A Method for Finding Nonlinear Approximation of Bifurcation Solutions of Some Nonlinear Differential Equations

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Abstract

This paper introduce a method for finding nonlinear approximation of the solutions of some nonlinear partial differential equations by using Lyapunov-Schmidt reduction. Also, it provides an example for finding nonlinear approximation of bifurcation of periodic solutions of Duffing equation.

Mathematics Subject Classification: 34K18; 34K10

Keywords: Bifurcation solutions; Local scheme of Lyapunov-Schmidt; Duffing equation

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Article Info: Received: June 10, 2013. Revised: August 20, 2013. Published online: September 15, 2013.
1 Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

\[ f(x, \lambda) = b, \quad x \in O \subset X, \quad b \in Y, \quad \lambda \in \mathbb{R}^n \]  \hspace{1cm} (1)

where \( f \) is a smooth Fredholm map of index zero and \( X, Y \) are Banach spaces and \( O \) is open subset of \( X \). For these problems, the method of reduction to finite dimensional equation,

\[ \theta(\xi, \lambda) = \beta, \quad \xi \in M, \quad \beta \in N \]  \hspace{1cm} (2)

can be used, where \( M \) and \( N \) are smooth finite dimensional manifolds.

A passage from equation (1) into equation (2) (variant local scheme of Lyapunov-Schmidt) with the conditions that equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, etc) can be found in [3], [11], [13], [14].

Suppose that \( f : \Omega \subset E \to F \) is a nonlinear Fredholm map of index zero. A smooth map \( f : \Omega \subset E \to F \) has variational property, if there exists a functional \( V : \Omega \subset E \to \mathbb{R} \) such that \( f = \text{grad}_HV \) or equivalently,

\[ \frac{\partial V}{\partial x}(x, \lambda)h = \langle f(x, \lambda), h \rangle_H, \quad \forall \ x \in \Omega, \ h \in E, \]

where \( \langle \cdot, \cdot \rangle_H \) is the scalar product in Hilbert space \( H \). In this case, the solutions of equation \( f(x, \lambda) = 0 \) are the critical points of functional \( V(x, \lambda) \). Suppose that \( f : E \to F \) is a smooth Fredholm map of index zero, \( E, F \) are Banach spaces and

\[ \frac{\partial V}{\partial x}(x, \lambda)h = \langle f(x, \lambda), h \rangle_H, \quad h \in E, \]

where \( V \) is a smooth functional on \( E \). Also it is assumed that \( E \subset F \subset H, H \) is a Hilbert space. By using a method of finite dimensional reduction (Local scheme of Lyapunov-Schmidt) the problem,

\[ V(x, \lambda) \to \text{extr} \quad x \in E, \ \lambda \in \mathbb{R}^n \]

can be reduced into equivalent problem

\[ W(\xi, \lambda) \to \text{extr} \quad \xi \in \mathbb{R}^n. \]
The function $W(\xi, \lambda)$ is called key function.

If $N = \text{span}\{e_1, ..., e_n\}$ is a subspace of $E$, where $e_1, ..., e_n$ is an orthonormal set in $H$, then the key function $W(\xi, \lambda)$ can be defined in the form of

$$W(\xi, \lambda) = \inf_{x: \langle x, e_i \rangle = \xi_i \forall i} V(x, \lambda), \quad \xi = (\xi_1, ..., \xi_n).$$

The function $W$ has all the topological and analytical properties of the functional $V$ (multiplicity, bifurcation diagram, etc.) [13]. The study of bifurcation solutions of functional $V$ is equivalent to the study of bifurcation solutions of key function. If $f$ has a variational property, then the equation

$$\theta(\xi, \lambda) = \text{grad}W(\xi, \lambda) = 0$$

is called bifurcation equation.

It is well known that in the method of Lyapunov-Schmidt, the space $E$ is decomposed into two orthogonal subspaces of the space $E$ and then every element $x \in E$ can be written in the unique form as a sum of two elements such that the solution of the equation (1) consists of the homogeneous solution and the particular solution. Sapronov and his group [2, 12] used the complement solution to find the function $W(\xi, \lambda)$ which denotes the linear Ritz approximation of the functional $V(x, \lambda)$. This paper introduce a method to find nonlinear Ritz approximation of the functional $V(x, \lambda)$, such a method is based on finding the particular solution of the equation (1).

\section{The Method}

Consider the nonlinear Fredholm operator of index zero $f : E \rightarrow F$ defined by the equation

$$f(u, \lambda) = 0, \quad \lambda \in R^n, \quad u \in \Omega \subset E, \quad (3)$$

where $E, F$ are real Banach spaces and $\Omega$ is an open subset of $E$. Assume that the operator $f$ has a variational property, i.e, there exists a functional $V : \Omega \subset E \rightarrow R$ such that $f = \text{grad}_H V$ where $\Omega$ is a bounded domain. The operator $f$ can be written as

$$f(u, \lambda) = Au + Nu = 0,$$
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where $A = \frac{\partial f}{\partial u}(u_0, \lambda)$ is a linear continuous Fredholm operator, $\frac{\partial f}{\partial u}(u_0, \lambda)$ the Frechet derivative of the operator $f$ at the point $u_0$ and $N$ the nonlinear operator. By using Lyapunov-Schmidt reduction, the decomposition is obtained below

$$E = M \oplus M^\perp$$
$$F = \tilde{M} \oplus \tilde{M}^\perp$$

where $M = \ker A$ is the null space of the operator $A$, $\dim M = dim \tilde{M} = n$ and $M^\perp$, $\tilde{M}^\perp$ are the orthogonal complements of the subspaces $M$ and $\tilde{M}$ respectively. If $e_1, e_2, ..., e_n$ is an orthonormal set in $H$ such that $Ae_i = \alpha_i(\lambda)e_i$, $\alpha_i(\lambda)$ is continuous function, $i = 1, ..., n$, then every element $u \in E$ can be represented in the unique form of

$$u = w + v, \quad w = \sum_{i=1}^{n} \xi_i e_i \in M, \quad M^\perp v \in M^\perp, \quad \xi_i = \langle u, e_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Hilbert space $H$. There exist projections $p : E \to M$ and $I - p : E \to M^\perp$ such that $w = pu$ and $(I - p)u = v$. Similarly, there exist projections $Q : F \to \tilde{M}$ and $I - Q : F \to \tilde{M}^\perp$ such that

$$f(u, \lambda) = Qf(u, \lambda) + (I - Q)f(u, \lambda)$$  \hspace{1cm} (4)

or

$$f(w + v, \lambda) = Qf(w + v, \lambda) + (I - Q)f(w + v, \lambda).$$

It follows that

$$Qf(w + v, \lambda) + (I - Q)f(w + v, \lambda) = 0,$$

and hence the result become

$$Qf(w + v, \lambda) = 0,$$
$$ (I - Q)f(w + v, \lambda) = 0.$$

The implicit function theorem implies that

$$W(\xi, \delta) = V(\Phi(\xi, \delta), \delta), \quad \xi = (\xi_1, \xi_2, ..., \xi_n)^T,$$

where $\deg W \geq 2$, then the linear Ritz approximation of the functional $V$ is a function $W$ defined by

$$W(\xi, \delta) = V\left(\sum_{i=1}^{n} \xi_i e_i, \delta \right) = W_0(\xi) + W_1(\xi, \delta),$$  \hspace{1cm} (5)
where $W_0(\xi)$ is a homogenous polynomial of order $n \geq 3$ such that $W_0(0) = 0$ and $W_1(\xi, \delta)$ is a polynomial function of degree less than $n$.

Let $q_1, q_2, ..., q_m$ be the coefficients of the quadratic terms of the function $W_1(\xi, \delta)$, then the function $W_1(\xi, \delta)$ can be written in the form of

$$W_1(\xi, \delta) = W_2(\xi, \delta) + \sum_{k=1}^{m} q_k \xi_k^2$$

where $\text{deg}W_2 = d, \ 2 < d < n$.

The nonlinear Ritz approximation of the functional $V$ is a function $W$ defined by

$$W(\xi, \delta) = V\left(\sum_{i=1}^{n} \xi_i e_i + \Phi\left(\sum_{i=1}^{n} \xi_i e_i, \delta\right)\right),$$

where $\Phi(w, \delta) = v(x, \xi, \delta), \ v \in N^\perp$.

To determine the nonlinear Ritz approximation of the functional $V$, Taylor’s expansion of the functions $\mu_k(\xi)$ and $v(x, \xi, \delta)$ is used by assuming the following

$$q_k = \hat{q}_k + \mu_k(\xi) = \hat{q}_k + \sum_{j=2}^{r} D^{(j)}_k(\xi), \quad k = 1, ..., m$$

$$v(x, \xi, \delta) = \sum_{j=2}^{r} B^{(j)}(\xi),$$

where $D^{(j)}_k(\xi)$ and $B^{(j)}(\xi)$ are homogenous polynomials of degree $j$ with coefficients $\mu_{ki}$ and $v_{ji}(x, \delta)$ respectively, $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

Since

$$Qf(u, \lambda) = \sum_{i=1}^{n} \langle f(u, \lambda), e_i \rangle e_i = 0$$

it follows that

$$\sum_{i=1}^{n} \langle Au + Nu, e_i \rangle e_i = 0.$$ 

Hence

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \langle Nu, e_i \rangle e_i = 0, \quad q_i = \alpha_i(\lambda)$$

or

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \left[ \int_{\Omega} N(w + v)e_i \right] e_i = 0. \quad (6)$$
From (4) it gets

$$(I - Q)f(u, \lambda) = f(u, \lambda) - Qf(u, \lambda),$$

but

$$A(w + v) + N(w + v) = 0 \quad (7)$$

it follows that

$$Av + N(w + v) + \sum_{i=1}^{n} q_i \xi_i e_i = 0.$$  

Substitute the values of $q_i$, $\mu_i(\xi)$ and $v(x, \xi, \delta)$ in (6) and (7) yields

$$\sum_{i=1}^{n} \left[ \hat{q}_i + \sum_{j=2}^{r} D_i^{(j)}(\xi) \right] \xi_i e_i +$$

$$+ \sum_{i=1}^{n} \left[ \int_{\Omega} N \left( \sum_{i=1}^{n} \xi_i e_i + \sum_{j=2}^{r} B^{(j)}(\xi) \right) e_i \right] e_i = 0, \quad (8)$$

$$A \left( \sum_{j=2}^{r} B^{(j)}(\xi) \right) + N \left( \sum_{i=1}^{n} \xi_i e_i + \sum_{j=2}^{r} B^{(j)}(\xi) \right) +$$

$$+ \sum_{i=1}^{n} \left( \hat{q}_i + \sum_{j=2}^{r} D_i^{(j)}(\xi) \right) \xi_i e_i = 0. \quad (9)$$

To determine the functions $v(x, \xi, \lambda)$ and $\mu_k(\xi)$ we equating the coefficients of $\hat{\xi} = \xi_1 \xi_2 \ldots \xi_n$ in the equation (8) to find the value of $\mu_{ki}$ and after some calculations of equation (9) it is obtained a linear ordinary differential equation in the variable $v_{ji}(x, \lambda)$. Solve the resulting equation one can find the value of $v_{ji}(x, \lambda)$.

### 3 Applications

In [8] the author introduced an example to find nonlinear approximation of bifurcation solutions of the fourth order differential equation,

$$\frac{d^4 u}{dx^4} + \alpha \frac{d^2 u}{dx^2} + \beta u + u^3 = 0.$$  

This equation also was studied by [1], [4], [5], [6], [7], [9], [10] with different nonlinear terms. The purpose of this study in hand is to apply the method
in the previous section to find the bifurcation of periodic solutions of Duffing equation of type

\[ \ddot{u} + \lambda u - u^3 = 0, \]  

(10)

with resonance 1:1. For simply, this equation is chosen because another equation may give more difficulty in the calculations and then the study could obtain more difficult key function. Suppose that \( f : E \to F \) is a nonlinear Fredholm operator of index zero from Banach space \( E \) to Banach space \( F \) defined by,

\[ f(u, \lambda) = \frac{d^2u}{dt^2} + \lambda u - u^3, \]

(11)

where \( E = \Pi^2([0, 2\pi], \mathbb{R}) \) is the space of all periodic continuous functions that have derivative of order at most two, \( F = \Pi([0, 2\pi], \mathbb{R}) \) is the space of all periodic continuous functions where \( u = u(t), \ t \in [0, 2\pi] \). In this case, the solutions of equation (10) is equivalent to the solutions of the operator equation given below

\[ f(u, \lambda) = 0. \]

(12)

It has been noticed that the operator \( f \) has a variational property, i.e. there exists a functional \( V \) such that \( f(u, \lambda) = \text{grad}_H V(u, \lambda) \) or equivalently,

\[ \frac{\partial V}{\partial u}(u, \lambda)h = \langle f(u, \lambda), h \rangle_H, \ \forall \ u \in \Omega, \ h \in E, \]

where \( \langle \cdot, \cdot \rangle_H \) is the scalar product in Hilbert space \( H \) and

\[ V(u, \lambda) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{(\dot{u})^2}{2} + \lambda u^2 - \frac{u^4}{4} \right) dt. \]

In this case, the solutions of equation (10) are the critical points of the functional \( V(u, \lambda) \), where the critical points of the functional \( V(u, \lambda) \) are the solutions of Euler-Lagrange equation

\[ \frac{\partial V}{\partial u}(u, \lambda)h = \frac{1}{2\pi} \int_0^{2\pi} (\ddot{u} + \lambda u - u^3)h \ dt = 0, \]

and \( \frac{\partial V}{\partial \lambda}(u, \lambda) \) is the Frechet derivative of the functional \( V(u, \lambda) \). Therefore, the study of equation (10) is equivalent to the study extremely problem,

\[ V(u, \lambda) \to \text{extr}, \quad u \in E. \]
The analysis of bifurcation can be found by using the method of Lyapunov-Schmidt to reduce the problem into finite dimensional space. By localized parameter,
\[ \lambda = \tilde{\lambda} + \delta_1, \quad \delta_1 \text{ is small parameter,} \]
the reduction leads to the function in two variables defined by
\[ W(\xi, \lambda) = \inf_{(u, e_i) = \xi_i, i=1,2} V(u, \lambda), \quad \xi = (\xi_1, \xi_2). \]

It is well known that in the reduction of Lyapunov-Schmidt, the function \( W(\xi, \lambda) \) is smooth. This function has all the topological and analytical properties of functional \( V \) [13]. For small \( \lambda \), there is one-to-one corresponding between the critical points of functional \( V \) and smooth function \( W \), preserving the type of critical points (multiplicity, index Morse, etc) [13]. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (12) has the following form:
\[ \ddot{h} + \lambda h = 0, \quad h \in E \]
The point \((0, \lambda) = (0, 1)\) is a bifurcation point of equation (10) [13].

The localized parameter
\[ \tilde{\lambda} = 1 + \delta_1, \]
leads to the bifurcation along the modes \( e_1 = c_1 \sin t, \quad e_2 = c_2 \cos t \), where \( \|e_1\| = \|e_2\| = 1, \ c_1 = c_2 = \sqrt{2} \). Let \( N = \text{Ker}(A) = \text{span}\{e_1, e_2\} \), where, \( A = f_u(0, \lambda) = \frac{d^2}{dt^2} + \lambda \), then the space \( E \) can be decomposed in direct sum of two subspaces, \( N \) and the orthogonal complement to \( N \),
\[ E = N \oplus \hat{E}, \quad \hat{E} = N^\perp \cap E = \{v \in E : v \perp N\}. \]

Similarly, the space \( F \) decomposed in direct sum of two subspaces, \( N \) and orthogonal complement to \( N \),
\[ F = N \oplus \hat{F}, \quad \hat{F} = N^\perp \cap F = \{v \in F : v \perp N\}. \]

There exist projections \( p : E \to N \) and \( I - p : E \to \hat{E} \) such that \( pu = w \) and \((I - p)u = v \), \((I \text{ is the identity operator})\). Hence, every vector \( u \in E \) can be written in the form,
\[ u = w + v, \quad w = \sum_{i=1}^{2} \xi_i e_i \in N, \quad N \perp v \in \hat{E}, \quad \xi_i = \langle u, e_i \rangle. \]
Similarly, there exist projections $Q : F \to N$ and $I - Q : F \to \hat{F}$ such that
\[
f(u, \lambda) = Qf(u, \lambda) + (I - Q)f(u, \lambda)
\] (13)

Accordingly, equation (12) can be written in the following form,
\[
Qf(w + v, \lambda) = 0,
\]
\[
(I - Q)f(w + v, \lambda) = 0.
\]

By the implicit function theorem, there exist a smooth map $\Phi : N \to \hat{E}$, such that
\[
W(\xi, \delta_1) = V(\Phi(\xi, \lambda), \delta_1).
\]

And then the linear Ritz approximation of the functional $V$ is a function $W$ given by,
\[
W(\xi, \delta_1) = V(\xi_1 e_1 + \xi_2 e_2, \delta_1) = \xi_1^4 + 4\xi_1^2 \xi_2^2 + \xi_2^4 + \frac{q_1}{2} \xi_1^2 + \frac{q_2}{2} \xi_2^2.
\]

The nonlinear Ritz approximation of the functional $V$ is a function $W$ given by
\[
W(\xi, \delta_1) = V(\xi_1 e_1 + \xi_2 e_2, \Phi(\xi_1 e_1 + \xi_2 e_2, \delta_1), \delta_1), \quad v(t, \xi, \lambda) = \Phi(w, \delta_1).
\]

To determine the nonlinear Ritz approximation of the functional $V$, the functions $v(t, \xi, \lambda) = O(\xi^3)$, $\mu(\xi) = O(\xi^2)$ and $\tilde{\mu}(\xi) = O(\xi^2)$ must be found in the form of power series in term of $\xi$, as follows:
\[
v(t, \xi, \lambda) = v_0(t, \lambda)\xi_1^3 + v_1(t, \lambda)\xi_1^2\xi_2 + v_2(t, \lambda)\xi_1\xi_2^2 + v_3(t, \lambda)\xi_2^3 + ..., \]
\[
\mu(\xi) = \mu_0\xi_1^2 + \mu_1\xi_1\xi_2 + \mu_2\xi_2^2 + ..., \]
\[
\tilde{\mu}(\xi) = \tilde{\mu}_0\xi_1^2 + \tilde{\mu}_1\xi_1\xi_2 + \tilde{\mu}_2\xi_2^2 + ..., \]
(14)

where $q_1 = \tilde{q}_1 + \mu(\xi_1, \xi_2)$, $q_2 = \tilde{q}_2 + \tilde{\mu}(\xi_1, \xi_2)$ and $\xi = (\xi_1, \xi_2)$. Equation (12) can be written in the form of
\[
f(u, \lambda) = Au + Tu = 0, \quad Tu = -u^3.
\]

Since,
\[
Qf(u, \lambda) = \sum_{i=1}^{2} (f(u, \lambda), e_i)e_i = 0.
\]
Then the result takes the form of
\[ \sum_{i=1}^{2} \langle Au + Tu, e_i \rangle e_i = 0, \]
and hence
\[ q_1 \xi_1 e_1 + q_2 \xi_2 e_2 - \left( \frac{1}{2\pi} \int_{0}^{2\pi} (\xi_1 e_1 + \xi_2 e_2 + v)^3 e_1 \, dt \right) e_1 \]
\[ - \left( \frac{1}{2\pi} \int_{0}^{2\pi} (\xi_1 e_1 + \xi_2 e_2 + v)^3 e_2 \, dt \right) e_2 = 0. \]  
(15)

From (13) and (15) it is obtained
\[ \ddot{v} + \lambda v - (\xi_1 e_1 + \xi_2 e_2 + v)^3 + q_1 \xi_1 e_1 + q_2 \xi_2 e_2 = 0. \]  
(16)

It follows that,
\[ \left[ (\bar{q}_1 + \mu(\xi_1, \xi_2)) \xi_1 - \xi_1^3 \frac{1}{2\pi} \int_{0}^{2\pi} e_1^4 dt - 3\xi_1 \xi_2 \frac{1}{2\pi} \int_{0}^{2\pi} e_1^3 e_2 \, dt \right] e_1 \]
\[ - 3\xi_1 \xi_2 \frac{1}{2\pi} \int_{0}^{2\pi} e_1^3 e_2 \, dt - \xi_2^3 \frac{1}{2\pi} \int_{0}^{2\pi} e_2^3 e_2 \, dt - \frac{1}{2\pi} \int_{0}^{2\pi} \theta_1(\xi_1, \xi_2, v) \, dt \right] e_1 \]
\[ + \left[ (\bar{q}_2 + \bar{\mu}(\xi_1, \xi_2)) \xi_2 - \xi_2^3 \frac{1}{2\pi} \int_{0}^{2\pi} e_2^3 e_2 \, dt - 3\xi_1 \xi_2 \frac{1}{2\pi} \int_{0}^{2\pi} e_1^3 e_2 \, dt \right] e_2 = 0, \]
\[ \ddot{v} + \lambda v - \xi_1^3 e_1^3 - 3\xi_1^2 \xi_2 e_1^2 e_2 - 3\xi_1 \xi_2^2 e_2 e_1^2 - \xi_2^3 e_2^3 - v^3 - 3v^2 \xi_1 e_1 \]
\[ - 3v^2 e_2 e_1 - 3v^2 \xi_2^2 e_1^2 - 6v \xi_1 \xi_2 e_1 e_2 - 3v \xi_2^2 e_2^2 \]
\[ + (\bar{q}_1 + \mu(\xi_1, \xi_2)) \xi_1 e_1 + (\bar{q}_2 + \bar{\mu}(\xi_1, \xi_2)) \xi_2 e_2 = 0. \]  
(17)

where
\[ \theta_1(\xi_1, \xi_2, v) = 3\xi_1^3 e_1^3 v + 6\xi_1 \xi_2 e_1^2 e_2 v + 3\xi_1 e_1^3 v^2 + 3\xi_2 e_2 e_1^2 v + 3\xi_2 e_1 e_1 v^2 + e_1 v^3 \]
\[ \theta_2(\xi_1, \xi_2, v) = 3\xi_2^3 e_2^3 v + 6\xi_2 \xi_1 e_2 e_1^2 v + 3\xi_2 e_1 e_2 v^2 + 3\xi_2 e_2^3 v + 3\xi_2 e_2 v^2 + e_2 v^3. \]

To determine the functions \( v(t, \xi, \lambda), \mu(\xi) \) and \( \bar{\mu}(\xi) \) first substituting (14) in (17) then find the coefficients \( \mu_0, \mu_1, \mu_2, \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2, v_0, v_1, v_2 \) and \( v_3 \) by equating the terms of \( \xi_1 \) and \( \xi_2 \) as follows.

Equating the coefficients of \( \xi_1^3 \), the following two equations has been found
\[ \left[ \mu_0 - \frac{1}{2\pi} \int_{0}^{2\pi} e_1^3 dt \right] e_1 - \left[ \frac{1}{2\pi} \int_{0}^{2\pi} e_2^3 dt \right] e_2 = 0, \]
\[ \ddot{v} + \lambda v - e_1^3 + \mu_0 e_1 = 0. \]  
(18)
From the first equation of (18) it is obtained that
\[ \mu_0 = \frac{3}{2}. \]
Substitute the value of \( \mu_0 \) in the second equation of (18) we have the following linear ODE,
\[ \ddot{v}_0 + \lambda v_0 - \frac{1}{\sqrt{2}} \sin 3t = 0. \]
(19)
Solve equation (19) the result became as follows
\[ v_0(t, \lambda) = \frac{\sin 3t}{\sqrt{2}(\lambda - 9)}. \]
Similarly, equating the coefficients of \( \xi_1^2 \xi_2 \) we get
\[ \begin{aligned}
\left[ \mu_1 - \frac{3}{2\pi} \int_0^{2\pi} e_1^3 e_2^2 dt \right] e_1 + \left[ \tilde{\mu}_0 - \frac{3}{2\pi} \int_0^{2\pi} e_1^2 e_2^2 dt \right] e_2 = 0,
\end{aligned} \]
\[ \begin{aligned}
\dot{v}_1 + \lambda v_1 - 3e_1^2 e_2 + \mu_1 e_1 + \tilde{\mu}_0 e_2 = 0.
\end{aligned} \]
(20)
From the first equation of (20) it is found that \( \mu_1 = 0 \) and \( \tilde{\mu}_0 = \frac{3}{2} \). Substitute these values in the second equation of (20) the result takes the form of
\[ \ddot{v}_1 + \lambda v_1 + \frac{3}{\sqrt{2}} \cos 3t = 0. \]
(21)
Solve equation (21) we have
\[ v_1(t, \lambda) = -\frac{3 \cos 3t}{\sqrt{2}(\lambda - 9)}. \]
Equating the coefficients of \( \xi_1 \xi_2^2 \) it is obtained that
\[ \begin{aligned}
\left[ \mu_2 - \frac{3}{2\pi} \int_0^{2\pi} e_1^2 e_2^2 dt \right] e_1 + \left[ \tilde{\mu}_1 - \frac{3}{2\pi} \int_0^{2\pi} e_1 e_2^2 dt \right] e_2 = 0,
\end{aligned} \]
\[ \begin{aligned}
\dot{v}_2 + \lambda v_2 - 3e_1 e_2^2 + \mu_2 e_1 + \tilde{\mu}_1 e_2 = 0.
\end{aligned} \]
(22)
From the first equation of (22) it is obtained that \( \tilde{\mu}_1 = 0 \) and \( \mu_2 = \frac{3}{2} \). Substitute these values in the second equation of (22) we get
\[ \ddot{v}_2 + \lambda v_2 + \frac{3}{\sqrt{2}} \sin 3t = 0. \]
(23)
Solve equation (23) the result became
\[ v_2(t, \lambda) = -\frac{3 \sin 3t}{\sqrt{2(\lambda - 9)}}. \]

Equating the coefficients of \( \xi_2^3 \), we have the following two equations,
\[
\begin{align*}
[\bar{\mu}_2 - \frac{1}{2\pi} \int_0^{2\pi} e_2^4 dt] e_2 + \left[ -\frac{1}{2\pi} \int_0^{2\pi} e_1 e_2^3 dt \right] e_1 &= 0, \\
\ddot{v}_3 + \lambda v_3 - e_2^3 + \bar{\mu}_2 e_2 &= 0. \tag{24}
\end{align*}
\]

From the first equation of (24) we have \( \bar{\mu}_2 = \frac{3}{2} \). Substitute the value of \( \bar{\mu}_2 \) in the second equation of (24) it is obtained the following linear ODE,
\[
\ddot{v}_3 + \lambda v_3 - \frac{1}{\sqrt{2}} \cos 3t = 0. \tag{25}
\]

Solve equation (25) we have
\[ v_3(t, \lambda) = \frac{\cos 3t}{\sqrt{2(\lambda - 9)}}. \]

Now substitute the values of \( \mu_0, \mu_1, \mu_2, \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2, v_0, v_1, v_2 \) and \( v_3 \) in (14) we have the nonlinear approximation solutions of equation (12) in the form of
\[
\begin{align*}
u(t, \xi) &= \sqrt{2} \xi_1 \sin t + \sqrt{2} \xi_2 \cos t + \frac{\sin 3t}{\sqrt{2(\lambda - 9)}} \xi_1^3 - \frac{3 \cos 3t}{\sqrt{2(\lambda - 9)}} \xi_2^3 \\
&- \frac{3 \sin 3t}{\sqrt{2(\lambda - 9)}} \xi_1 \xi_2^2 + \frac{\cos 3t}{\sqrt{2(\lambda - 9)}} \xi_2^3 + O(\xi^5) , \\
q_1 &= \bar{q}_1 + \frac{3}{2} \xi_1^2 + \frac{3}{2} \xi_2^2 + O(\xi^3), \\
q_2 &= \bar{q}_2 + \frac{3}{2} \xi_1^2 + \frac{3}{2} \xi_2^2 + O(\xi^3), \\
\xi &= (\xi_1, \xi_2). \tag{26}
\end{align*}
\]

By using (26)-(26) the following result has been stated.

**Theorem 3.1.** The key function of the functional \( V \) has the following form,
\[
\tilde{W}(\xi, \delta) = \xi_1^{12} + \xi_2^{12} + \lambda_1 \xi_1^{10} \xi_2^2 + \lambda_2 \xi_1^{10} \xi_2^2 + \lambda_3 \xi_1^{10} \xi_2^2 + \lambda_4 \xi_1^4 \xi_2^8 + \lambda_5 \xi_1^6 \xi_2^6 \\
+ \lambda_6 \xi_1^6 \xi_2^6 + \lambda_7 \xi_2^8 + \lambda_8 \xi_1^2 \xi_2^6 + \lambda_9 \xi_1^2 \xi_2^6 + \lambda_{10} \xi_1^4 \xi_2^4 + \lambda_{11} \xi_1^6 + \lambda_{12} \xi_2^6 \\
+ \lambda_{13} \xi_1^2 \xi_2^4 + \lambda_{14} \xi_1^2 \xi_2^4 + \lambda_{15} \xi_1^4 + \lambda_{16} \xi_2^4 + \lambda_{17} \xi_1^2 \xi_2^2 + \lambda_{18} \xi_1^2 \\
+ \lambda_{19} \xi_2^2 + O(|\xi|^{12}) + O(|\delta|) \quad O(|\delta|), \tag{27}
\]
where \( \lambda_i = \lambda_i(\lambda) \).
Function (27) has all the topological and analytical properties of functional $V$. Also, the function is symmetric in the variables $\xi_1$ and $\xi_2$ ($\tilde{W}(\xi_1, \xi_2) = \tilde{W}(-\xi_1, -\xi_2)$) it have 121 critical points. The point $u(t) = \xi_1 e_1 + \xi_2 e_2 + v(t, \xi, \lambda)$ is a critical point of the functional $V(u, \lambda)$ if and only if the point $\xi$ is a critical point of the function $\tilde{W}(\xi, \delta)$ [13]. This means that the existence of the solutions of equation (12) depends on the existence of the critical points of the functional $V(u, \lambda)$ and then on the existence of the critical points of the function $\tilde{W}(\xi, \delta)$. From this notation, the nonlinear approximation of the solutions of equation (12) corresponding to each critical point of the function $\tilde{W}(\xi, \delta)$ can be found. The spreading of the critical points of the function $\tilde{W}(\xi, \delta)$ depending on the change of parameter $\lambda$ will be discussed another paper.

Acknowledgements. I would like to thank the reviewer for good reading and useful suggestions.

References


