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A New Iterative Scheme for Approximating Common Fixed Points of Two Nonexpansive Mappings in Banach Space

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Abstract

In this paper, we introduce the modified iterations of Mann's type for nonexpansive mappings to have the strong convergence in a uniformly convex Banach space. We study approximation of common fixed point of nonexpansive mappings in Banach space by using a new iterative scheme.

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1 Introduction

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is *nonexpansive mapping*[1] if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed point of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $Fix(T) \neq \emptyset$.

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_n = Tx_{n-1}$ for $n \geq 1$ with x_0 arbitrary, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is nonexpansive), then Picard's successive iteration fail, in general, to converge. Instead, Mann's iteration method [10] prevails. Mann's method, an averaged process in nature, generates a sequence $\{x_n\}$ recursively via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen.

Mann's iteration method (1.1) has been proved to be a powerful method for solving nonlinear operator equations involving nonexpansive mapping, asymptotically nonexpansive mapping, and other kinds of nonlinear mapping; see [2, 3, 6-8, 10, 12, 13, 15, 16] and the references therein.

It is known that Mann's iteration method (1.1) is in general not strongly convergent [5] for nonexpansive mappings. So to get strong convergence, one has to modify the iteration method (1.1). In this regard, we will show in Section 3.

Motivated and inspired by the research going on in these fields, we suggest and analyze now new modified Mann's iteration for finding the common fixed point of the nonexpansive mappings in Banach space. We propose the modified Mann's iteration and consider the strong convergence of the approximate solutions for nonexpansive in Banach space.

We suggest and analyze the following two iterative methods:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ z_n = \gamma_n x_n + (1-\gamma_n)Sx_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)z_n, \ n \geq 0, \end{cases} \tag{1.2}$$

and if there exists two sequences $\{x'_n\}$ and $\{x''_n\}$ generated by

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = \beta_n x'_n + (1-\beta_n)Tx'_n, \\ z_n = \gamma_n x''_n + (1-\gamma_n)Sx''_n, \\ x'_{n+1} = \alpha_n z_n + (1-\alpha_n)y_n, \\ x''_{n+1} = \alpha_n y_n + (1-\alpha_n)z_n, \ n \geq 0, \end{cases} \tag{1.3}$$

where $\alpha_n, \beta_n, \gamma_n$ are constants satisfying certain conditions.

We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. We use F to denote the set of common fixed point of the mappings T and S.

2 Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section.

Lemma 2.1. [12] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ *exists;*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2. [14] *Suppose that E is a uniformly convex Banach space and $0 < t_n < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. [15] *A mapping $T{:}C{\to}C$ with nonempty fixed point set F in C will be said to satisfy Condition (I):*

If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

Lemma 2.4. [18] *Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $\varphi : [0,\infty] \to [0,\infty], \varphi(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $\lambda \in [0,1]$ and $x, y \in E$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Note that the inequality in Lemma 2.4 is known as *$Xu's$ inequality*. It follows from Lemma 2.4 that we get the following lemma can be found in [4].

Lemma 2.5. *Given a number $r > 0$. Let E is a uniformly convex Banach space then there exists a continuous strictly increasing function $\varphi : [0,\infty] \to [0,\infty]$ with $\varphi(0) = 0$, such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu\varphi(\|x-y\|)$$

for all $\lambda, \mu, \gamma \in [0,1]$ and $x, y, z \in E$ such that $\|x\| \leq r$, $\|y\| \leq r$ and $\|z\| \leq r$.

Lemma 2.6. [9] *Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, then $\liminf_{n\to\infty} \beta_n = 0$.*

3 Convergence to a common fixed point of nonexpansive mappings

3.1 There exists one sequence $\{x_n\}$

In this part, we prove our main theorem for finding a common fixed point of nonexpansive mappings in Banach space in the case of one sequence.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T and S be two nonexpansive self mappings of C satisfy Condition (I) and $F \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1), such that $\sum \alpha_n < \infty$, $\sum \gamma_n\beta_n = \infty$, $\sum(1-\gamma_n) < \infty$ for all $n \geq 1$.*

Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm (1.2), then $\{x_n\}_{n=0}^{\infty}$ strongly converges to a common fixed point of T and S.

Proof. First, we observe that $\{x_n\}$ is bounded, if we take an arbitrary fixed point q of F, noting that $\|y_n - q\| \leq \|x_n - q\|$ and $\|z_n - q\| \leq \|x_n - q\|$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n y_n + (1-\alpha_n)z_n - q\| \\ &\leq \alpha_n\|y_n - z_n\| + \|z_n - q\| \\ &\leq \alpha_n\|y_n - q\| + \alpha_n\|z_n - q\| + \|z_n - q\| \\ &\leq (1+2\alpha_n)\|x_n - q\|. \end{aligned} \tag{3.1}$$

By Lemma 2.1 and $\sum \alpha_n < \infty$, thus $\lim_{n\to\infty} \|x_n - q\|$ exists. Denote

$$\lim_{n\to\infty} \|x_n - q\| = c.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Now

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n y_n + (1-\alpha_n)z_n - q\| \\ &= \|\alpha_n(y_n - z_n) + (z_n - q)\| \\ &\leq \alpha_n\|y_n - z_n\| + \|z_n - q\|. \end{aligned}$$

By $\sum \alpha_n < \infty$ and the boundedness of $\{z_n\}$ and $\{y_n\}$, we obtain

$$\lim_{n\to\infty} \|x_n - q\| \leq \liminf_{n\to\infty} \|z_n - q\|. \tag{3.2}$$

Since $\|z_n - q\| \leq \|x_n - q\|$, which implies that

$$\limsup_{n\to\infty} \|z_n - q\| \leq \lim_{n\to\infty} \|x_n - q\|, \tag{3.3}$$

so that (3.2) and (3.3) give

$$\lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|x_n - q\| = c.$$

Moreover, $\|Sx_n - q\| \le \|x_n - q\|$ implies that

$$\limsup_{n\to\infty} \|Sx_n - q\| \le c.$$

Thus

$$\begin{aligned} c = \lim_{n\to\infty} \|z_n - q\| &= \lim_{n\to\infty} \|\gamma_n x_n + (1-\gamma_n)Sx_n - q\| \\ &= \lim_{n\to\infty} \|\gamma_n (x_n - q) + (1-\gamma_n)(Sx_n - q)\|, \end{aligned}$$

given by Lemma 2.2 that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.4}$$

By (3.1) and $\sum \alpha_n < \infty$, then we have

$$\begin{aligned} \|x_{n+m} - q\| &\le (1 + 2\alpha_{n+m-1})\|x_{n+m-1} - q\| \\ &\le e^{2\alpha_{n+m-1}} \|x_{n+m-1} - q\| \\ &\le e^{2\alpha_{n+m-1}} e^{2\alpha_{n+m-2}} \|x_{n+m-2} - q\| \\ &\le \dots \\ &\le e^{2\sum_{i=n}^{n+m-1} \alpha_i} \|x_n - q\|. \end{aligned}$$

That is

$$\|x_{n+m} - q\| \le M \|x_n - q\|, \tag{3.5}$$

where $M = e^{2\sum_{i=n}^{n+m-1} \alpha_i}$, for all $m, n \ge 1$, for all $q \in F$ and for $M > 0$.

Next we prove that $\{x_n\}_{n=0}^{\infty}$ is Cauchy sequence.

Since $q \in F$ arbitrarily and $\lim_{n\to\infty} \|x_n - q\|$ exists, consequently $d(x_n, F)$ exists by Lemma 2.3. From the Lemma 2.3 and (3.4), we get

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|x_n - Sx_n\| = 0$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfy $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Let $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, F) = 0$ and therefore exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) \leq \frac{\varepsilon}{2M},$$

in particular,

$$d(x_{n_0}, F) \leq \frac{\varepsilon}{2M}.$$

There must exist $p_1 \in F$, such that

$$d(x_{n_0}, p_1) \leq \frac{\varepsilon}{2M}.$$

From (3.5), it can be obtained that when $n \geq n_0$,

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\leq 2M\|x_{n_0} - p_1\| \\
&\leq 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

This implies $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a closed convex subset C of a Banach space E. Thus, it must converges to a point in C, let $\lim_{n\to\infty} x_n = p$.

For all $\epsilon > 0$, as $\lim_{n\to\infty} x_n = p$, thus there exists a number n_1 such that when $n_2 \geq n_1$,

$$\|x_{n_2} - p\| \leq \frac{\epsilon}{4}. \tag{3.6}$$

In fact, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that using the n_2 above, when $n \geq n_2$, we get

$$d(x_n, F) \leq \frac{\epsilon}{8}.$$

In particular, $d(x_{n_2}, F) \leq \frac{\epsilon}{8}$. Thus, there must exist $\bar{p} \in F$, such that

$$\|x_{n_2} - \bar{p}\| = d(x_{n_2}, \bar{p}) = \frac{\epsilon}{8}. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned}
\|Sp - p\| &= \|Sp - \bar{p} + Sx_{n_2} - \bar{p} + \bar{p} - x_{n_2} + x_{n_2} - p + \bar{p} - Sx_{n_2}\| \\
&\leq \|Sp - \bar{p}\| + \|x_{n_2} - \bar{p}\| + \|x_{n_2} - p\| + 2\|Sx_{n_2} - \bar{p}\| \\
&\leq \|p - \bar{p}\| + 3\|x_{n_2} - \bar{p}\| + \|x_{n_2} - p\| \\
&\leq \|x_{n_2} - p\| + \|x_{n_2} - \bar{p}\| + 3\|x_{n_2} - \bar{p}\| + \|x_{n_2} - p\| \\
&= 4\|x_{n_2} - \bar{p}\| + 2\|x_{n_2} - p\| \\
&\leq \frac{4\epsilon}{8} + \frac{2\epsilon}{4} = \epsilon.
\end{aligned}$$

As ϵ is an arbitrary positive number, thus, $Sp = p$.

Let

$$u_{n+1} = \gamma_n x_n + \beta_n T x_n + (1 - \beta_n - \gamma_n) T x_n.$$

Then we have

$$\begin{aligned}
&\|u_{n+1} - q\|^2 \\
&= \|\gamma_n x_n + \beta_n T x_n + (1 - \beta_n - \gamma_n) T x_n - q\|^2 \\
&= \|\gamma_n (x_n - q) + \beta_n (T x_n - q) + (1 - \beta_n - \gamma_n)(T x_n - q)\|^2 \\
&\leq \gamma_n \|x_n - q\|^2 + \beta_n \|T x_n - q\|^2 + (1 - \beta_n - \gamma_n)\|T x_n - q\|^2 - \gamma_n \beta_n \varphi \|x_n - T x_n\| \\
&\leq \|x_n - q\|^2 - \gamma_n \beta_n \varphi \|x_n - T x_n\|,
\end{aligned}$$

and hence

$$\gamma_n \beta_n \varphi \|x_n - T x_n\| \leq \|x_n - q\|^2 - \|u_{n+1} - q\|^2$$

for $q \in F$. Summing from n=1 to ∞, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \gamma_n \beta_n \varphi \|x_n - T x_n\| &= \sum_{n=1}^{\infty} (\|x_n - q\|^2 - \|u_{n+1} - q\|^2) \\
&= \sum_{n=1}^{\infty} (\|x_n - q\| + \|u_{n+1} - q\|)(\|x_n - q\| - \|u_{n+1} - q\|) \\
&\leq 2K \sum_{n=1}^{\infty} \|u_{n+1} - x_n\| \\
&\leq 2K \sum_{n=1}^{\infty} (1 - \gamma_n) \|x_n - T x_n\| \\
&\leq 4K^2 \sum_{n=1}^{\infty} (1 - \gamma_n),
\end{aligned}$$

where $K = \sup_{n \in \mathbb{N}} \{\|x_n - q\|\}$, from $\sum_{n=1}^{\infty} (1 - \gamma_n) < \infty$, we get

$$\sum_{n=1}^{\infty} \gamma_n \beta_n \varphi \|x_n - T x_n\| \leq 4K^2 \sum_{n=1}^{\infty} (1 - \gamma_n) < \infty.$$

Since $\sum_{n=1}^{\infty} \gamma_n \beta_n = \infty$, from Lemma 2.6, we get $\liminf_{n \to \infty} \varphi \|T x_n - x_n\| = 0$. Hence

$$\liminf_{n \to \infty} \|T x_n - x_n\| = 0. \tag{3.8}$$

Since T is a nonexpansive mapping, we have

$$\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&= \|Tx_{n+1} - \alpha_n y_n - (1-\alpha_n)z_n\| \\
&= \|Tx_{n+1} - Tx_n + Tx_n + \alpha_n Tx_n - \alpha_n Tx_n - \alpha_n y_n - (1-\alpha_n)z_n\| \\
&\leq \|Tx_{n+1} - Tx_n\| + (1-\alpha_n)\|Tx_n - z_n\| + \alpha_n\|Tx_n - y_n\| \\
&\leq \|x_{n+1} - x_n\| + (1-\alpha_n)\|Tx_n - z_n\| + \alpha_n\|Tx_n - y_n\| \\
&= \|\alpha_n y_n + (1-\alpha_n)z_n - x_n\| + (1-\alpha_n)\|Tx_n - z_n\| + \alpha_n\|Tx_n - y_n\| \\
&\leq \alpha_n\|y_n - x_n\| + (1-\alpha_n)\|z_n - x_n\| + (1-\alpha_n)\|Tx_n - z_n\| + \alpha_n\|Tx_n - y_n\| \\
&= \alpha_n\|\beta_n x_n + (1-\beta_n)Tx_n - x_n\| + (1-\alpha_n)\|\gamma_n x_n + (1-\gamma_n)Sx_n - x_n\| \\
&\quad + (1-\alpha_n)\|Tx_n - \gamma_n x_n - (1-\gamma_n)Sx_n\| + \alpha_n\|Tx_n - \beta_n x_n - (1-\beta_n)Tx_n\| \\
&\leq \alpha_n(1-\beta_n)\|Tx_n - x_n\| + (1-\alpha_n)(1-\gamma_n)\|Sx_n - x_n\| + \alpha_n\beta_n\|Tx_n - x_n\| \\
&\quad + \gamma_n(1-\alpha_n)\|Tx_n - x_n\| + (1-\alpha_n)(1-\gamma_n)\|Tx_n - Sx_n\| \\
&\leq \alpha_n(1-\beta_n)\|Tx_n - x_n\| + (1-\gamma_n)\|Sx_n - x_n\| + \alpha_n\beta_n\|Tx_n - x_n\| \\
&\quad + (1-\alpha_n)\|Tx_n - x_n\| + (1-\gamma_n)\|Tx_n - Sx_n\| \\
&\leq \|Tx_n - x_n\| + (1-\gamma_n)(\|Sx_n - x_n\| + \|Tx_n - Sx_n\|).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\gamma_n) < \infty$, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|Tx_n - x_n\|$ exists. Therefore, from (3.8), we get

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

Then using the same argument we can show that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of T and S. $\square$

Remark 3.1. *It is to be noted that (1.2) reduces to (1.1) when T or S equal to I.*

3.2 There exists two sequences $\{x'_n\}$ and $\{x''_n\}$

In this part, we prove our main theorem for finding a common fixed point of nonexpansive mappings in Banach space in the case of two sequences.

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T and S be two nonexpansive self mappings of C satisfy Condition (I) and $F \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1), such that $\sum \alpha_n < \infty$, $\sum \beta_n < \infty$, $\beta_n > \gamma_n$ for all $n \geq 1$.*

Define two sequences $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ in C by the algorithm (1.3), then $\{x'_n\}_{n=0}^{\infty}$ and $\{x''_n\}_{n=0}^{\infty}$ strongly converge to a common fixed point of T and S.

Proof. By the boundedness of C, we obverse that both $\{x'_n\}$ and $\{x''_n\}$ are bounded, if we take an arbitrary fixed point q of F, noting that $\|y_n - q\| \leq \|x'_n - q\|$ and $\|z_n - q\| \leq \|x''_n - q\|$, we have

$$\begin{aligned} \|x'_{n+1} - q\| &= \|\alpha_n z_n + (1-\alpha_n) y_n - q\| \\ &\leq \alpha_n \|z_n - y_n\| + \|y_n - q\| \\ &\leq \alpha_n \|y_n - q\| + \alpha_n \|z_n - q\| + \|y_n - q\| \\ &\leq (1+\alpha_n)\|x'_n - q\| + \alpha_n \|x''_n - q\|. \end{aligned} \tag{3.9}$$

By Lemma 2.1 and $\sum \alpha_n < \infty$, thus $\lim_{n\to\infty} \|x'_n - q\|$ exists. Denote

$$\lim_{n\to\infty} \|x'_n - q\| = c'.$$

Similarly, we have

$$\lim_{n\to\infty} \|x''_n - q\| = c''.$$

Since both $\{x'_n\}$ and $\{x''_n\}$ are bounded, we get $\{y_n\}$ and $\{z_n\}$ are bounded. Now

$$\begin{aligned} \|x'_{n+1} - q\| &= \|\alpha_n z_n + (1-\alpha_n) y_n - q\| \\ &= \|\alpha_n (z_n - y_n) + (y_n - q)\| \\ &\leq \alpha_n \|z_n - y_n\| + \|y_n - q\|. \end{aligned}$$

By $\sum \alpha_n < \infty$, we obtain

$$\lim_{n\to\infty} \|x'_n - q\| \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{3.10}$$

Since $\|y_n - q\| \leq \|x'_n - q\|$, which implies that

$$\limsup_{n\to\infty} \|y_n - q\| \leq \lim_{n\to\infty} \|x'_n - q\|, \tag{3.11}$$

so that (3.10) and (3.11) give

$$\lim_{n\to\infty}\|y_n - q\| = \lim_{n\to\infty}\|x'_n - q\| = c'.$$

Moreover, $\|Tx'_n - q\| \le \|x'_n - q\|$ implies that

$$\limsup_{n\to\infty}\|Tx'_n - q\| \le c'.$$

Thus

$$\begin{aligned} c' = \lim_{n\to\infty}\|y_n - q\| &= \lim_{n\to\infty}\|\beta_n x'_n + (1-\beta_n)Tx'_n - q\| \\ &= \lim_{n\to\infty}\|\beta_n(x'_n - q) + (1-\beta_n)(Tx'_n - q)\|, \end{aligned}$$

given by Lemma 2.2 that

$$\lim_{n\to\infty}\|Tx'_n - x'_n\| = 0. \tag{3.12}$$

By (3.9) and $\sum \alpha_n < \infty$, then we have

$$\begin{aligned} \|x'_{n+m} - q\| &\le (1+\alpha_{n+m-1})\|x'_{n+m-1} - q\| + \alpha_{n+m-1}\|x''_{n+m-1} - q\| \\ &\le e^{\alpha_{n+m-1}}\|x'_{n+m-1} - q\| + \alpha_{n+m-1}\|x''_{n+m-1} - q\| \\ &\le e^{\alpha_{n+m-1}}e^{\alpha_{n+m-2}}\|x_{n+m-2} - q\| \\ &\quad + e^{\alpha_{n+m-1}}(\alpha_{n+m-2}\|x''_{n+m-2} - q\| + \alpha_{n+m-1}\|x''_{n+m-1} - q\|) \\ &\le \dots \\ &\le e^{\sum_{i=n}^{n+m-1}\alpha_i}\|x'_n - q\| + e^{\sum_{i=n}^{n+m-1}\alpha_i}\cdot\sum_{i=n}^{n+m-1}\alpha_i\|x''_i - q\|. \end{aligned}$$

That is

$$\|x'_{n+m} - q\| \le L(\|x'_n - q\| + \sum_{i=n}^{n+m-1} s_i), \tag{3.13}$$

where $L = e^{\sum_{i=n}^{n+m-1}\alpha_i}$, $s_i = \alpha_i\|x''_i - q\|$. for all $m, n \ge 1$, for all $q \in F$.

Next we prove that $\{x'_n\}_{n=0}^{\infty}$ is Cauchy sequence.

Since $q \in F$ arbitrarily and $\lim_{n\to\infty}\|x'_n - q\|$ exists, consequently $d(x'_n, F)$ exists by Lemma 2.3. From the Lemma 2.3 and (3.12), we get

$$\lim_{n\to\infty} f(d(x'_n, F)) \le \lim_{n\to\infty}\|x'_n - Tx'_n\| = 0$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfy $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\lim_{n\to\infty} d(x'_n, F) = 0.$$

Let $\varepsilon > 0$, since $\lim_{n\to\infty} d(x'_n, F) = 0$ and $\sum_{i=0}^{\infty} s_i < \infty$, therefore it exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x'_n, F) \leq \frac{\varepsilon}{2L},$$

in particular,

$$d(x'_{n_0}, F) \leq \frac{\varepsilon}{2L}.$$

There must exist $p_1 \in F$, such that

$$d(x'_{n_0}, p_1) \leq \frac{\varepsilon}{2L}.$$

From (3.13), it can be obtained that when $n \geq n_0$,

$$\begin{aligned}
\|x'_{n+m} - x'_n\| &\leq \|x'_{n+m} - p_1\| + \|x'_n - p_1\| \\
&\leq 2L\|x'_{n_0} - p_1\| \\
&\leq 2L \cdot \frac{\varepsilon}{2L} = \varepsilon.
\end{aligned}$$

This implies $\{x'_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a closed convex subset C of a Banach space E. Thus, it must converges to a point in C, let $\lim_{n\to\infty} x'_n = x'$.

For all $\epsilon > 0$, as $\lim_{n\to\infty} x'_n = x'$, thus there exists a number n_1 such that when $n_2 \geq n_1$,

$$\|x'_{n_2} - x'\| \leq \frac{\epsilon}{4}. \tag{3.14}$$

In fact, $\lim_{n\to\infty} d(x'_n, F) = 0$ implies that using the n_2 above, when $n \geq n_2$, we get

$$d(x'_n, F) \leq \frac{\epsilon}{8}.$$

In particular, $d(x'_{n_2}, F) \leq \frac{\epsilon}{8}$. Thus, there must exist $\bar{x}' \in F$, such that

$$\|x'_{n_2} - \bar{x}'\| = d(x'_{n_2}, \bar{x}') = \frac{\epsilon}{8}. \tag{3.15}$$

From (3.14) and (3.15), we get

$$\begin{aligned}\|Tx' - x'\| &= \|Tx' - \bar{x}' + Tx'_{n_2} - \bar{x}' + \bar{x}' - x'_{n_2} + x'_{n_2} - x' + \bar{x}' - Tx'_{n_2}\| \\ &\le \|Tx' - \bar{x}'\| + \|x'_{n_2} - \bar{x}'\| + \|x'_{n_2} - x'\| + 2\|Tx'_{n_2} - \bar{x}'\| \\ &\le \|x' - \bar{x}'\| + 3\|x'_{n_2} - \bar{x}'\| + \|x'_{n_2} - x'\| \\ &\le \|x'_{n_2} - x'\| + \|x'_{n_2} - \bar{x}'\| + 3\|x'_{n_2} - \bar{x}'\| + \|x'_{n_2} - x'\| \\ &= 4\|x'_{n_2} - \bar{x}'\| + 2\|x'_{n_2} - x'\| \\ &\le \frac{4\epsilon}{8} + \frac{2\epsilon}{4} = \epsilon.\end{aligned}$$

As ϵ is an arbitrary positive number, thus, $Tx' = x'$.

Similarly, we have $\lim_{n\to\infty} \|Sx''_n - x''_n\| = 0$, and then $Sx'' = x''(x''_n \to x''$ as $n \to \infty)$

Finally, we prove $x' = x''$.

Let $w_{n+1} = \alpha_n Tx'_n + (1-\alpha_n)Sx''_n$ and $\|x''_n - q\| \ge \|x'_n - q\|$ for all $n \ge 1$. Then

$$\begin{aligned}\|w_{n+1} - q\| &= \alpha_n\|x'_n - q\| + (1-\alpha_n)\|x''_n - q\| \\ &\le \max\{\|x'_n - q\|, \|x''_n - q\|\} \\ &\le \|x''_n - q\|.\end{aligned}$$

Now,

$$\begin{aligned}\|x''_{n+1} - q\| &= \|\alpha_n y_n + (1-\alpha_n)z_n - q\| \\ &= \|\alpha_n\beta_n x'_n + \alpha_n(1-\beta_n)Tx'_n + (1-\alpha_n)\gamma_n x''_n + (1-\alpha_n)(1-\gamma_n)Sx''_n - q\| \\ &\le \beta_n\|\alpha_n x'_n + (1-\alpha_n)x''_n\| + \|\alpha_n Tx'_n + (1-\alpha_n)Sx''_n - q\| \\ &= \beta_n\|\alpha_n x'_n + (1-\alpha_n)x''_n\| + \|w_{n+1} - q\|,\end{aligned}$$

since $\sum \beta_n < \infty$ and the boundedness of $\{x'_n\}$ and $\{x''_n\}$, we get

$$c'' \le \lim_{n\to\infty} \|w_{n+1} - q\|.$$

Then we have

$$\lim_{n\to\infty} \|w_{n+1} - q\| = c''.$$

Moreover, $\|Tx'_n - q\| \le \|x'_n - q\|$ and $\|Sx''_n - q\| \le \|x''_n - q\|$, imply that

$$\limsup_{n\to\infty} \|Tx'_n - q\| \le c'' \text{ and } \limsup_{n\to\infty} \|Sx''_n - q\| \le c''.$$

Thus,

$$c'' = \lim_{n\to\infty} \|w_{n+1} - q\| = \|\alpha_n T x'_n + (1-\alpha_n) S x''_n - q\|$$
$$= \|\alpha_n (T x'_n - q) + (1-\alpha_n)(S x''_n - q)\|,$$

given by Lemma 2.2 that

$$\lim_{n\to\infty} \|S x''_n - T x'_n\| = 0. \tag{3.16}$$

So

$$\begin{aligned}
&\lim_{n\to\infty} \|x'_{n+1} - x''_{n+1}\| \\
&= \lim_{n\to\infty} (2\alpha_n - 1)\|z_n - y_n\| \\
&= \lim_{n\to\infty} (2\alpha_n - 1)\|\gamma_n (x''_n - S x''_n) - \beta_n (x'_n - T x'_n) + (S x''_n - T x'_n)\|,
\end{aligned}$$

so we obtain $\lim_{n\to\infty} \|x'_{n+1} - x''_{n+1}\| = 0$ for (3.12) and (3.16), it means $x' = x''$. This completes the proof. □

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