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Some characterization results in the calculus of variations in the degenerate case

K. Kouhaila[1], E. Azroul[2] and A. Benkirane[3]

Abstract

In this article, we prove an approximation result in weighted Sobolev spaces and we give an application of this approximation result to a necessary condition in the calculus of variations.

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1 Introduction

On a bounded domain $\Omega \subset I\!R^N$, we consider the functionals of the kind

$$J(u) = \int_\Omega a(x, u, \nabla u)dx.$$

[1] Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar-Mahraz, B.P 1796 Atlas Fez, Morroco.
[2] Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar-Mahraz, B.P 1796 Atlas Fez, Morroco.
[3] Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar-Mahraz, B.P 1796 Atlas Fez, Morroco.

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The major problem in the calculation of variations is to find the elements u checking in the boundary conditions required by the nature of the problem and minimizing the functional J.

Since the nineteenth century, more work were done in this direction, noting the contributions of Lagrange, Riemann, Weierstrass, Jacobi, Hamilton, etc.

The methods developped by them are termed conventional methods.

In the early twentieth century, various techniques have been introduced by Hilbert and Lebesgue in connection with the study of Dirichlet integrals.

Later these techniques and methods have been generalized by Tonelli and are now known as direct methods in calculus of variations.

Developed after several authors, these direct methods such as: Marcellini, Sbordone, Dacorogna, etc.

In the L^p case the search of suffisent conditions to secure those functionals attain an extrem value has a long history (see [3]).The most important problem is to verify the weak lower semicontinuity of those functionals with respect to the space involved .This usually involves hypothesis that the integrand a is convex with respect to the gradient . In 1992 R.Landes in [3] studied the revese problem at a fixed level set and an to many situations have been showed that if J is weakely lower semi-continous at one fixed (nonvoid) level set then this partical level set is an extrem value of J or the defining function a is convex in the gradient. The above statement for a as function of u (or of x and u) is not hard to prove (see[3]) but when $a = a(x, \nabla u)$ or $a = a(x, u, \nabla u)$ this is due to an approximation result in Sobolev-spaces .
In 2001 E.Azroul and A.Benkirane studied the same work that R.Landes in the case of Orlicz-Sobolev spaces $W^1L_M(\Omega)$.
Since this approximation is important for possible application in calculus of variations , one of the main purpose in this paper is to extend the above approximation result to the setting of weighted Sobolev spaces $W^{1,p}(\Omega, \omega)$, this is the objective of the first part of this paper. The second part, is devoted to the application of this approximation. However we prove when $a = a(x, \nabla u)$ that if J is weakly lower semi-continuous at one fixed level set H_τ in the space $W^{1,p}(\Omega, \omega)$, then this particular level set is an extreme value of J or the function a is convex with respect to the gradient .

2 Functional prerequis

In this section, we present some definitions and well-known about weighted-Sobolev spaces (standard references are in [1], [5], [8] and [11]).

2.1 Weighted Lebesgue spaces $L^p(\Omega,\gamma)$

Let p the real numbers such that $1 \prec p \prec \infty$, and γ the weight function . We define the weighted lebesgue space $L^p(\Omega,\gamma)$ by $L^p(\Omega,\gamma) = \left\{u, u\gamma^{\frac{1}{p}} \in L^p(\Omega)\right\}$ under this norm:

$$\|u\|_{p,\gamma} = \int_{\Omega} |u(x)|^p \, \gamma(x)dx)^{\frac{1}{p}}. \tag{1}$$

2.2 Weighted-Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^N$, and let $\{w_i, i = 0,1,2..,N\}$ be a family of weight functions.

We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega_0)$ with weak derivatives $\frac{\partial u}{\partial x_i} \in L^p(\Omega,\omega_i)$ for $i = 1,..,N$.The weighted Sobolev space $W^{1,p}(\Omega,\omega)$ is a normed linear space if equiped with the norm:

$$\|u\|_{1,p,\omega} = (\int_{\Omega} |u(x)|^p \, w_0(x)dx + \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}(x)\right|^p \omega_i(x)dx)^{\frac{1}{p}}. \tag{2}$$

Theorem 2.1. (cf.[5])

i) *Let $1 \prec p \prec \infty$ and suppose that the weight functions w_i satisfy*

$$w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega), i = 0,1,...,N. \tag{3}$$

Then $W^{1,p}(\Omega,\omega)$ is a uniformly convex (and hence reflexive) Banach space.

ii) *If we additionally suppose that also*

$$w_i \in L^1_{loc}(\Omega), i = 0,1,...,N, \tag{4}$$

then C_0^∞ *is a subset of* $W^{1,p}(\Omega,\omega)$, *and we can introduce the space* $W_0^{1,p}(\Omega,\omega)$ *as the closure of* C_0^∞ *with respect to the norm* $\|.\|_{1,p,\omega}$.

2.3 Imbedding results

Example 2.2. Let us consider, the wheigted Sobolev space $W^{1,p}(\Omega,\omega)$, with a special choice of the family w :

$$w_0(x) \equiv 1,\ w_i(x) = \gamma(x) \text{ for } i = 1,...,N.$$

In this case, the space $W^{1,p}(\Omega,\gamma)$=$W^{1,p}(\Omega,\omega)$ is normed by

$$\|u\|_{1,p,\gamma} = (\int_\Omega |u(x)|^p\,dx + \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^p \gamma(x)dx)^{\frac{1}{p}}. \tag{5}$$

Let us suppose that the weight function γ satisfies the conditions (2.3) and (2.4) also the condition

$$\gamma^{-s} \in L^1(\Omega). \tag{6}$$

for a certain $s \succ 0$ which will be specified later. Introducing the parameter p_s by

$$p_s = \tfrac{ps}{s+1} \prec p$$

then, we have the following imbeddings

$$W^{1,p}(\Omega,\gamma) \hookrightarrow L^r(\Omega). \tag{7}$$

where

$$1 \le r \le p_s^* = \tfrac{Np_s}{N-p_s} = \tfrac{Nps}{N(s+1)-ps} \text{ for } ps \prec N(s+1),$$

and $r \ge 1$ is arbitrary for $ps \ge N(s+1)$(cf.[5]. theorem 1.2 (a), (b)). Moreover, we have the compact imbedding

$$W^{1,p}(\Omega,\gamma) \hookrightarrow\hookrightarrow L^r(\Omega). \tag{8}$$

provided $1 \le r \prec p_s^*$.
In particular, we have $p_s^* \succ p$ if $s \succ \frac{N}{p}$, and consequently,

$$W^{1,p}(\Omega,\gamma) \hookrightarrow\hookrightarrow L^p(\Omega) \quad for \quad s \succ \frac{N}{p}. \tag{9}$$

In this case we consider the family of weight functions $\omega = \{w_i, i = 0, 1, 2.., N\}$ with $w_0 = 1$. Let us suppose that w_i satisfies

$$\text{There exists } \; s \in \left]\frac{N}{p}, \infty\right[\bigcap \left[\frac{1}{p-1}, \infty\right[\text{ such that } w_i^{-s} \in L^1(\Omega), \forall \; 1 \leq i \leq N. \tag{10}$$

Theorem 2.3. (cf.[5]) *If the conditions (3),(4) and (10) are satisfied, then the space* $W_0^{1,p}(\Omega, \omega)$ *is reflexive and the following compact imbeding*

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^p(\Omega) \tag{11}$$

is verified.

3 Approximation result

In this section, we consider the family of weight functions $\omega = \{w_i, i = 0, 1, 2.., N\}$ with $w_0(x) \equiv 1$. Let us w_i satisfies the conditions (3), (4) and (10).

Theorem 3.1. *Let* Ω *be a bounded domain in* $I\!R^N$. *If* $u \in W^{1,p}(\Omega, \omega)$, $1 \prec p \prec \infty$, *then for almost all* $x_0 \in \Omega$, *there exists* $u_\lambda \in W^{1,p}(\Omega, \omega)$, *such that*

i) $u_\lambda \to u$ *in* $W^{1,p}(\Omega, \omega)$,

ii) $u_\lambda \equiv c(x_0, \lambda)$ *in* $B(x_0, \lambda)$.

Proof of Theorem 3-1:
Let Φ_λ be a C_0^∞ cut-off function with support in $B(0, 2\lambda)$ such that $\Phi_\lambda \equiv 1$ in $B(0, \lambda)$ and $|\nabla \Phi_\lambda| \leq \frac{2}{\lambda}$.

Let x_0 be a Lebesgue point of the function u in Ω , hence we can take $c(x_0, \lambda) = u(x_0)$.
We define in Ω the function u_λ by

$$u_\lambda(x) = u(x)(1 - \Phi_\lambda(x - x_0)) + u(x_0)\Phi_\lambda(x - x_0). \tag{12}$$

First it's clear that $u_\lambda \in W^{1,p}(\Omega, \omega)$.
In fact: Since $u \in W^{1,p}(\Omega, \omega)$, then

$$\int_\Omega |u(x)|^p \, dx \prec \infty, \quad \text{and} \quad \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^p \omega_i(x) dx \prec \infty, \quad \text{for } 1 \leq i \leq N.$$

Since $x \mapsto x^p$ is a convex function, then

$$\begin{aligned} &\int_\Omega |u_\lambda(x)|^p \, dx \\ &\quad \leq 2^{p-1} \int_\Omega |u(x)(1 - \Phi_\lambda(x - x_0))|^p \, dx + 2^{p-1} \int_\Omega |u(x_0)\Phi_\lambda(x - x_0)|^p dx \\ &\quad \leq k_1 \int_\Omega |u(x)|^p \, dx + 2^{p-1} |u(x_0)|^p \int_{B(0,2\lambda)} |\Phi_\lambda(x - x_0)|^p \, dx \prec \infty, \end{aligned}$$

where $k_1 = 2^{p-1} \sup_{B(0,2\lambda)} |1 - \Phi_\lambda(x - x_0)|^p$.

It remains to show that,

$$\frac{\partial u_\lambda}{\partial x_i} \in L^p(\Omega, \omega_i), \quad 1 \leq i \leq N.$$

By a simple calculation we find that,

$$\frac{\partial u_\lambda}{\partial x_i} = \frac{\partial u(x)}{\partial x_i}(1 - \Phi_\lambda(x - x_0)) + (u(x_0) - u(x)) \frac{\partial \Phi_\lambda(x - x_0)}{\partial x_i}.$$

Then,

$$\begin{aligned} rcl \int_\Omega \left| \frac{\partial u_\lambda}{\partial x_i} \right|^p \omega_i(x) dx \quad &\leq \quad 2^{p-1} \int_\Omega \left| (1 - \Phi_\lambda(x - x_0)) \frac{\partial u(x)}{\partial x_i} \right|^p \omega_i(x) dx \\ &\quad + 2^{p-1} \int_\Omega \left| (u(x) - u(x_0)) \frac{\partial \Phi_\lambda(x - x_0)}{\partial x_i} \right|^p \omega_i(x) dx \\ (*) \quad &\leq \quad k_1 \int_\Omega \left| \frac{\partial u(x)}{\partial x_i}) \right|^p \omega_i(x) dx \\ &\quad + 2^{p-1} \int_\Omega \left| (u(x) - u(x_0)) \frac{\partial \Phi_\lambda(x - x_0}{\partial x_i}) \right|^p \omega_i(x) dx, \end{aligned}$$

where k_1 is the same above constant.

Since $u \in W^{1,p}(\Omega, \omega)$, then the first term on the right side of the inequality (*) is finite.

In addition we will show in Lemma 3.2, that,

$$I'_\lambda = \int_{\Omega_{2\lambda_0}} \int_{B(y,2\lambda)} \frac{|u(x) - u(y)|^p}{\lambda^p} \omega_i(x) dx \, dy \prec \infty.$$

Then,

$$\int_{B(y,2\lambda)} \frac{|u(x)-u(y)|^p}{\lambda^p}\omega_i(x)dx \prec \infty \text{ a.e } y,$$

which implies that the second term is also finite.

Thus $u_\lambda \in W^{1,p}(\Omega,\omega)$.

It is clear by using the Lebesgue theorem that

$$u_\lambda \to u \quad in \quad L^p(\Omega) \quad \text{as} \quad \lambda \to 0. \tag{13}$$

Therefore, it remains to show that,

$$\frac{\partial u_{\lambda_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad in \quad L^p(\Omega,\omega_i), \quad 1 \leq i \leq N, \tag{14}$$

for the sequence λ_k with $\lambda_k \to 0$ as $k \to \infty$.
By a simple calculation we find that ,

$$\tfrac{\partial(u-u_\lambda)(x)}{\partial x_i} = \tfrac{\partial u(x)}{\partial x_i}\Phi_\lambda(x-x_0) + \tfrac{\partial \Phi_\lambda(x-x_0)}{\partial x_i}(u(x)-u(x_0))$$

and by the convexity of the function $x \mapsto x^p$ we can write

$$\begin{aligned}\int_\Omega \left|\frac{\partial(u-u_\lambda)(x)}{\partial x_i}\right|^p \omega_i(x)dx \;&\leq\; 2^{p-1}\int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\Phi_\lambda(x-x_0)\right|^p \omega_i(x)dx \\ &+\; 2^{p-1}\int_\Omega \left|(u(x)-u(x_0)\frac{\partial\Phi_\lambda(x-x_0)}{\partial x_i}\right|^p \omega_i(x)dx.\end{aligned}$$

By virtue of Lebesgue theorem, the first term in the right expression of the above inequality converges to zero as $\lambda \to 0$, so it remains to show that :

$$\int_\Omega \left|(u(x)-u(x_0))\frac{\partial\Phi_\lambda(x-x_0)}{\partial x_i}\right|^p \omega_i(x)dx \to 0 \quad \text{as} \quad \lambda \to 0. \tag{15}$$

For this we use the following lemma.

Lemma 3.2. *For almost all $x_0 \in \Omega$, there exists a sequence $\lambda_k \succ 0$ with $\lambda_k \to 0$ as $k \to \infty$ such as*

$$\int_{B(x_0,2\lambda)} \frac{|u(x)-u(x_0)|^p}{\lambda_k^p}\omega_i(x)dx \to 0 \textit{ as } k \to \infty \textit{ for }, 1 \leq i \leq N.$$

Using the above lemma we conclude directly, and hence the proof of Theorem 3.1 is achieved.

Proof of Lemma 3.2. Let $x_0 \in \Omega$. For each $t \succ 0$, we define the set $\Omega_t = \{x \in \Omega; dist(x, \partial\Omega) \succ t\}$.

Let $\lambda_0 \succ 0$. For $\lambda \prec \lambda_0$, we consider the function $\psi_\lambda :\ \Omega_{2\lambda_0} \longrightarrow I\!R$ defined by

$$\psi_\lambda(y) = \int_{B(y,2\lambda)} \frac{|u(x) - u(y)|^p}{\lambda^p} \omega_i(x) dx. \tag{16}$$

Since $\psi_\lambda(y) = \displaystyle\int_\Omega \frac{|u(x) - u(y)|^p}{\lambda^p} \omega_i(x) \chi_{B(y,2\lambda)} dx$, then the function $\psi_\lambda : \Omega_{2\lambda_0} \rightarrow I\!R$ is mesurable ; χ_F, as usual denotes the charateristic function of the set F.

For all $\lambda_0 \succ 0$, we shall show that :

$$|\psi_\lambda(y)| \rightarrow 0 \quad \text{in} \quad L^1(\Omega_{2\lambda_0}) \quad \text{as} \quad \lambda \rightarrow 0, \quad \lambda \prec \lambda_0. \tag{17}$$

This obviously implies the statement of Lemma 3.2, (because if (17) is satisfied, then there is a subsequence λ_k converges at 0 as $k \rightarrow \infty$ and such that $\psi_{\lambda_k}(y) \rightarrow 0$ a.e. in $\Omega_{2\lambda_0}$).

Since λ_0 is arbitrary, then the previous convergence is true a.e. in Ω.

To verify (17), we denotes by $u_\delta = u * \Psi_\delta$ the mollification of u, where $\Psi_\delta \in D(I\!R^N)$, $\Psi_\delta = 1$ for $|x| \geq \delta$, $\Psi_\delta \geq 0$ and $\displaystyle\int_{I\!R^N} \Psi_\delta(x) dx = 1$. Hence, Ψ_δ is well defined in $\Omega_{2\lambda_0}$ for $\delta \prec \lambda_0$ and we have

$$\begin{aligned}
\int_{\Omega_{2\lambda_0}} |\psi_\lambda(y)|\, dy &= \int_{\Omega_{2\lambda_0}} \int_{B(y,2\lambda)} \frac{|u(x) - u(x_0)|^p}{\lambda^p} \omega_i(x) dx\ dy \\
&\leq lim_{\delta \rightarrow 0} \int_{\Omega_{2\lambda_0}} \int_{B(0,2\lambda)} \frac{|u_\delta(y - x) - u_\delta(y)|^p}{\lambda^p} \omega_i(x) dx\ dy
\end{aligned}$$

Since u_δ is continuously differentiable, we may estimate

$$I_\lambda = \int_{\Omega_{2\lambda_0}} \int_{B(0,2\lambda)} \frac{|u_\delta(y - x) - u_\delta(y)|^p}{\lambda^p} \omega_i(x) dx\ dy$$

In fact, we have

$$\begin{aligned}
I_\lambda &\leq \int_{\Omega_{2\lambda_0}} \int_{B(0,2\lambda)} (\lambda^{-p} \int_0^1 |\nabla u_\delta(y - tx)|^p \, |x|^p \, dt)\ \omega_i(x)\ dx\ dy \\
&\leq 2^p \int_{\Omega_{2\lambda_0}} \int_{B(0,2\lambda)} \int_0^1 |\nabla u_\delta(y - tx)|^p \, dt \omega_i(x) dx\ dy
\end{aligned}$$

Then,

$$
\begin{aligned}
I_\lambda &\leq \int_{\Omega_{2\lambda_0}}\int_{B(0,2\lambda)}\int_0^1 |\nabla u_\delta(y-tx)|^p\,\omega_i(x)\;dt\;dx\;dy\\
&\overset{(S)}{=} 2^p\int_0^1\int_{\Omega_{2\lambda_0}}\int_{B(0,2\lambda)}\left|\int_{B(0,\delta)}\nabla u(y-tx-z)\Psi_\delta(z)dz\right|^p \omega_i(x)\;dt\;dx\;dy\\
&\overset{(\Gamma)}{\leq} 2^p\int_0^1\int_{\Omega_{2\lambda_0}}\int_{B(0,2\lambda)}\int_{B(0,\delta)} |k_1\nabla u(y-tx-z)|^p\,dz\;\omega_i(x)dt\;dx\;dy\\
&= 2^pk_2\int_0^1\int_{B(0,2\lambda)}\int_{B(0,\delta)}(\int_{\Omega_{2\lambda_0}} |\nabla u(y-tx-z)|^p\,dy)\omega_i(x)dt\;dx\;dz\\
&\leq 2^pk_2\int_0^1\int_{B(0,2\lambda)}\int_{B(0,\delta)} \|\nabla u\|_p^p\,\omega_i(x)dt\;dx\;dz\\
&\leq 2^pk_2\,\|\nabla u\|_p^p\int_{B(0,2\lambda)}\int_{B(0,\delta)} dz\omega_i(x)dx\\
&\leq 2^pk_2(\frac{\sigma_N}{N})\delta^N\,\|\nabla u\|_p^p\int_{B(0,2\lambda)}\omega_i(x)dx\\
&\leq 2^pk_3(\frac{\sigma_N}{N})\lambda^N\,\|u\|_p^p\quad(\text{because}\quad\lambda\succ\delta)
\end{aligned}
$$

for some positive constants k_1,k_2 ,and k_3 (σ_N denotes the measure of the unit sphere in $I\!R^N$). Then, we obtain

$$I_\lambda \to 0 \quad \text{as} \quad \lambda\to 0.$$

Then it follows for $\lambda_0\succ 0$ that

$$\int_{\Omega_{2\lambda_0}}|\psi_\lambda(y)|\,dy\to 0\quad\text{as}\quad\lambda\to 0,\quad\lambda\prec\lambda_0,$$

which allows to conclude for almost every $x_0\in\Omega$, we have $\psi_{\lambda_k}(x_0)\to 0$ as $k\to\infty$. To justify (S), we recall that in $\Omega_{2\lambda_0}$ the differentiation and the mollification commute for $\delta\prec\lambda\prec\lambda_0$. (Γ) is in application of Jensen's inequality, which proves the statement of Lemma 3.2.

Remark 3.3. *In particular case when $\omega_i(x)\equiv 1$, for $0\leq i\leq N$ we obtain the statement of [2, Lemma 2.1], and in the Orlicz-Sobolev spaces we find Theorem 1 of [4].*

4 Characterization results

Let Ω be a bounded domain in $I\!R^N$, let $\omega = \{w_i, i = 0, 1, 2.., N\}$ be a family of weight functions with $w_0(x) \equiv 1$, such that w_i, $i = 1, ..., N$ satisfies (3), (4) and (10).

We consider the functional of kind

$$J(u) = \int_{\Omega} a(x, \nabla u)dx. \tag{18}$$

Where $J : W^{1,p}(\Omega, \omega) \rightarrow I\!R$ is continuous and $a : \Omega \times I\!R^N \rightarrow I\!R$ is a Carathéodory function satisfying

$$|a(x, \xi)| \leq T(x)M(|\xi|). \tag{19}$$

for some nondecreasing function $M : I\!R \rightarrow I\!R$ and some $T(x) \in L^1(\Omega)$.

For each τ we write H_τ for the level set of the functional J, i.e. $H_\tau = \{u \in W^{1,p}(\Omega, \omega) : J(u) = \tau\}$.

And for $\overline{H}_\tau^w$ for the closure of H_τ in $W^{1,p}(\Omega, \omega)$ for the weak topology.

Definition 4.1. *A functional $J : W^{1,p}(\Omega, \omega) \rightarrow I\!R$ is called weakly lower semicontinuous at a level set H_τ. If $J(u) \leq \tau$ for all $u \in \overline{H}_\tau^w$.*

Remark 4.2. *Note that this definition does not imply that $J_{/\overline{H}_\tau^w}$ is weakly lower semicontinuous.*

Theorem 4.3. *Let Ω be a bounded domain in $I\!R^N$, and let $\omega = \{w_i, i = 0, 1, 2.., N\}$ be a family of weight functions with $w_0(x) \equiv 1$, such that w_i, $i = 1, ..., N$ satisfies (3), (4) and (10).*

Let $J : W^{1,p}(\Omega, \omega) \rightarrow I\!R$ be a continuous functional defined as (18), with the Carathéodory function $a : \Omega \times I\!R^N \rightarrow I\!R$ satisfying (19).
If J is weakly lower semicontinous at nonvoid level set H_τ, then we have the alternative:
Either τ is an extreme value of J or for almost all $x \in \Omega$ the function $a(x, .)$ is convex.

Remark 4.4. *Note that when $a = a(u)$ or $a = a(x, u)$, we can easily adapt the same argument of [3].*

Proof of Theorem 4.3: Let assume that the level set τ is not an extreme value of J, then we shall show that

$$a(x, \alpha\xi + (1-\alpha)\xi^*) \leq \alpha a(x,\xi) + (1-\alpha)a(x,\xi^*)$$

for all $\alpha \in [0,1]$, for all $\xi, \xi^* \in I\!R^N$ and for $a.e.x \in \Omega$.

We can assume that $\tau = 0$ and that in $W^{1,p}(\Omega,\omega)$ there are two functions $\hat{h}_1$ and $\hat{h}_2$ such that $J(\hat{h}_1) \prec -\epsilon_0$ and $J(\hat{h}_2) \succ \epsilon_0$ for some $\epsilon_0 \succ 0$.

Let x_0 be a Lebesgue point of $a(x,\xi)$ for all $\xi \in I\!\!Q^N$. We can assume that $x_0 = 0$.

Using the continuity of the fonctional J and Theorem 3.1, there is a ball $B(0,R_0) \subset \Omega$ and there are $\bar{f}$, $\bar{f}_1$ and $\bar{f}_2$ (see [3]) such that,

$$\nabla\bar{f} = \nabla\bar{f}_1 = \nabla\bar{f}_2 = 0 \quad \text{on} \quad B(0,R_0), \tag{20}$$

$$J(\bar{f}_1) \prec \frac{7}{8}\epsilon_0, \quad J(\bar{f}_2) \succ \frac{7}{8}\epsilon_0 \quad \text{and} \quad \left|J(\bar{f})\right| \prec \frac{1}{8}\epsilon_0. \tag{21}$$

Furthermore for all function $\bar{h}$ satisfying $\left|J(\bar{h})\right| \prec \frac{7}{8}\epsilon_0$ there is $t_i \in [0,1]$ with $i = i(\bar{a}) \in \{1,2\}$ such that the function $\bar{\phi} = \bar{h} + t_i(\bar{f}_i - \bar{h})$ lies in the level set H_0, i.e. $J(\bar{\phi}) = 0$.

Let us now fix $\alpha \in [0,1] \bigcap I\!\!Q$ and $\xi, \xi^* \in I\!\!Q^N$. We define the sequence of functions

$$\hat{\phi}_n(x) = <\xi^*, x> + \int_0^{<\xi-\xi^*,x>} g_\alpha(nt)dt,$$

where $<,>$ denotes the usual inner product in $I\!R^N$ and

$$g_\alpha(x) = \begin{cases} 1 & \text{if} \quad 0 \prec t \prec \alpha \\ 0 & \text{if} \quad \alpha \prec t \prec 1 \end{cases}$$

We recall the fact that (see [3])

$$g_n(x) \rightharpoonup^* \alpha \qquad \text{in} \quad L^\infty(\Omega)$$

and

$$(1 - g_n(x)) \rightharpoonup^* (1-\alpha) \qquad \text{in} \quad L^\infty(\Omega)$$

It's clear that

$$\nabla\hat{\phi}_n(x) = \xi^* + (\xi - \xi^*)g_\alpha(n < \xi - \xi^*, x >),$$

from the convergence almost everywhere $\hat{\phi}_n(x) \to \hat{\phi}_0(x)$ we have convergence

$$\hat{\phi}_n \rightharpoonup \hat{\phi}_0 \qquad \text{in} \quad W^{1,p}(\Omega,\omega),$$

where

$$\hat{\phi}_0(x) =< \alpha\xi + (1-\alpha)\xi^*, x >$$

Let $\varphi : I\!R \to I\!R$ be a C^∞-function with support in the interval $(-1,1)$ and $\varphi(t) = 1$ for all $|t| \prec \frac{1}{2}$. Defining $\bar{\phi}_R(x)$ by $\bar{\phi}_R(x) = \varphi(\frac{|x|}{R})\hat{\phi}_0(x)$ for all $R \succ 0$, we calculate

$$\nabla\bar{\phi}_R(x) = \varphi'(\tfrac{|x|}{R})\tfrac{|x|}{R}\hat{\phi}_0(x) + \varphi(\tfrac{|x|}{R})\nabla\hat{\phi}_0(x)$$

Moreover, the function $\bar{\phi}_R(x) = \varphi(\frac{|x|}{R})\hat{\phi}_0(x)$ satisfying the properties (see [3, Proposition 3.1]):

$$\left|\nabla\bar{\phi}_R(x)\right| \leq c \quad \text{in} \quad \Omega. \tag{22}$$

$$\int_{B(0,R)} a(x, \nabla\bar{\phi}_R(x))dx \to 0 \quad \text{as} \quad R \to 0. \tag{23}$$

Note that (22) is used to prove (23).

Next we consider the sequence $\hat{\phi}_n(x)$ in a ball $B(0,r)$, say. We shall show that is possible alter each element of the sequence $\hat{\phi}_n(x)$ in the manner that it coincides with limit $\hat{\phi}_0(x)$ in the bundary.

The following lemma is generalization of [3, Proposition 3.2] in weigthed Sobolev spaces.

Lemma 4.5. *There is a sequence $h_n(x)$ in $W^{1,p}(\Omega,\omega)$ such that:*

i) $h_n(x) = \hat{\phi}_0(x) =< \alpha\xi + (1-\alpha)\xi^*, x >$ *in* $\partial B(0,r)$

ii) $h_n - \hat{\phi}_n \to 0$ *in* $W^{1,p}(\Omega,\omega)$ *as* $n \to \infty$

iii) $h_n \rightharpoonup \hat{\phi}_0$ *in* $W^{1,p}(\Omega,\omega)$ *for the weak topology.*

iv) $\|\nabla h_n\|_\infty + \left\|\nabla\hat{\phi}_n\right\|_\infty \leq c$

v) $\left|\int_{B(0,r)} a(x,\nabla\hat{\phi}_n)dx - \int_{B(0,r)} a(x,\nabla h_n)dx\right| \to 0$ *as* $n \to \infty$

vi) $\int_{B(0,r)} a(x,\nabla h_n)dx \to 0$ *as* $r \to 0$ *uniformly in* n.

Now, we are in a position to complete the proof of Theorem 4.3. For $R \leq R_0$ and $r = \frac{R}{2}$,we define the sequence:

$$\widehat{f}_n(x) = \begin{cases} \overline{f}(x) & \text{if} \quad x \in \Omega \backslash B(0,R), \\ \overline{f}(x) + \overline{\phi}_R(x) & \text{if} \quad x \in B(0,R) \backslash B(0,r), \\ \overline{f}(x) + h_n(x) & \text{if} \quad x \in B(0,r); \end{cases}$$

which converges in $W^{1,p}(\Omega, \omega)$ for the weak topology to

$$\widehat{f}_0(x) = \begin{cases} \overline{f}(x) & \text{for} \quad x \in \Omega \backslash B(0,R), \\ \overline{f}(x) + \overline{\phi}_{_R}(x) & \text{for} \quad x \in B(0,R). \end{cases}$$

We accont of (22) and (23) and Lemma 4.5, (as in [3] and [2]). We have for $R \succ 0$ small enough $\left| J(\overline{f}_n) \right| \prec \frac{7}{8}\epsilon_0$ for all n. Hence for any n, we find numbers $t_n \in [0,1]$ and $i_n \in \{1,2\}$, such that for $f_n = \widehat{f}_n + t_n(\overline{f}_{i_n} - \widehat{f}_n)$ we have $J(f_n) = 0$.

Now choosing a subsequence t_n such that $t_n \to t_0$ and $i_n = i$; $i \in \{1,2\}$, we have

$$f_n \rightharpoonup f_0 \quad \text{in} \quad W^{1,p}(\Omega, \omega) \quad \text{for the weak topolgy.}$$

Because, of the continuity of J with strong topology of $W^{1,p}(\Omega, \omega)$, we have

$$lim_{n \to \infty} J(\overline{f} + t_n(\overline{f}_{i_n} - \overline{f})) = J(\overline{f} + t_0(\overline{f}_i - \overline{f})),$$

and by constuction

$$a(x, \nabla(\overline{f} + t_n(\overline{f}_i - \overline{f})) = a(x,0) \text{ in } B(0,R)$$

because

$$\nabla \overline{f} = \nabla \overline{f}_1 = \nabla \overline{f}_2 = 0 \ \text{ in } \ B(0,R)$$

Yielding,

$$lim_{n \to \infty} \int_{B(0,R)} a(x, \nabla f_n(x)) dx \geq \int_{B(0,R)} f(x, \nabla f_0(x)) dx.$$

Since $f_n = f_0$ in $B(0,R) \backslash B(0,r)$, $r = \frac{R}{2}$, we finally get

$$\begin{aligned} \int_{B(0,r)} a(x, \alpha\xi + (1-\alpha)\xi^*) dx &= \int_{B(0,r)} a(x, \nabla f_0(x)) dx \\ &\leq lim_{n \to \infty} \int_{B(0,r)} a(x, \nabla f_n)(x)) dx \\ &= lim_{n \to \infty} \int_{B(0,R)} a(x, \nabla h_n(x)) dx \\ &= \alpha \int_{B(0,r)} a(x, \xi) dx + (1-\alpha) \int_{B(0,r)} a(x, \xi^*) dx. \end{aligned}$$

Since the above inequality can be obtained for all $B(0,r)$ with radius $r \prec \frac{R}{2}$, we conclude that $a(x_0, \alpha\xi + (1-\alpha)\xi^*) \leq \alpha f(x_0,\xi) + (1-\alpha)f(x_0,\xi^*)$ for all $\alpha \in [0,1] \cap I\!\!Q$ and all $\xi, \xi^* \in Q^N$. It then follows by the continuity of $a(x,\xi)$ with respect to ξ, that the above inequality holds for all $\lambda \in [0,1]$ and all $\xi, \xi^* \in I\!\!R^N$.

Proof of Lemma 4.5: Let $\tilde{k}_\delta$ be a C^∞ -function with support in $[-1,1]$ such that $\tilde{k}_\delta(t) = 1$ for all $|t| \prec 1-\delta$ and $\left|\tilde{k}_\delta'\right| \prec \frac{2}{\delta}$ for all t.

Defining the function $k_\delta(x) = \tilde{k}_\delta(\frac{|x|}{r})$ and $h_{n,\delta}(x) = \hat{\phi}_0(x) + k_\delta(x)(\hat{\phi}_n(x) - \hat{\phi}_0(x))$

we have the following inequality

$$|\nabla k_\delta(x)| \left|\hat{\phi}_n(x) - \hat{\phi}_0(x)\right| \leq O(n^{-1})\frac{1}{\delta}\chi_{supp}(\nabla k_\delta). \tag{24}$$

$$\left|\nabla(\hat{\phi}_n(x) - \hat{\phi}_0(x))\right|(1-k_\delta(x)) \leq c' r(|\xi^*| + |\xi|)(1-k_\delta(x)). \tag{25}$$

$$\left\|h_{n,\delta} - \hat{\phi}_n\right\|_{1,p,\omega}^p \leq O(\delta) + c\sum_{i=1}^N \int_{B(0,r)} \left|\nabla((\hat{\phi}_n(x) - \hat{\phi}_0(x))(1-k_\delta(x)))\right|^p \omega_i(x)dx. \tag{26}$$

for some positive constants c and c'.

For (24) and (25) see the proof of [3 ,proposition 3.2].

Assume now that(26) is true , thus we get

$$k_\delta(x) = \begin{cases} 0 & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 1 & \text{in} \quad B(0,(1-\delta)r) \\ \tilde{k_\delta}(\frac{|x|}{r}) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

which implies that

$$h_{n,\delta}(x) - \hat{\phi}_n(x) = \begin{cases} \hat{\phi}_0(x) - \hat{\phi}_n(x) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ (1-\tilde{k_\delta}(\frac{|x|}{r}))(\hat{\phi}_0(x) - \hat{\phi}_n(x)) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

and

$$\nabla((h_{n,\delta} - \hat{\phi}_n)(x)) = \begin{cases} \nabla(\hat{\phi}_0(x) - \hat{\phi}_n(x)) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ \nabla(\tilde{k_\delta}(\frac{|x|}{r}))((\hat{\phi}_n - \hat{\phi}_0)(x)) & \\ +(1-\tilde{k_\delta}(\frac{|x|}{r}))\nabla((\hat{\phi}_0 - \hat{\phi}_n)(x)) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

Hence, we have the estimate

$$\begin{aligned}
&\int_\Omega \left|h_{n,\delta}-\hat{\phi}_n\right|^p dx+\sum_{i=1}^N\int_\Omega\left|\frac{\partial}{\partial x_i}(h_{n,\delta}-\hat{\phi}_n)\right|^p\omega_i(x)dx\\
&\le O(\delta)+c\sum_{i=1}^N\int_{B(0,r)\setminus B(0,(1-\delta)r)}\left|\nabla((\hat{\phi}_n(x)-\hat{\phi}_0(x))(1-k_\delta(x)))\right|^p\omega_i(x)dx\\
&\le O(\delta)+c(c_1O(n^{-1})\frac{1}{\delta})^p\sum_{i=1}^N\int_{B(0,r)\setminus B(0,(1-\delta)r)}\omega_i(x)dx\\
&\le O(\delta)+Ncc_2(c_1O(n^{-1})\frac{1}{\delta})^p,
\end{aligned}$$

with $c_2=\max\limits_{i=1,..N}\int_\Omega\omega_i(x)dx$ (because $\omega_i\in L^1_{loc}(\Omega)$).
Selecting numbers δ_n such that $O(n^{-1})\frac{1}{\delta_n}=1$, this implies that $O(\delta_n)=O(n^{-1})$ and $\delta_n\to 0$ as $n\to\infty$.
then, we conclude that,

$$\left\|h_{n,\delta}-\hat{\phi}_n\right\|^p_{1,p,\omega}\le O(n^{-1})+Ncc_2(c_1O(n^{-1})\frac{1}{\delta})^p$$

which conveges to 0 as $n\to\infty$. We define the functions $h_n=h_{n,\delta}$ and we have

$$\left\|h_{n,\delta}-\hat{\phi}_n\right\|_{1,p,\omega}\to 0 \text{ as } n\to 0.$$

Which gives (ii) in lemma 4.5 and

$$h_n-\hat{\phi}_0=(h_n-\hat{\phi}_n)+(\hat{\phi}_n-\hat{\phi}_0)\rightharpoonup 0 \text{ in } W^{1,p}(\Omega,\omega) \text{ for the weak topology}$$

(because $(\hat{\phi}_n-\hat{\phi}_0)\rightharpoonup 0$ in $W^{1,p}(\Omega,\omega)$.

The properties i), iv) and vi) are satisfied by the definition of h_n. Now, we return to show the inequality (26). In fact, we can write

$$\begin{aligned}
&\int_\Omega \left|h_{n,\delta}-\hat{\phi}_n\right|^p dx+\sum_{i=1}^N\int_\Omega\left|\frac{\partial}{\partial x_i}(h_{n,\delta}-\hat{\phi}_n)\right|^p\omega_i(x)dx\\
&=\int_{\Omega\setminus\overline{B}(0,r)}\left|\hat{\phi}_n-\hat{\phi}_0\right|^p dx+\int_{\overline{B}(0,r)}(\left|\hat{\phi}_n-\hat{\phi}_0\right|(1-k_\delta))^p dx\\
&+\sum_{i=1}^N\int_{\Omega\setminus\overline{B}(0,r)}\left|\frac{\partial}{\partial x_i}(\hat{\phi}_n-\hat{\phi}_0)\right|^p\omega_i(x)dx+\sum_{i=1}^N\int_{\overline{B}(0,r)}\left|\frac{\partial}{\partial x_i}((\hat{\phi}_n-\hat{\phi}_0)(1-k_\delta))\right|^p\omega_i(x)dx\\
&\le\int_{\Omega\setminus\overline{B}(0,r)}\left|\hat{\phi}_n-\hat{\phi}_0\right|^p dx+\int_{\overline{B}(0,r)}(\left|\hat{\phi}_n-\hat{\phi}_0\right|(1-k_\delta))^p dx
\end{aligned}$$

$$+\sum_{i=1}^{N}\int_{\Omega\setminus\overline{B}(0,r)}\left|\frac{\partial}{\partial x_i}(\hat{\phi}_n-\hat{\phi}_0)\right|^p\omega_i(x)dx+\sum_{i=1}^{N}\int_{\overline{B}(0,r)}\left|\nabla((\hat{\phi}_n-\hat{\phi}_0)(1-k_\delta))\right|^p\omega_i(x)dx.$$

Since $(1-k_\delta(x))\to 0$ a.e in $\overline{B}(0,r)$

and

$$\int_{\Omega\setminus\overline{B}(0,r)}\left|\hat{\phi}_n-\hat{\phi}_0\right|^p dx+\sum_{i=1}^{N}\int_{\Omega\setminus\overline{B}(0,r)}\left|\frac{\partial}{\partial x_i}(\hat{\phi}_n-\hat{\phi}_0)\right|^p\omega_i(x)dx\to 0 \text{ as } n\to\infty,$$

then we conclude that

$$\left\|h_{n,\delta}-\hat{\phi}_n\right\|_{1,p,\omega}^p\leq$$
$$O(\delta)+c\sum_{i=1}^{N}\int_{B(0,r)}\left|\nabla((\hat{\phi}_n(x)-\hat{\phi}_0(x))(1-k_\delta(x)))\right|^p\omega_i(x)dx,$$

which implies the inequality (26).

Proposition 4.6. *The sequence of function $\hat{\phi}_n$ defined by,*

$$\hat{\phi}_n(x)=<\xi^*,x>+\int_0^{<\xi-\xi^*,x>}g_\alpha(nt)dt$$

satisfying the following propertie:

(i) $\hat{\phi}_n(x)\to\hat{\phi}_0(x)$ *for almost all* $x\in\Omega$ *where* $\hat{\phi}_0(x)=<\alpha\xi+(1-\alpha)\xi^*,x>$

(ii) $\hat{\phi}_n\to\hat{\phi}_0$ *in* $W^{1,p}(\Omega,\omega)$

Proof of Proposition 4.6: It is obvious to show i) (see [3]).
Now show ii) Since Ω bounded and

$$g_n(x)\rightharpoonup^*\alpha\qquad\text{in}\qquad L^\infty(\Omega)$$

then

$$\begin{aligned}
\int_{\Omega}\left|(\hat{\phi_n}-\hat{\phi_0})(x)\right|^p dx &= \int_{\Omega}\left|\int_0^{\langle\xi-\xi^*,x\rangle} g_\alpha(nt)dt-\alpha\left\langle\xi-\xi^*,x\right\rangle\right|^p dx \\
&= \int_{\Omega}\left|\int_0^{\langle\xi-\xi^*,x\rangle}(g_\alpha(nt)-\alpha)dt\right|^p dx \\
&\leq \frac{1}{n^p}\int_{\Omega}\left|\int_0^{\frac{\langle\xi-\xi^*,x\rangle}{n}}(g_\alpha(t)-\alpha)dt\right|^p dx \\
&\leq k\tfrac{1}{n^p}
\end{aligned}$$

then, $\hat{\phi_n}\to\hat{\phi_0}$ in $L^p(\Omega)$

$$\begin{aligned}
\int_{\Omega}\left|\frac{\partial(\hat{\phi_n}-\hat{\phi_0})(x)}{\partial x_i}\right|^p w_i(x)dx &\leq \int_{\Omega}\left|\nabla(\hat{\phi_n}-\hat{\phi_0})(x)\right|^p w_i(x)dx \\
&\leq \int_{\Omega}\left|(\xi-\xi^*)(g_\alpha(n\left\langle\xi-\xi^*,x\right\rangle)-\alpha)\right|^p w_i(x)dx \\
&\leq \left|(\xi-\xi^*)\right|^p\int_{\Omega}\left|(g_\alpha(n\left\langle\xi-\xi^*,x\right\rangle)-\alpha)\right|^p w_i(x)dx
\end{aligned}$$

By using the lebesgue theorem $\int_{\Omega}\left|(g_\alpha(n\left\langle\xi-\xi^*,x\right\rangle)-\alpha)\right|^p w_i(x)dx \to 0$ as $n\to\infty$,then

$$\tfrac{\partial\hat{\phi_n}}{\partial x_i}\to\tfrac{\partial\hat{\phi_0}}{\partial x_i} \quad \text{in} \quad L^p(\Omega,w_i) \quad \text{for } i=1,...,N. \text{ Then}$$

$$\hat{\phi_n}\to\hat{\phi_0} \quad \text{in} \quad W^{1,p}(\Omega,\omega)\hat{c}_0$$

Corollary 4.7. *Under the same assumptions as in Theorem 4.1 suppose that there is a nonvoid weakly closed level set H_τ. If τ is not an extreme value of J, then the function $a(x,\nabla u(x))$ is affine in the gradient.*

Remark 4.8. *In the particular case when $w_i(x)=1$ for $i=0,1,...,N,$ we obtain the statement of [3, Theorem 3-1], and in the Orlicz-Soblev spaces we found the theorem 6 of [4].*

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