{\alpha x : -\infty < \alpha < 1\}, \quad \bar{S}_x = \{\alpha x : -\infty < \alpha \leq 1\}.$$

Let $\mathcal{M}$ be a σ-algebra in X, containing the sets $\bar{S}_x$ for all $x \in X$, and let μ be a positive σ-finite measure on $\mathcal{M}$.

For a measure space $(X, \mathcal{M})$, $Ca(X, \mathcal{M})$ denotes the space of all countably additive scalar functions on $\mathcal{M}$. Define a norm $\|\cdot\|$ on $Ca(X, \mathcal{M})$ by

$$\|\lambda\| = |\lambda|(X), \tag{3.1}$$

where $|\lambda|$ is a total variation measure of λ and is given by

$$|\lambda|(X) = \sup \sum_{i=1}^{\infty} |\lambda(E_i)|, \ E_i \subset X, \tag{3.2}$$

where the supremum is taken over all possible partitions $\{E_i : i \in \mathbb{N}\}$ of X. It is known that $Ca(X, \mathcal{M})$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (3.1).

Let $X_0 \subset X$ and $x_0 \in X_0$ be fixed, and let $\mathcal{M}_0$ be the smallest σ-algebra in X_0 containing $\bar{S}_{x_0} - S_{x_0}$ and the sets $\bar{S}_x$ for $x \in X_0 - S_{x_0}$.

Given a $\lambda \in Ca(X_0, \mathcal{M}_0)$ with $\lambda \ll \mu$, consider the abstract measure differential equation (AMDE) in (1.1)

$$\frac{d\lambda}{d\mu} = f(x, \lambda(\bar{S}_x)),$$

where $\frac{d\lambda}{d\mu}$ is a distributional derivative of λ, $f : S_x \times (-a, a) \to \mathbb{R}$, and f is D_{HK}-integral for each $\lambda \in Ca(X_0, \mathcal{M}_0)$.

Definition 3.1. Let $\alpha_0 \in (-a, a)$, a measure $\lambda \in Ca(X_0, \mathcal{M}_0)$ will be called a solution of AMDE (1.1) on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$ if

(1)$\lambda(\bar{S}_{x_0}) = \alpha_0$,

(2)$\lambda(E) \in (-a, a), \ E \in \mathcal{M}_0$,

(3)$\lambda(E) = (D_{HK}) \int_E f(x, \bar{S}_x) d\mu_0$ for $E \subset X_0 - S_{x_0}$, where μ_0 is the restriction of μ on $\mathcal{M}_0$.

In the following section we shall prove the main existence theorem for AMDE (1.1) under suitable condition on f. We will use the following form of the Leray-Schauder's nonlinear alternative. See Dugundji and Granas [7] for details.

Theorem 3.1. Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Banach space X centered at the origin 0 of radius r, for some $r > 0$. Let $T : B[0, r] \to X$ be a completely continuous operator. Then either

(1) the operator equation $Tx = x$ has a solution in $B[0, r]$, or

(2) there exists an $u \in X$ with $\|u\| = r$ such that $u = pTu$ for some $0 < p < 1$.

4 Existence Theorem

In this section, we shall study the AMDE in (1.1)

$$\frac{d\lambda}{d\mu} = f(x, \lambda(\bar{S}_x)),$$

where $\frac{d\lambda}{d\mu}$ denotes the distributional derivative of $\lambda \in Ca(X, \mathcal{M})$, and f is a distribution. Through this section, we denote by D_{HK} the space of D_{HK}-integrable functions and by $'(*)\int'$ the *-integral.

We now impose some assumptions on the distribution f.

(D_1) $f(\cdot, \lambda(\cdot))$ is D_{HK}-integrable for every fixed $\lambda \in Ca(X, \mathcal{M})$,

(D_2) $f(x, \cdot)$ is continuous for all $x \in S_x$,

(D_3) there exist $f_-, f_+ \in D_{HK}$, for all $x \in S_x$, we have

$$f_- \preceq f(x, \cdot) \preceq f_+.$$

We are now ready to give our main result.

Theorem 4.1. Let $\alpha_0 \in (-a, a), x_0 \in X_0$. If the distribution f in AMDE (1.1) satisfies assumptions $(D_1) - (D_3)$, then there exist a solution of (1.1) on X_0 with initial date $[\bar{S}_{x_0}, \alpha_0]$.

Proof Let $M = \sup_E\{|(D_{HK})\int_E f_-|\ ,\ |(D_{HK})\int_E f_+|\}$, $E \subset X_0 - S_{x_0}$ $(E \in \mathcal{M}_0)$, then

$$|(D_{HK})\int_E f_-| \le M, \qquad |(D_{HK})\int_E f_+| \le M. \tag{4.2}$$

By (2.2), (4.2) and condition (D_3), we get

$$|(D_{HK})\int_E f| \le M.$$

Consider the space $Ca(X_0, \mathcal{M}_0)$ where $\mathcal{M}_0$ is the smallest σ-algebra containing $\bar{S}_{x_0} - S_{x_0}$ and all the sets of the form $\bar{S}_x$ for $x \in X_0 - S_{x_0}$. Let $k = |\alpha_0| + M$ and $r > k$, r is a real number. Let $B[0, r]$ be a closed ball in $Ca(X_0, \mathcal{M}_0)$, and for all $\lambda \in B[0, r]$, we have the properties

$$\lambda(\bar{S}_{x_0}) = \alpha_0,$$

and

$$\|\lambda\| \le k.$$

Moreover for each $\lambda \in B[0,r]$, we have

$$|\lambda(E)| \leq |\lambda|(E) \leq \|\lambda\| \leq k, \qquad E \in \mathcal{M}_0. \tag{4.3}$$

Let T be the mapping defined on $B[0,r]$ by

$$(T\lambda)(E) = \begin{cases} \alpha_0, & for \;\; E = \bar{S}_{x_0}, \\ (D_{HK}) \int_E f(x, \lambda(\bar{S}_x)) d\mu, & for \;\; E \subset X_0 - S_{x_0}. \end{cases}$$

Then $T\lambda \in Ca(X_0, \mathcal{M}_0)$ and

$$\begin{aligned} \|T\lambda\| &= |\alpha_0| + (D_{HK}) \int_{X_0 - S_{x_0}} |f(x, \lambda(\bar{S}_x))| d|\mu| \\ &\leq |\alpha_0| + M \\ &= k. \end{aligned}$$

Therefore, $T\lambda \in B[0,r]$, then $T(B[0,r]) \subset B[0,r]$, thus $T(B[0,r])$ is uniformly bounded in $Ca(X_0, \mathcal{M}_0)$.

Let $\lambda_1, \lambda_2 \in B[0,r]$, by (4.3), we have

$$(T\lambda_1 - T\lambda_2)(E) = \begin{cases} 0, & for\; E = \bar{S}_{x_0}, \\ (D_{HK}) \int_E [f(x, \lambda_1(\bar{S}_x)) - f(x, \lambda_2(\bar{S}_x))] d\mu, & for\; E \subset X_0 - S_{x_0}. \end{cases}$$

By condition (D_2), we know, for each $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$, when $|\lambda_1(\bar{S}_x) - \lambda_2(\bar{S}_x)| < \delta$, we have

$$|f(x, \lambda_1) - f(x, \lambda_2)| < \varepsilon.$$

Then

$$\|T\lambda_1 - T\lambda_2\| = (D_{HK}) \int_E |f(x, \lambda_1(\bar{S}_x)) - f(x, \lambda_2(\bar{S}_x))| d|\mu| \quad \leq |\mu|(X_0 - S_{x_0}) \cdot \varepsilon.$$

This shows that $T(B[0,r])$ is equiuniformly continuous in $Ca(X_0, \mathcal{M}_0)$. In view of the Ascoli-Arzelà theorem, $T(B[0,r])$ is a relatively compact subset of $Ca(X_0, \mathcal{M}_0)$.

We now need to prove that T is continuous. Let $\{\lambda_n\}$ be a sequence in $B[0,r]$, and $\lambda_n \to \lambda$. Then according condition (D_2),

$$f(\cdot, \lambda_n) \to f(\cdot, \lambda), \;\; as \; n \to \infty$$

Thus, by Lemma 2.8. and

$$\lim_{m \to \infty} (D_{HK}) \int_E f(x, \lambda_m(\bar{S}_x)) \mu = (D_{HK}) \int_E f(x, \lambda(\bar{S}_x)) d\mu.$$

Therefore, $\lim_{m\to\infty} T\lambda_m = T\lambda$, which implies that T is continuous. Thus T is a compact mapping. Hence an application of Theorem 3.1 yields that either $x = Tx$ has a solution or the operator equation $x = pTx$ has a solution u with $\|u\| = r$ for some $0 < p < 1$. We shall show that this later assertion is not possible. We assume the contrary. Then we have

$$u(E) = pTu(E) = \begin{cases} p\alpha_0, & for \quad E = \bar{S}_{x_0}, \\ p(D_{HK})\int_E f(x, u(\bar{S}_x))d\mu, & for \quad E \subset S_{rx_0} - S_{x_0}. \end{cases}$$

Hence

$$\begin{aligned} |u(E)| &= |p\alpha_0| + |p(D_{HK})\int_E |f(x, u(\bar{S}_x))d\mu| \\ &\leq |\alpha_0| + |(D_{HK})\int_E |f(x, u(\bar{S}_x))d\mu| \\ &\leq |\alpha_0| + M \\ &= k. \end{aligned}$$

This futher implies that

$$\|u\| = |u|(E) \leq k.$$

Substituting $\|u\| = r$ in the above inequality, this yields $r \leq k$, which is a contradiction to the inequality $r > k$.

Hence the operator equation $\lambda = T\lambda$ has a solution. Consequently the AMDE (1.1) has a solution λ_0, λ_0 is then the solution of AMDE (1.1) on X_0 with initial date $[\bar{S}_{x_0}, \alpha_0]$. This completes the proof. $\square$

Remark 4.2. In AMDE (1.1), if f satisfies the conditions in Theorem 1 in [1] that there exists a μ-integrable function w such that $|f(x, \alpha)| \leq w(x)$ uniformly in $\alpha \in (-a, a)$, and f satisfies a Lipschitz condition in α, then f satisfies Theorem 4.1, thus Theorem 4.1 is an extension of Theorem 1 in [1].

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