Journal of Applied Mathematics & Bioinformatics, vol.3, no.3, 2013, 173-186
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2013

A Method for Finding Nonlinear Approximation of Bifurcation Solutions of Some Nonlinear Differential Equations

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Abstract

This paper introduce a method for finding nonlinear approximation of the solutions of some nonlinear partial differential equations by using Lyapunov-Schmidt reduction. Also, it provides an example for finding nonlinear approximation of bifurcation of periodic solutions of Duffing equation.

Mathematics Subject Classification: 34K18; 34K10
Keywords: Bifurcation solutions; Local scheme of Lyapunov-Schmidt; Duffing equation

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Article Info: *Received* : June 10, 2013. *Revised* : August 20, 2013
Published online : September 15, 2013

1 Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

$$f(x,\lambda) = b, \quad x \in O \subset X, \quad b \in Y, \quad \lambda \in R^n \tag{1}$$

where f is a smooth Fredholm map of index zero and X, Y are Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi,\lambda) = \beta, \quad \xi \in M, \quad \beta \in N \tag{2}$$

can be used, where M and N are smooth finite dimensional manifolds.

A passage from equation (1) into equation (2) (variant local scheme of Lyapunov -Schmidt) with the conditions that equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, etc) can be found in [3], [11], [13], [14].

Suppose that $f : \Omega \subset E \to F$ is a nonlinear Fredholm map of index zero. A smooth map $f : \Omega \subset E \to F$ has variational property, if there exists a functional $V : \Omega \subset E \to R$ such that $f = grad_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H, \quad \forall\ x \in \Omega, \quad h \in E,$$

where $\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space H. In this case, the solutions of equation $f(x,\lambda) = 0$ are the critical points of functional $V(x,\lambda)$. Suppose that $f : E \to F$ is a smooth Fredholm map of index zero, E, F are Banach spaces and

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H, \quad h \in E,$$

where V is a smooth functional on E. Also it is assumed that $E \subset F \subset H$, H is a Hilbert space. By using a method of finite dimensional reduction (Local scheme of Lyapunov-Schmidt) the problem,

$$V(x,\lambda) \to extr \quad x \in E, \quad \lambda \in R^n$$

can be reduced into equivalent problem

$$W(\xi,\lambda) \to extr \quad \xi \in R^n.$$

The function $W(\xi, \lambda)$ is called key function.

If $N = span\{e_1, ..., e_n\}$ is a subspace of E, where $e_1, ..., e_n$ is an orthonormal set in H, then the key function $W(\xi, \lambda)$ can be defined in the form of

$$W(\xi, \lambda) = \inf_{x:\langle x, e_i\rangle = \xi_i \ \forall \ i} V(x, \lambda), \quad \xi = (\xi_1, ..., \xi_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, *etc.*) [13]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function. If f has a variational property, then the equation

$$\theta(\xi, \lambda) = gradW(\xi, \lambda) = 0$$

is called bifurcation equation.

It is well known that in the method of Lyapunov-Schmidt, the space E is decomposed into two orthogonal subspaces of the space E and then every element $x \in E$ can be written in the unique form as a sum of two elements such that the solution of the equation (1) consists of the homogeneous solution and the particular solution. Sapronov and his group [2, 12] used the complement solution to find the function $W(\xi, \lambda)$ which denotes the linear Ritz approximation of the functional $V(x, \lambda)$. This paper introduce a method to find nonlinear Ritz approximation of the functional $V(x, \lambda)$, such a method is based on finding the particular solution of the equation (1).

2 The Method

Consider the nonlinear Fredholm operator of index zero $f : E \to F$ defined by the equation

$$f(u, \lambda) = 0, \qquad \lambda \in R^n, \quad u \in \Omega \subset E, \tag{3}$$

where E, F are real Banach spaces and Ω is an open subset of E. Assume that the operator f has a variational property, i.e, there exists a functional $V : \Omega \subset E \to R$ such that $f = grad_H V$ where Ω is a bounded domain. The operator f can be written as

$$f(u, \lambda) = Au + Nu = 0,$$

where $A = \frac{\partial f}{\partial u}(u_0, \lambda)$ is a linear continuous Fredholm operator, $\frac{\partial f}{\partial u}(u_0, \lambda)$ the Frechet derivative of the operator f at the point u_0 and N the nonlinear operator. By using Lyapunov-Schmidt reduction, the decomposition is obtained below

$$E = M \oplus M^{\perp}$$

$$F = \tilde{M} \oplus \tilde{M}^{\perp}$$

where $M = kerA$ is the null space of the operator A, $dimM = dim\tilde{M} = n$ and $M^{\perp}$, $\tilde{M}^{\perp}$ are the orthogonal complements of the subspaces M and $\tilde{M}$ respectively. If $e_1, e_2, ..., e_n$ is an orthonormal set in H such that $Ae_i = \alpha_i(\lambda)e_i$, $\alpha_i(\lambda)$ is continuous function, $i = 1, ..., n$, then every element $u \in E$ can be represented in the unique form of

$$u = w + v, \quad w = \sum_{i=1}^{n} \xi_i e_i \in M, \quad M \perp v \in M^{\perp}, \quad \xi_i = \langle u, e_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Hilbert space H. There exist projections $p : E \to M$ and $I - p : E \to M^{\perp}$ such that $w = pu$ and $(I - p)u = v$. Similarly, there exist projections $Q : F \to \tilde{M}$ and $I - Q : F \to \tilde{M}^{\perp}$ such that

$$f(u, \lambda) = Qf(u, \lambda) + (I - Q)f(u, \lambda) \tag{4}$$

or

$$f(w + v, \lambda) = Qf(w + v, \lambda) + (I - Q)f(w + v, \lambda).$$

It follows that

$$Qf(w + v, \lambda) + (I - Q)f(w + v, \lambda) = 0,$$

and hence the result become

$$Qf(w + v, \lambda) = 0,$$

$$(I - Q)f(w + v, \lambda) = 0.$$

The implicit function theorem implies that

$$W(\xi, \delta) = V(\Phi(\xi, \delta), \delta), \quad \xi = (\xi_1, \xi_2, ..., \xi_n)^{\top},$$

where $degW \geq 2$, then the linear Ritz approximation of the functional V is a function W defined by

$$W(\xi, \delta) = V\Big(\sum_{i=1}^{n} \xi_i e_i, \delta\Big) = W_0(\xi) + W_1(\xi, \delta), \tag{5}$$

where $W_0(\xi)$ is a homogenous polynomial of order $n \geq 3$ such that $W_0(0) = 0$ and $W_1(\xi, \delta)$ is a polynomial function of degree less than n.

Let $q_1, q_2, ..., q_m$ be the coefficients of the quadratic terms of the function $W_1(\xi, \delta)$, then the function $W_1(\xi, \delta)$ can be written in the form of

$$W_1(\xi, \delta) = W_2(\xi, \delta) + \sum_{k=1}^{m} q_k \xi_k^2$$

where $degW_2 = d, \;\; 2 < d < n$.

The nonlinear Ritz approximation of the functional V is a function W defined by

$$W(\xi, \delta) = V\Big(\sum_{i=1}^{n} \xi_i e_i + \Phi(\sum_{i=1}^{n} \xi_i e_i, \delta), \delta\Big),$$

where $\Phi(w, \delta) = v(x, \xi, \delta), \quad v \in N^{\perp}$.

To determine the nonlinear Ritz approximation of the functional V, Taylor's expansion of the functions $\mu_k(\xi)$ and $v(x, \xi, \delta)$ is used by assuming the following

$$q_k = \hat{q}_k + \mu_k(\xi) = \hat{q}_k + \sum_{j=2}^{r} D_k^{(j)}(\xi), \qquad k = 1, ..., m$$

$$v(x, \xi, \delta) = \sum_{j=2}^{r} B^{(j)}(\xi),$$

where $D_k^{(j)}(\xi)$ and $B^{(j)}(\xi)$ are homogenous polynomials of degree j with coefficients μ_{ki} and $v_{ji}(x, \delta)$ respectively, $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

Since

$$Qf(u, \lambda) = \sum_{i=1}^{n} \langle f(u, \lambda), e_i \rangle e_i = 0$$

it follows that

$$\sum_{i=1}^{n} \langle Au + Nu, e_i \rangle e_i = 0.$$

Hence

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \langle Nu, e_i \rangle e_i = 0, \qquad q_i = \alpha_i(\lambda)$$

or

$$\sum_{i=1}^{n} q_i \xi_i e_i + \sum_{i=1}^{n} \Big[\int_{\Omega} N(w + v) e_i \Big] e_i = 0. \tag{6}$$

From (4) it gets

$$(I-Q)f(u,\lambda) = f(u,\lambda) - Qf(u,\lambda),$$

but

$$A(w+v) + N(w+v) = 0 \tag{7}$$

it follows that

$$Av + N(w+v) + \sum_{i=1}^{n} q_i \xi_i e_i = 0.$$

Substitute the values of q_i, $\mu_i(\xi)$ and $v(x,\xi,\delta)$ in (6) and (7) yields

$$\sum_{i=1}^{n}\Big[\hat{q}_i + \sum_{j=2}^{r} D_i^{(j)}(\xi)\Big]\xi_i e_i + \\ + \sum_{i=1}^{n}\Big[\int_\Omega N\Big(\sum_{i=1}^{n}\xi_i e_i + \sum_{j=2}^{r} B^{(j)}(\xi)\Big)e_i\Big]e_i = 0, \tag{8}$$

$$A\Big(\sum_{j=2}^{r} B^{(j)}(\xi)\Big) + N\Big(\sum_{i=1}^{n}\xi_i e_i + \sum_{j=2}^{r} B^{(j)}(\xi)\Big) + \\ + \sum_{i=1}^{n}\Big(\hat{q}_i + \sum_{j=2}^{r} D_i^{(j)}(\xi)\Big)\xi_i e_i = 0. \tag{9}$$

To determine the functions $v(x,\xi,\lambda)$ and $\mu_k(\xi)$ we equating the coefficients of $\hat{\xi} = \xi_1\xi_2...\xi_n$ in the equation (8) to find the value of μ_{ki} and after some calculations of equation (9) it is obtained a linear ordinary differential equation in the variable $v_{ji}(x,\lambda)$. Solve the resulting equation one can find the value of $v_{ji}(x,\lambda)$.

3 Applications

In [8] the author introduced an example to find nonlinear approximation of bifurcation solutions of the fourth order differential equation,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0.$$

This equation also was studied by [1], [4], [5], [6], [7], [9], [10] with different nonlinear terms. The purpose of this study in hand is to apply the method in the previous section to find the bifurcation of periodic solutions of Duffing equation of type

$$\ddot{u} + \lambda u - u^3 = 0, \tag{10}$$

with resonance 1:1. For simply, this equation is chosen because another equation may give more difficulty in the calculations and then the study could obtain more difficult key function. Suppose that $f : E \to F$ is a nonlinear Fredholm operator of index zero from Banach space E to Banach space F defined by,

$$f(u, \lambda) = \frac{d^2u}{dt^2} + \lambda u - u^3, \tag{11}$$

where $E = \Pi^2([0, 2\pi], R)$ is the space of all periodic continuous functions that have derivative of order at most two, $F = \Pi([0, 2\pi], R)$ is the space of all periodic continuous functions where $u = u(t),\ t \in [0, 2\pi]$. In this case, the solutions of equation (10) is equivalent to the solutions of the operator equation given below

$$f(u, \lambda) = 0. \tag{12}$$

It has been noticed that the operator f has a variational property, i.e. there exists a functional V such that $f(u, \lambda) = grad_H V(u, \lambda)$ or equivalently,

$$\frac{\partial V}{\partial u}(u, \lambda)h = \langle f(u, \lambda), h\rangle_H, \quad \forall\ u \in \Omega, \quad h \in E,$$

where $(\langle \cdot, \cdot\rangle_H$ is the scalar product in Hilbert space $H)$ and

$$V(u, \lambda) = \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{(\dot{u})^2}{2} + \lambda\frac{u^2}{2} - \frac{u^4}{4}\Big)dt.$$

In this case, the solutions of equation (10) are the critical points of the functional $V(u, \lambda)$, where the critical points of the functional $V(u, \lambda)$ are the solutions of Euler-Lagrange equation

$$\frac{\partial V}{\partial u}(u, \lambda)h = \frac{1}{2\pi}\int_0^{2\pi} (\ddot{u} + \lambda u - u^3)h\ dt = 0,$$

and $\frac{\partial V}{\partial u}(u, \lambda)$ is the Frechet derivative of the functional $V(u, \lambda)$. Therefore, the study of equation (10) is equivalent to the study extremely problem,

$$V(u, \lambda) \to extr, \qquad u \in E.$$

The analysis of bifurcation can be found by using the method of Lyapunov-Schmidt to reduce the problem into finite dimensional space. By localized parameter,

$$\lambda = \tilde{\lambda} + \delta_1, \qquad \delta_1 \text{ is small parameter},$$

the reduction leads to the function in two variables defined by

$$W(\xi, \lambda) = \inf_{\langle u,\ e_i \rangle = \xi_i,\ i=1,2} V(u, \lambda), \quad \xi = (\xi_1, \xi_2).$$

It is well known that in the reduction of Lyapunov-Schmidt, the function $W(\xi, \lambda)$ is smooth. This function has all the topological and analytical properties of functional V [13]. For small λ, there is one-to-one corresponding between the critical points of functional V and smooth function W, preserving the type of critical points (multiplicity, index Morse, *etc*) [13]. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (12) has the following form:

$$\ddot{h} + \lambda h = 0, \quad h \in E$$

The point $(0, \lambda) = (0, 1)$ is a bifurcation point of equation (10) [13].

The localized parameter

$$\hat{\lambda} = 1 + \delta_1,$$

leads to the bifurcation along the modes $e_1 = c_1 sint, \quad e_2 = c_2 cost$, where $\|e_1\| = \|e_2\| = 1$, $c_1 = c_2 = \sqrt{2}$. Let $N = Ker(A) = span\{e_1, e_2\}$, where, $A = f_u(0, \lambda) = \frac{d^2}{dt^2} + \lambda$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus \hat{E}, \qquad \hat{E} = N^{\perp} \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space F decomposed in direct sum of two subspaces, N and orthogonal complement to N,

$$F = N \oplus \hat{F}, \qquad \hat{F} = N^{\perp} \cap F = \{v \in F : v \perp N\}.$$

There exist projections $p : E \to N$ and $I - p : E \to \hat{E}$ such that $pu = w$ and $(I - p)u = v$, (I is the identity operator). Hence, every vector $u \in E$ can be written in the form,

$$u = w + v, \quad w = \sum_{i=1}^{2} \xi_i e_i \in N, \quad N \perp v \in \hat{E}, \quad \xi_i = \Big\langle u\,,\, e_i \Big\rangle.$$

Similarly, there exist projections $Q : F \to N$ and $I - Q : F \to \hat{F}$ such that

$$f(u,\lambda) = Qf(u,\lambda) + (I-Q)f(u,\lambda) \tag{13}$$

Accordingly, equation (12) can be written in the following form,

$$Qf(w+v,\lambda) = 0,$$

$$(I-Q)f(w+v,\lambda) = 0.$$

By the implicit function theorem, there exist a smooth map $\Phi : N \to \hat{E}$, such that

$$W(\xi,\delta_1) = V(\Phi(\xi,\lambda),\delta_1).$$

And then the linear Ritz approximation of the functional V is a function W given by,

$$W(\xi,\delta_1) = V(\xi_1\ e_1 + \xi_2\ e_2,\delta_1) = \xi_1^4 + 4\xi_1^2\xi_2^2 + \xi_2^4 + \frac{q_1}{2}\xi_1^2 + \frac{q_2}{2}\xi_2^2.$$

The nonlinear Ritz approximation of the functional V is a function W given by

$$W(\xi,\delta_1) = V(\xi_1\ e_1 + \xi_2\ e_2, \Phi(\xi_1\ e_1 + \xi_2\ e_2,\delta_1),\delta_1), \qquad v(t,\xi,\lambda) = \Phi(w,\delta_1).$$

To determine the nonlinear Ritz approximation of the functional V, the functions $v(t,\xi,\lambda) = O(\xi^3)$, $\mu(\xi) = O(\xi^2)$ and $\tilde{\mu}(\xi) = O(\xi^2)$ must be found in the form of power series in term of ξ, as follows:

$$\begin{aligned} &v(t,\xi,\lambda) = v_0(t,\lambda)\xi_1^3 + v_1(t,\lambda)\xi_1^2\xi_2 + v_2(t,\lambda)\xi_1\xi_2^2 + v_3(t,\lambda)\xi_2^3 + ..., \\ &\mu(\xi) = \mu_0\xi_1^2 + \mu_1\xi_1\xi_2 + \mu_2\xi_2^2 + ..., \\ &\tilde{\mu}(\xi) = \tilde{\mu}_0\xi_1^2 + \tilde{\mu}_1\xi_1\xi_2 + \tilde{\mu}_2\xi_2^2 + ..., \end{aligned} \tag{14}$$

where $q_1 = \tilde{q}_1 + \mu(\xi_1,\xi_2)$, $q_2 = \tilde{q}_2 + \tilde{\mu}(\xi_1,\xi_2)$ and $\xi = (\xi_1,\xi_2)$. Equation (12) can be written in the form of

$$f(u,\lambda) = Au + Tu = 0, \qquad Tu = -u^3.$$

Since,

$$Qf(u,\lambda) = \sum_{i=1}^{2}\langle f(u,\lambda), e_i\rangle e_i = 0.$$

Then the result takes the form of

$$\sum_{i=1}^{2}\langle Au+Tu, e_i\rangle e_i = 0,$$

and hence

$$q_1\xi_1 e_1 + q_2\xi_2 e_2 - \Big(\frac{1}{2\pi}\int_0^{2\pi}(\xi_1 e_1+\xi_2 e_2+v)^3 e_1\ dt\Big)e_1 \\ - \Big(\frac{1}{2\pi}\int_0^{2\pi}(\xi_1 e_1+\xi_2 e_2+v)^3 e_2\ dt\Big)e_2 = 0. \tag{15}$$

From (13) and (15) it is obtained

$$\ddot{v} + \lambda v - (\xi_1 e_1 + \xi_2 e_2 + v)^3 + q_1\xi_1 e_1 + q_2\xi_2 e_2 = 0. \tag{16}$$

It follows that,

$$\Big[(\tilde{q}_1 + \mu(\xi_1,\xi_2))\xi_1 - \xi_1^3\frac{1}{2\pi}\int_0^{2\pi} e_1^4 dt - 3\xi_1^2\xi_2\frac{1}{2\pi}\int_0^{2\pi} e_1^3 e_2 dt \\ - 3\xi_1\xi_2^2\frac{1}{2\pi}\int_0^{2\pi} e_1^2 e_2^2 dt - \xi_2^3\frac{1}{2\pi}\int_0^{2\pi} e_1 e_2^3 dt - \frac{1}{2\pi}\int_0^{2\pi}\theta_1(\xi_1,\xi_2,v)dt\Big]e_1 \\ + \Big[(\tilde{q}_2 + \tilde{\mu}(\xi_1,\xi_2))\xi_2 - \xi_1^3\frac{1}{2\pi}\int_0^{2\pi} e_1^3 e_2 dt - 3\xi_1^2\xi_2\frac{1}{2\pi}\int_0^{2\pi} e_1^2 e_2^2 dt \\ - 3\xi_1\xi_2^2\frac{1}{2\pi}\int_0^{2\pi} e_1 e_2^3 dt - \xi_2^3\frac{1}{2\pi}\int_0^{2\pi} e_2^4 dt - \frac{1}{2\pi}\int_0^{2\pi}\theta_2(\xi_1,\xi_2,v)dt\Big]e_2 = 0, \\ \ddot{v} + \lambda v - \xi_1^3 e_1^3 - 3\xi_1^2\xi_2 e_1^2 e_2 - 3\xi_1\xi_2^2 e_1 e_2^2 - \xi_2^3 e_2^3 - v^3 - 3v^2\xi_1 e_1 \\ - 3v^2\xi_2 e_2 - 3v\xi_1^2 e_1^2 - 6v\xi_1\xi_2 e_1 e_2 - 3v\xi_2^2 e_2^2 \\ + (\tilde{q}_1 + \mu(\xi_1,\xi_2))\xi_1 e_1 + (\tilde{q}_2 + \tilde{\mu}(\xi_1,\xi_2))\xi_2 e_2 = 0. \tag{17}$$

where

$$\theta_1(\xi_1,\xi_2,v) = 3\xi_1^2 e_1^3 v + 6\xi_1\xi_2 e_1^2 e_2 v + 3\xi_1 e_1^2 v^2 + 3\xi_2^2 e_1 e_2^2 v + 3\xi_2 e_1 e_1 v^2 + e_1 v^3 \\ \theta_2(\xi_1,\xi_2,v) = 3\xi_1^2 e_1^2 e_2 v + 6\xi_1\xi_2 e_1 e_2^2 v + 3\xi_1 e_1 e_2 v^2 + 3\xi_2^2 e_2^3 v + 3\xi_2 e_2^2 v^2 + e_2 v^3.$$

To determine the functions $v(t,\xi,\lambda)$, $\mu(\xi)$ and $\tilde{\mu}(\xi)$ first substituting (14) in (17) then find the coefficients $\mu_0,\mu_1,\mu_2,\tilde{\mu}_0,\tilde{\mu}_1,\tilde{\mu}_2,v_0,v_1,v_2$ and v_3 by equating the terms of ξ_1 and ξ_2 as follows.

Equating the coefficients of ξ_1^3, the following two equations has been found

$$\Big[\mu_0 - \frac{1}{2\pi}\int_0^{2\pi} e_1^4 dt\Big]e_1 - \Big[\frac{1}{2\pi}\int_0^{2\pi} e_1^3 e_2 dt\Big]e_2 = 0, \\ \ddot{v} + \lambda v - e_1^3 + \mu_0 e_1 = 0. \tag{18}$$

From the first equation of (18) it is obtained that

$$\mu_0 = \frac{3}{2}.$$

Substitute the value of μ_0 in the second equation of (18) we have the following linear ODE,

$$\ddot{v}_0 + \lambda v_0 - \frac{1}{\sqrt{2}}\, sin3t = 0. \tag{19}$$

Solve equation (19) the result became as follows

$$v_0(t, \lambda) = \frac{sin3t}{\sqrt{2}(\lambda - 9)}.$$

Similarly, equating the coefficients of $\xi_1^2\xi_2$ we get

$$\begin{aligned}
&\Big[\mu_1 - \frac{3}{2\pi}\int_0^{2\pi} e_1^3 e_2 dt\Big] e_1 + \Big[\tilde{\mu}_0 - \frac{3}{2\pi}\int_0^{2\pi} e_1^2 e_2^2 dt\Big] e_2 = 0,\\
&\ddot{v}_1 + \lambda v_1 - 3e_1^2 e_2 + \mu_1 e_1 + \tilde{\mu}_0 e_2 = 0.
\end{aligned} \tag{20}$$

From the first equation of (20) it is found that $\mu_1 = 0$ and $\tilde{\mu}_0 = \frac{3}{2}$. Substitute these values in the second equation of (20) the result takes the form of

$$\ddot{v}_1 + \lambda v_1 + \frac{3}{\sqrt{2}}\, cos3t = 0. \tag{21}$$

Solve equation (21) we have

$$v_1(t, \lambda) = -\frac{3\ cos3t}{\sqrt{2}(\lambda - 9)}.$$

Equating the coefficients of $\xi_1\xi_2^2$ it is obtained that

$$\begin{aligned}
&\Big[\mu_2 - \frac{3}{2\pi}\int_0^{2\pi} e_1^2 e_2^2 dt\Big] e_1 + \Big[\tilde{\mu}_1 - \frac{3}{2\pi}\int_0^{2\pi} e_1 e_2^3 dt\Big] e_2 = 0,\\
&\ddot{v}_2 + \lambda v_2 - 3e_1 e_2^2 + \mu_2 e_1 + \tilde{\mu}_1 e_2 = 0.
\end{aligned} \tag{22}$$

From the first equation of (22) it is obtained that $\tilde{\mu}_1 = 0$ and $\mu_2 = \frac{3}{2}$. Substitute these values in the second equation of (22) we get

$$\ddot{v}_2 + \lambda v_2 + \frac{3}{\sqrt{2}}\, sin3t = 0. \tag{23}$$

Solve equation (23) the result became

$$v_2(t,\lambda) = -\frac{3\ sin3t}{\sqrt{2}(\lambda-9)}.$$

Equating the coefficients of ξ_2^3, we have the following two equations,

$$\begin{aligned}
&\Big[\tilde{\mu}_2 - \frac{1}{2\pi}\int_0^{2\pi} e_2^4 dt\Big]e_2 + \Big[-\frac{1}{2\pi}\int_0^{2\pi} e_1 e_2^3 dt\Big]e_1 = 0,\\
&\ddot{v}_3 + \lambda v_3 - e_2^3 + \tilde{\mu}_2 e_2 = 0. \qquad (24)
\end{aligned}$$

From the first equation of (24) we have $\tilde{\mu}_2 = \frac{3}{2}$. Substitute the value of $\tilde{\mu}_2$ in the second equation of (24) it is obtained the following linear ODE,

$$\ddot{v}_3 + \lambda v_3 - \frac{1}{\sqrt{2}}\ cos3t = 0. \qquad (25)$$

Solve equation (25) we have

$$v_3(t,\lambda) = \frac{cos3t}{\sqrt{2}(\lambda-9)}.$$

Now substitute the values of $\mu_0, \mu_1, \mu_2, \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, v_0, v_1, v_2$ and v_3 in (14) we have the nonlinear approximation solutions of equation (12) in the form of

$$\begin{aligned}
u(t,\xi) &= \sqrt{2}\ \xi_1\ sint + \sqrt{2}\ \xi_2\ cost + \frac{sin3t}{\sqrt{2}(\lambda-9)}\ \xi_1^3 - \frac{3\ cos3t}{\sqrt{2}(\lambda-9)}\ \xi_1^2\xi_2\\
&\quad - \frac{3\ sin3t}{\sqrt{2}(\lambda-9)}\ \xi_1\xi_2^2 + \frac{cos3t}{\sqrt{2}(\lambda-9)}\ \xi_2^3 + O(\xi^5),\\
q_1 &= \tilde{q}_1 + \frac{3}{2}\xi_1^2 + \frac{3}{2}\xi_2^2 + O(\xi^3), \quad q_2 = \tilde{q}_2 + \frac{3}{2}\xi_1^2 + \frac{3}{2}\xi_2^2 + O(\xi^3),\\
\xi &= (\xi_1, \xi_2). \qquad (26)
\end{aligned}$$

By using (26)-(26) the following result has been stated.

Theorem 3.1. *The key function of the functional V has the following form,*

$$\begin{aligned}
\tilde{W}(\xi,\delta) \;=\; & \xi_1^{12} + \xi_2^{12} + \lambda_1\xi_1^2\xi_2^{10} + \lambda_2\xi_1^{10}\xi_2^2 + \lambda_3\xi_1^8\xi_2^4 + \lambda_4\xi_1^4\xi_2^8 + \lambda_5\xi_1^6\xi_2^6\\
& + \lambda_6\xi_1^8 + \lambda_7\xi_2^8 + \lambda_8\xi_1^2\xi_2^6 + \lambda_9\xi_1^6\xi_2^2 + \lambda_{10}\xi_1^4\xi_2^4 + \lambda_{11}\xi_1^6 + \lambda_{12}\xi_2^6\\
& + \lambda_{13}\xi_1^2\xi_2^4 + \lambda_{14}\xi_1^4\xi_2^2 + \lambda_{15}\xi_1^4 + \lambda_{16}\xi_2^4 + \lambda_{17}\xi_1^2\xi_2^2 + \lambda_{18}\xi_1^2\\
& + \lambda_{19}\xi_2^2 + o(|\xi|^{12}) + O(|\xi|^{12})O(|\delta|), \qquad (27)
\end{aligned}$$

where $\lambda_i = \lambda_i(\lambda)$.

Function (27) has all the topological and analytical properties of functional V. Also, the function is symmetric in the variables ξ_1 and ξ_2 ($\tilde{W}(\xi_1, \xi_2) = \tilde{W}(-\xi_1, -\xi_2)$) it have 121 critical points. The point $u(t) = \xi_1 e_1 + \xi_2 e_2 + v(t, \xi, \lambda)$ is a critical point of the functional $V(u, \lambda)$ if and only if the point ξ is a critical point of the function $\tilde{W}(\xi, \delta)$ [13]. This means that the existence of the solutions of equation (12) depends on the existence of the critical points of the functional $V(u, \lambda)$ and then on the existence of the critical points of the function $\tilde{W}(\xi, \delta)$. From this notation, the nonlinear approximation of the solutions of equation (12) corresponding to each critical point of the function $\tilde{W}(\xi, \delta)$ can be found. The spreading of the critical points of the function $\tilde{W}(\xi, \delta)$ depending on the change of parameter λ will be discussed another paper.

Acknowledgements. I would like to thank the reviewer for good reading and useful suggestions.

References

[1] B.C. Bardin and S.D. Furta, *Periodic Travelling Waves of an Infinite Beam on a Nonlinear elastic support*, Institute B f r Mechanik, Universitet Stuttgart, Institutsbericht IB-36, Januar 2001.

[2] B.M. Darinskii, C.L. Tcarev and Yu.I. Sapronov, *Bifurcation of Extremely of Fredholm Functionals*, Voronezh, 2004.

[3] B.V. Loginov, *Theory of Branching Nonlinear Equations in Theconditions of Invariance Group*, Tashkent, Fan, 1985.

[4] M.A. Abdul Hussain, *Corner Singularities of Smooth Functions in the Analysis of Bifurcations Balance of the Elastic Beams and Periodic Waves*, Ph. D. thesis, Voronezh, Russia, 2005.

[5] M.A. Abdul Hussain, Bifurcation Solutions of Boundary Value Problem, *Journal of Vestnik Voronezh*, Voronezh State University, Russia, **1**, (2007), 162-166.

[6] M.A. Abdul Hussain, Existence of Bifurcation Solutions of Nonlinear Wave Equation of Elastic Beams, *Journal of Mathematical Models and Operator Equations*, Voronezh State University, Russia, **5**, (2008), 149-157.

[7] M.A. Abdul Hussain, Bifurcation Solutions of Elastic Beams Equation with Small Perturbation, *Int. J. Math. Anal.*, Ruse, **3** (18), (2009), 879-888.

[8] M.A. Abdul Hussain, Two Modes Bifurcation Solutions of Elastic Beams Equation with Nonlinear Approximation, *Communications in Mathematics and Applications journal*, India, **1**(2), (2010), 123-131.

[9] M.A. Abdul Hussain, Two-Mode Bifurcation in Solution of a Perturbed Nonlinear Fourth Order Differential Equation, *Archivum Mathematicum (BRNO)*, Tomus, Czech Republic, **48**, (2012), 27-37.

[10] M.J. Mohammed, *Bifurcation Solutions of Nonlinear Wave Equation*, M.Sc. thesis, Basrah Univ., Iraq, 2007.

[11] M.M. Vainberg and V.A. Trenogin, *Theory of Branching Solutions of Nonlinear Equations*, M. Science, 1969.

[12] V.R. Zachepa and Yu.I. Sapronov, *Local Analysis of Fredholm Equation*, Voronezh, 2002.

[13] Yu.I. Sapronov, Regular Perturbation of Fredholm Maps and Theorem about odd Field, *Works Dept. of Math.*, Voronezh Univ., **10**, (1973), 82-88.

[14] Yu.I. Sapronov, Finite Dimensional Reduction in the Smooth Extremely Problems, *Uspehi math., Science*, **51**(1), (1996), 101- 132.