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On Second-Order Fritz John Type Duality for Variational Problems

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Abstract

Second-order dual to a variational problem is formulated. This dual uses the Fritz John type necessary optimality conditions instead of the Karush-Kuhn-Tucker type necessary optimality conditions and thus, does not require a constraint qualification. Weak, strong, Mangasarian type strict-converse, and Huard type converse duality theorems between primal and dual problems are established under appropriate generalized second-order invexity conditions. A pair of second-order dual variational problems with natural boundary conditions is constructed, and it is briefly indicated that duality results for this pair can be validated analogously to those for the earlier models dealt with in this research. Finally, it is pointed out that our results can be viewed as the dynamic generalizations of those for nonlinear programming problems, already treated in the literature.

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1 Introduction

Second-order duality in mathematical programming problem has been widely investigated in the recent past. Mangasarian [1] formulated second-order dual to nonlinear programming problem and gave duality result under assumptions that are involved and rather difficult to verify, Subsequently Mond [2] established the duality results for non-linear programming problem under simpler assumption. Subsequently many researchers investigated second-order duality under invexity and generalized invexity conditions. Duality for continuous programming problem has been studied by a number of researcher researches. Mond and Hanson [3] were the first to consider a class of constraint variational problems and study first order-duality for such problem. Motivated with the results of [3], a number of duality theorems appeared in the literature.

Chen [4] was the first to identify second-order dual formulated for a constrained variational problem and established various duality results under an involved invexity like assumptions. Recently, Husain et al [5] have presented Mond-Weir type duality for the problem of [6] and by introducing continuous-time version of second-order invexity and generalized second-order invexity, validated various duality results. Earlier Weir and Mond studied duality for nonlinear programming problem using Fritz John type optimality conditions instead of Karush-Kuhn-Tucker optimality conditions and thus their duality results do not require a constraint qualification.

In this research we study Fritz type second-order duality using Fritz John type optimality conditions and validate various duality theorems under the assumption of second-order pseudoinvexity and second-order quasi-invexity. A pair of Fritz John type second-order dual variational problems with natural boundary condition by ignoring fixed point condition is formulated and a close relationship between our duality results and those of Husain et al [7] is briefly outlined.

2 Pre-Requisites

Let $I=[a,b]$ be a real interval, $\phi: I \times R^n \times R^n \rightarrow R$ and $\psi: I \times R^n \times R^n \rightarrow R^m$ be twice continuously differentiable functions. In order to consider $\phi(t, x(t), \dot{x}(t))$, where $x: I \rightarrow R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x *and* $\phi_{\dot{x}}$, the first order of ϕ with respect to $x(t)$ and $\dot{x}(t)$, respectively, that is,

$$\phi_x = \left(\frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, \cdots, \frac{\partial\phi}{\partial x^n}\right)^T, \quad \phi_{\dot{x}} = \left(\frac{\partial\phi}{\partial \dot{x}^1}, \frac{\partial\phi}{\partial \dot{x}^2}, \cdots, \frac{\partial\phi}{\partial \dot{x}^n}\right)^T$$

Denote by ϕ_{xx} the Hessian matrix of ϕ, and ψ_x the $m \times n$ Jacobian matrix respectively, that is, with respect to $x(t)$, that is $\phi_{xx} = \left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right)$, $i, j = 1, 2, ..., n$, ψ_x the $m \times n$ Jacobian matrix

$$\psi_x = \begin{pmatrix} \frac{\partial\psi^1}{\partial x^1} & \frac{\partial\psi^1}{\partial x^2} \cdots & \frac{\partial\psi^1}{\partial x^n} \\ \frac{\partial\psi^2}{\partial x^1} & \frac{\partial\psi^2}{\partial x^2} \cdots & \frac{\partial\psi^2}{\partial x^n} \\ \vdots & \vdots & \vdots \\ \frac{\partial\psi^m}{\partial x^1} & \frac{\partial\psi^m}{\partial x^2} \cdots & \frac{\partial\psi^m}{\partial x^n} \end{pmatrix}_{m\times n}.$$

The symbols $\phi_{\dot{x}}$, $\phi_{\dot{x}x}$, $\phi_{x\dot{x}}$ and $\psi_{\dot{x}}$ have analogous representations.

Designate by X, the space of piecewise smooth functions $x: I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \int_a^t u(s)\,ds,$$

Thus $\frac{d}{dt} = D$ except at discontinuities.

We incorporate the following definitions which are required in the subsequent analysis.

Definition2.1 (Second-order Pseudoinvex) If the functional $\int_I \phi(t, x, \dot{x})\,dt$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt \geq 0$$
$$\Rightarrow \int_I \phi(t, x, \dot{x})\,dt \geq \int_I \left\{\phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\} dt,$$

or

$$\int_I \phi(t, x, \dot{x})\,dt \leq \int_I \left\{\phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\} dt$$
$$\Rightarrow \int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt \leq 0$$

then $\int_I \phi(t, x, \dot{x})\,dt$ is said to be second-order pseudoinvex with respect to η. Where $G = \phi_{xx} - 2D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}}$, and $\beta \in C(I, R^n)$, the space of n-dimensional continuous vector functions.

Definition 2.2 (Strictly-pseudoinvex) If the functional $\int_I \phi(t, x, \dot{x})\,dt$ satisfies

$$\int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t) \right\} dt \geq 0$$

$$\Rightarrow \int_I \phi(t, x, \dot{x})\, dt > \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t) \right\} dt,$$

then $\int_I \phi(t, x, \dot{x})\, dt$ is said to be second-order strictly- pseudoinvex with respect to η.

Definition2.3 (Semi-strictly pseudoinvex) If $(SGF_r CD)$ and $g : I \times R^n \times R^n \to R^m$, $\int_I y^T(t)\, g\, dt$ will be semi-strictly pseudoinvex with respect to η, if $\int_I y^T(t)\, g\, dt$ is strictly pseudoinvex for all $y(t) \geq 0,\ t \in I$, $y(t) \neq 0,\ t \in I$.

Definition2.4. (Second- order Quasi-invex) If the functional $\int_I \phi(t, x, \dot{x})\, dt$ satisfies

$$\int_I \phi(t, x, \dot{x})\, dt \leq \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t) \right\} dt$$

$$\Rightarrow \int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G(t)\beta(t) \right\} dt \leq 0,$$

Then $\int_I \phi(t, x, \dot{x})\, dt$ is said to be second-order quasi-invex with respect to η.

Remark 2.1 If φ does not depend explicitly on t, then the above definitions reduce to those given in [8] for static cases.

Consider the following class of nondifferentiable continuous programming problem studied in [9]:

(CP$_0$): Minimize $\int_I \left\{ f\left(t, x(t), \dot{x}(t)\right) \right\} dt$

Subject to

$$x(a)=0=x(b),$$
$$g(t,x(t),\dot{x}(t))\leq 0, \quad t\in I$$
$$h(t,x(t),\dot{x}(t))=0, \quad t\in I$$

where, (i) f, g and h are twice differentiable functions from $I\times R^n\times R^n$ into R, R^m and R^k respectively, and

(ii) $B(t)$ is a positive semi definite $n\times n$ matrix with $B(\cdot)$ continuous on I.

The following proposition gives the Fritz John type optimality conditions which are derived in [9].

Proposition 2.1 (Fritz-John Conditions) If $\bar{x}$ is an optimal solution of (CP_0) and $h_x(.,x(.),\dot{x}(.))$ maps on the closed subspace of $C(I,R^n)$, then there exist Lagrange multipliers $r\in R$, and piecewise smooth $\bar{y}:I\to R^m$ such that

$$r\, f_x(t,\bar{x}(t),\dot{\bar{x}}(t))+\bar{y}(t)^T g_x(t,\bar{x}(t),\dot{\bar{x}}(t))+\bar{\mu}(t)^T h_x(t,\bar{x}(t),\dot{\bar{x}}(t))$$
$$=D\left[rf_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))+\bar{y}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))+\bar{\mu}(t)^T h_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\right] \quad t\in I,$$
$$\bar{y}(t)^T g(t,\bar{x}(t),\dot{\bar{x}}(t))=0, \quad t\in I$$
$$(r,\bar{y}(t))\geq 0, \quad t\in I$$
$$(r,\bar{y}(t))\neq 0, \quad t\in I$$

If $r=1$, then $\bar{x}$ is called a normal solution and the above conditions reduce to Karush-Kuhn-Tucker conditions.

Ignoring the equality constraint in (CP_0), consider the following variational problem:

(CP): Minimize $\int_I f(t,x,\dot{x})dt$

Subject to

$$x(a)=0=x(b) \tag{1}$$
$$g(t,x,\dot{x})\leq 0 \tag{2}$$

Chen [4] formulated the following Wolf type dual (CD_1) to (CP):

(CD_1): Maximize $\int_I \left\{ \left(f(t,u,\dot{u}) + y(t)g(t,u,\dot{u}) \right) - \frac{1}{2}\beta(t)G\beta(t) \right\} dt$

Subject to

$$u(a) = 0, u(b) = 0$$

$$f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,x,\dot{x}) \right) + G\beta(t) = 0,\ t \in I$$

$$y(t) \geq 0, t \in I,$$

where

$$G = f_{uu}(t,u,\dot{u}) + y(t)^T g_{xu}(t,u,\dot{u}) - 2D\left[f_{u\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_{\dot{u}} \right]$$
$$+ D^2\left(\left(f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}} \right)_{\dot{u}} \right) - D^3\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}} \right)_{\ddot{u}},$$

and $\beta(t) \in R^n,\ t \in I$

Using the invexity-like assumptions on the functions that constitute the primal problem, Chen [4] derived second-order, strong and converse duality results for the above pair of problem (CP) as (CD_1). Recently in order to relax invexity requirement on the function, further, Husain et al [5] formulated the following Mond-Weir type second-order dual to (CP) which is given below and established various duality theorems:

$(M-WCD)$: Maximize $\int_I (f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F\, \beta(t))\, dt$

Subject to

$$u(a) = 0,\ u(b) = 0$$

$$f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right)$$
$$+ \left(F + H \right)\beta(t) = 0,\quad t \in I$$

$$\int_I \left(y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)\, H\, \beta(t) \right) dt \geq 0$$

$$y(t) \geq 0,\ t \in I$$

where

$$F = f_{uu}(t,u,\dot{u}) - 2Df_{u\dot{u}}(t,u,\dot{u}) + D^2 f_{\dot{u}\dot{u}}(t,u,\dot{u}) - D^3 f_{\dot{u}\ddot{u}}(t,u,\dot{u}), t \in I$$

and

$$H = y(t)^T g_{uu}(t,u,\dot{u}) - 2D\left(y(t)^T g_{u\dot{u}}(t,u,\dot{u})\right) + D^2\left(y(t)^T g_{\dot{u}\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} - D^3\left(y(t)^T g_{\dot{u}}\right)_{\ddot{u}}$$

Husain et al [5] establish weak duality theorem under the assumption that $\int_I f(t,.,.)dt$ is second-order pseudoinvex and $\int_I y(t)^T g(t,.,.)dt$ is second-order quasi-invex with respect to η. They proved strong duality for the pair of Mond-weir type dual continuous programming problem, using the Karush-Kuhn-Tucker type necessary conditions at the optimal for the primal (CP) and hence regularity condition was needed at the optimal point of the problem (CP).

In this research a second-order dual and a generalized dual to (CP) are proposed and establish duality theorems using Fritz John necessary conditions at the optimal point for the primal (CP). Thus the requirement for a constraint qualification or regularity conditions is eliminated.

3 Fritz John Type second-order duality

We formulate the following Fritz John type second-order dual to problem (CP):

(SF_rCD): Maximize $\int_I \{f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F(t)\beta(t)\}dt$

Subject to

$$u(a) = 0 = u(b)$$

$$r\ f_u + y(t)^T g_u - D(r\ f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (rF + H)\beta(t) = 0,\ t \in I \tag{3}$$

$$\int_I (y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t))\,dt \geq 0 \tag{4}$$

$$(r, y(t)) \geq 0, \quad t \in I$$
$$(r, y(t)) \neq 0, \quad t \in I \tag{5}$$

Theorem 3.1 (Weak Duality) Let $x \in X$ be feasible to (CP) and $\left(r, u(t), y(t), \beta(t)\right)$ be feasible to $\left(SF_r CD\right)$. If $\int_I f\,dt$ be second-order pseudoinvex and $\int_I y\left(t\right)^T g\,dt$ is second-order semi-strictly pseudoinvex with respect to the same η.

Then

$$\inf(CP) \geq Sup\left(SF_r CD\right)$$

Proof: Let x be feasible for (CP) and $\left(r, u(t), y(t), \beta(t)\right)$ of $(SF_r CD)$.

Then, suppose $\int_I f(t,x,\dot{x})dt < \int_I (f(t,u,\dot{u}) - \frac{1}{2}\beta\left(t\right)^T F\beta\left(t\right))\,dt$.

This, because of second-order pseudoinvexity of $\int_I f(t,.,.)dt$ with respect to η yields,

$$\int_I \{\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta^T F \ \beta(t)\}dt < 0$$

This gives

$$\int_I r\{\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta^T F\beta(t)\}dt \leq 0 \tag{6}$$

with strict inequality in (6) if $r > 0$.

From the constraint of $\left(CP\right)$ and $\left(SF_r CD\right)$, we have

$$\int_I y(t)^T g(t,x,\dot{x})\,dt \leq \int_I (y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta\left(t\right)^T H\,\beta\left(t\right))\,dt.$$

This, because of second-order semi-strict pseudoinvexity of $\int_I y(t)^T g(t,x,\dot{x})\ dt$, with respect to η implies,

$$\int_I \{\eta^T (y(t)^T g_u) + (D\eta)^T (y(t) g_{\dot{u}})^T + \eta^T H \beta(t)\} dt \leq 0 \tag{7}$$

with strict inequality in (7) if some $y^i(t) > 0,\ t \in I,\ \ i \in \{1, 2, 3, 4, ..., m\}$.

Combining (6) and (7), we obtain

$$0 > \int_I \left[\eta^T (rf_u + y(t)^T g_u) + (D\eta)^T \left(rf_{\dot{u}} + y(t) g_{\dot{u}}\right) + \eta^T \left(rF + H\right)\beta(t)\right] dt,$$

$$0 > \int_I \eta^T \left[(rf_u + y(t)^T g_u) - D\left(rf_{\dot{u}} + y(t) g_{\dot{u}}\right) + \left(rF + H\right)\beta(t)\right] dt$$

$$+ \eta^T \left(rf_{\dot{u}} + y(t)^T g_{\dot{u}}\right)\Bigg|_{t=a}^{t=b},$$

(by integrating by parts)

$$= \int_I \eta \left[(rf_u + y(t)^T g_u) - D\left(rf_{\dot{u}} + y(t) g_{\dot{u}}\right) + \left(rF + H\right)\beta(t)\right] dt\ ,$$

$$\left(U \sin g\ \ \eta = 0,\ t = a\ and\ \ t = b\right)$$

That is,

$$\int_I \eta^T [(rf_u + y(t)^T g_u) - D\left(rf_{\dot{u}} + y(t) g_{\dot{u}}\right) + \left(rF + H\right)\beta(t)] dt < 0,$$

contradicting the equality constraint (3) of $\left(SF_r CD\right)$.

Hence $\int_I f(t, x, \dot{x}) dt \geq \int_I (f(t, u, \dot{u}) - \frac{1}{2}\beta(t)^T F \beta(t)) dt,$

giving

$$\inf\ \left(CP\right) \geq Sup\ \left(SF_r CD\right).$$

Theorem 3.2 (Strong duality) If $\bar{x}$ is optimal to (CP), then there exist $\bar{r} \in R$ and piecewise smooth $\bar{y} : I \to R^m$ such that $(\bar{r}, \bar{x}, \bar{y})$ is feasible for $(SF_r CD)$ and the corresponding values of (CP) and $(SF_r CD)$ are equal. If also,

$\int_I f(t,..,)\,dt$ is second-order pseudoinvex and $\int_I y(t)^T g(t,.,.)\,dt$ is semi-strictly pseudoinvex. Then $\bar{x}$ and $(\bar{r},\bar{x},\bar{y})$, are optimal solution of (CP) and (SF_rCD), respectively.

Proof: Since $\bar{x}$ is optimal to (CP) by Proposition 2.1, there exist $\bar{r}\in R$ and piecewise smooth $\bar{y}: I \to R^m$ such that

$$(\bar{r}\, f_x + \bar{y}(t)^T g_x) - D(\bar{r}\, f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}}) = 0,\ t\in I$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t\in I$$

$$(\bar{r},\bar{y}(t)) \geq 0,\ t\in I$$

$$(\bar{r}, y(t)) \neq 0,\quad t\in I$$

This implies that $(\bar{x},\bar{r},\bar{y},\beta(t)=0),\ t\in I$ is feasible for (SF_rCD). Equality of objective functionals is obvious. In view of second-order pseudoinvexity of $\int_I f(t,.,.)\,dt$ and second-order semi-strict pseudoinvexity of $\int_I y(t)^T g(t,..,)\,dt$ with respect to η, optimality follows by Theorem 3.1.

4 Generalized Second-order Fritz John Duality

In this section, we generalized the dual (SF_rCD).

Let $M=\{1,2,3,...,m\}$, $I_\alpha \subset M$, $\alpha = 0,1,2,3,...,k$ with $I_\alpha \cap I_\beta = \phi$, $(\alpha\neq\beta)$ and $\bigcup_{\alpha=0}^{k} I_\alpha = M$. Let $K=\{0,1,2,3,...,k\}$ and $N\subset K$.

The following is the generalized second-order Fritz John dual to (CP):

(SGF_rCD): Minimize $\int_I f(t,u,\dot{u})\,dt$

Subject to

$$u(a)=0=u(b) \tag{8}$$

$$(r\, f_x + y(t)^T g_u) - D(r\, f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (r\,F + H)\beta(t) = 0,\ t \in I \tag{9}$$

$$\sum_{i\in I_\alpha}\int_I y(t)g(t,u,\dot{u})dt \geq 0, \quad \alpha = 0,1,2,3,...,k, \tag{10}$$

$$(r, y(t)) \geq 0, \quad (r, y^i(t),\ i \in I_\alpha,\ \alpha \in N) \neq 0, \quad t \in I. \tag{11}$$

Theorem 4.1 (Weak duality) If $\int_I f\left(t,u,\dot{u}\right)dt$ is pseudoinvex

$\sum_{i\in I\alpha}\int_I y^i\left(t\right)g^i\left(t,..,\right)dt,\ \alpha \in N$ second-order semi-strictly pseudoinvex,

$\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,..,..)dt,\ \alpha \in K \setminus N$ second-order quasi-invex with respect to the same η. Then

$$\inf\left(CP\right) \geq sup\left(SGF_rCD\right).$$

Proof: Let x be feasible to (CP) and $(u, r, \overline{y}, \beta)$ be feasible for $\left(SGF_rCD\right)$.

Suppose

$$\int_I f\left(t,x,\dot{x}\right)dt < \int_I (f\left(t,u,\dot{u}\right) - \frac{1}{2}\beta\left(t\right)^T F\beta\left(t\right))\,dt.$$

This, by second-order pseudoinvexity of $\int_I f\left(t,u,\dot{u}\right)dt$,

$$\int_I \{\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta^T F\beta(t)\}dt < 0$$

Thus

$$\int_I r\{\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta^T F\beta(t)\}dt \leq 0 \tag{12}$$

with strict inequality in (12) if $r > 0$,

$$\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,x,\dot{x})dt \leq \sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,u,\dot{u})dt, \quad \alpha \in N$$

By second-order semi-strict pseudoinvexity of $\sum_{i\in I_\alpha}\int_I y^i(t)g^i dt,\ \alpha \in N,$

$$\sum_{i\in I_\alpha}\int_I\left\{\eta^T\left(y^i(t)g^i{}_u\right)+(D\eta)^T(y^i g_{\dot u})+\eta^T H^i\beta(t)\right\}dt\le 0,\quad \alpha\in N, \tag{13}$$

where

$$H^i=\left(y^i(t)\,g^i_u\right)_u-2D\left(y^i(t)\,g^i_u\right)_{\dot u}-D^2\left(y^i(t)\,g^i_{\dot u}\right)_{\dot u}-D^3\left(y^i(t)\,g^i_{\dot u}\right)_{\ddot u}$$

with strict inequality in (5), if $y^i(t)>0,\ t\in I,\ i\in I_\alpha,\ \alpha\in N.$
Also

$$\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,.,..,.)dt\le\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,u,\dot u)dt,\ \alpha\in K\setminus N$$

By second-order quasi-invexity condition $\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,.,..,.)dt,\ \alpha\in K\setminus N$, this implies that

$$\sum_{i\in I_\alpha}\int_I\left\{\eta^T(y^i(t)g^i)+(D\eta)^T(y^i(t)g^i{}_u)+\eta^T H^i\beta(t)\right\}dt\le 0,\quad \alpha\in K\setminus N \tag{14}$$

Combining (12), (13) and (14) we have

$$\int_I\{\eta^T(r\ f_u+y(t)^T g_u)+(D\eta)^T(r\ f_{\dot u}+y(t)^T g_{\dot u})+\eta^T(rF+H)\beta(t)\}\,dt<0$$

This, as earlier, yields

$$\int_I\{\eta^T\left(\ r\ f_u+y(t)^T g_u\right)-D\left(r\ f_{\dot u}+y(t)^T g_{\dot u}\right)+\left(rF+H\right)\beta(t)\}\,dt<0\,.$$

This contradicts the dual constraint of (SGF$_r$CD). Hence the conclusion follows.

Theorem 4.2 (Strong duality) If $\bar x$ is an optimal solution of (CP) then there exist $\bar r\in R$ and piecewise smooth $\bar y:I\to R^m$ such that $\left(\bar r,\bar x,\bar y,\beta\left(t\right)=0\right)$ is feasible for (SGF$_r$CD) and the corresponding value of functions of (CP) and (SGF$_r$CD) are equal. If, also, $\int_I f(t,.,.)dt$ is second-order pseudoinvex, $\sum_{i\in I_\alpha}\int_I y^i(t)g^i dt,\ \alpha\in N$ is second-order semi-strictly pseudo-invex and $\sum_{i\in I_\alpha}\int_I y^i(t)g^i dt,\alpha\in K\setminus N$ is second-order quasi-invex with respect to the same η, then $\bar x$ and $(\bar r,\bar x,\bar y,\beta)$, are optimal solution of (CD) and (SF$_r$CD) respectively.

Proof: Since $\bar{x}$ is an optimal solution of (CP), by Proposition 2.1, there exist $\bar{r} \in R$ and piecewise smooth $\bar{y} : I \to R^m$ such that

$$(\bar{r}\ f_x + y^T(t)g_x) - D(\bar{r}\ f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}}) = 0,\ t \in I$$

$$\bar{y}^T(t)g(t, x, \dot{\bar{x}}) = 0,\ \ t \in I$$

$$(\bar{r}, \bar{y}(t)) \geq 0,\ t \in I$$

$$(\bar{r}, \bar{y}(t)) \neq 0,\ t \in I$$

$$y(t)^T g(t, x, \bar{x}) = \sum_{\substack{i \in I_\alpha \\ \alpha \in N}} y^i(t)g^i(t, x, \bar{x}) + \sum_{\substack{i \in I_\alpha \\ \alpha \in K/N}} y^i(t)g^i(t, x, \bar{x}) = 0,$$

implies

$$\sum_{i \in I_\alpha} \int_I y^i(t)g^i(t, x, \bar{x})dt = 0, \alpha = 0, 1, 2, ..., k.$$

$$\sum_{\substack{i \in I_\alpha \\ \alpha \in K/N}} y^i(t)g^i(t, x, \bar{x}) = 0, \alpha \in K \setminus N\ .$$

This implies that $\left(\bar{x}, \bar{r}, \bar{y}, \beta(t) = 0\right)$ is feasible for $\left(SGF_rCD\right)$. Equality follows since the objective functionals are the same in the formulations.

Optimality of $\left(\bar{x}, \bar{r}, \bar{y}, \beta(t) = 0\right)$ for $\left(SGF_rCD\right)$ follows by Theorem 4.1. The following is the Mangasarian type [10] strict converse duality.

Theorem 4.3 (Strict Converse Duality) Assume that

(i) $\int_I f\left(t, x, \dot{x}\right)dt$ is second-order strict pseudo-invex, $\sum_{i \in I_\alpha} \int y^i(t)g^i dt,\ \ \alpha \in N$ are second-order semi-strictly pseudoinvex and $\sum_{i \in I_\alpha} \int y^i(t)g^i dt,\ \alpha \in K \setminus N$ are second-order quasi-invex with respect to the same η.

(ii) (CP) has an optimal solution $\bar{x}$. If $\left(\bar{u}, \bar{r}, \bar{y}, \beta\left(t\right)\right)$ is an optimal solution of $\left(SGF_rCD\right)$,

Then $\bar{u} = \bar{x}$ i.e., $\bar{u}$ is an optimal solution of (CP).

Proof: We assume that $\bar{u}(t) \neq \bar{x}(t)$ and exhibit a contradiction. Since $\bar{x}$ is optimal to (CP), by Theorem 3.5 there exists $(\bar{r}, \bar{y})$ with piecewise $\bar{r} \in R$ and piecewise smooth $\bar{y}: I \to R^m$ such that $(\bar{r}, \bar{x}, \bar{y}, \beta(t))$ is optimal to $(SGF_r CD)$. Since $\left(\bar{u}, \bar{r}, \bar{y}, \bar{\beta}\right)$ is an optimal solution of $(SGF_r CD)$, it implies that

$$\int_I f\left(t, \bar{x}, \dot{\bar{x}}\right) dt = \int_I f\left(t, \bar{u}, \dot{\bar{u}}\right) dt.$$

This, by the second-order strict pseudoinvexity of $\int_I f(.)\, dt$, gives

$$\int_I \left(\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta F \beta(t)\right) dt < 0.$$

Multiplying this by $\bar{r}$, we have

$$\int_I \bar{r}\left(\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta F \beta(t)\right) dt \leq 0 \tag{15}$$

with strict inequality (15) in the above with $\bar{r} > 0$.

Also $\sum_{i \in I_\alpha} \int_I \bar{y}^i(t) g^i(t, \bar{x}, \dot{\bar{x}}) dt \leq \sum_{i \in I_\alpha} \int_I \bar{y}^i(t) g^i(t, \bar{u}, \dot{\bar{u}}) dt, \quad \alpha = 0, 1, 2, 3, ..., k.$

This, because of second-order semi-strict pseudoinvexity of

$$\sum_{i \in I_\alpha} \int_I \bar{y}^i(t) g^i(t, ..., ..) dt, \quad \alpha \in N.$$

We have

$$\sum_{i \in I_\alpha} \int_I \begin{bmatrix} \eta^T \left(\bar{y}^i(t) g_u{}^i(t, \bar{u}, \dot{\bar{u}})\right) + (D\eta)^T \left(\bar{y}^i(t) g_u^i(t, \bar{u}, \dot{\bar{u}})\right) \\ + \eta^T H^i(t) \beta(t) \end{bmatrix} dt \leq 0, \quad \alpha = 0, 1, 2, ..., k \tag{16}$$

with strict inequality in the above if $\bar{y}^i(t) > 0,\ i \in I_\alpha,\ \alpha \in N.$

Combining (15) and (16) together with

$$\int_I \{\eta^T (\bar{r}\ f_u + y(t)^T g_{\dot{u}}) + (D\eta)^T (\bar{r}\ f_{\dot{u}} + y(t)^T g_u) + \eta^T (\bar{r}\ F + H) \beta(t)\} dt < 0$$

which, as earlier analysis, gives,

$$\int_I \eta^T \left[(\bar{r} f_u + y(t)^T g_u) - D(\bar{r}\ f_{\dot{u}} + y(t)^T g_{\dot{u}}) + \eta^T (\bar{r}\ F + H) \beta(t) \right] dt < 0$$

contradicting the feasibility of $(\bar{r}, \bar{u}, \bar{y}, \beta(t))$ of $(SF_r CD)$. Hence $\bar{x} = \bar{u}$.

5 Converse Duality

In this section, we shall establish Huard type [10] converse duality for the pair of dual problems (CP) and (SF_rCD) considered in earlier section.

Theorem 5.1 (Converse Duality) Let $(r, \bar{x}, \bar{y}, \beta)$ be an optimal solution of (SF_rCD). If for each $t \in I$,

(A_1) the vectors $\{F_i, H_i,\ i = 1, 2, 3, \dots, n\}$ are linearly independent where F_i and H_i are the i^{th} row of the matrices F and H respectively, and

(A_2) $y(t)^T g_x - D\, y(t)^T g_{\dot{x}} + H\,\beta(t) \neq 0,\ t \in I$

(A_3) (a) $\int \beta(t)^T (H + (y(t)^T g_x)_x)\,\beta(t)\,dt > 0$ and $\int \beta(t)^T (y(t)^T g_x) dt \geq 0$

or

(b) $\int_I \beta(H + (y(t)^T g_x)_x)\,\beta(t) dt < 0$ and $\int_I \beta(t)^T (y(t)^T g_x) dt \leq 0$.

Then $\bar{x}(t)$ is feasible for (CP) and the two objective functional have the same value. Also, if, Theorem 3.1 holds for all feasible solutions of (CP) and (SF_rCD), then $\bar{x}$ is an optimal solution of (CP).

Proof: Since (x, r, y, β) is an optimal solution of (SF_rCD), by Proposition 2.1, there exist, $\tau \in R,\ \omega \in R,\ \xi \in R,$ piecewise smooth $\theta : I \to R^m$ and $\eta : I \to R^n$ such that

$$\tau \begin{bmatrix} (f_x - Df_{\dot{x}}) - \frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_x + \frac{1}{2}D\left(\beta(t)^T F\beta(t)\right)_{\dot{x}} \\ -\frac{1}{2}D^2\left(\beta(t)^T F\beta(t)\right)_{\ddot{x}} + \frac{1}{2}D^3\left(\beta(t)^T F\beta(t)\right)_{\dddot{x}} - \frac{1}{2}D^4\left(\beta(t)^T F\beta(t)\right)_{\ddddot{x}} \end{bmatrix}$$

$$+\theta(t)^T \begin{bmatrix} r(f_{xx} - Df_{x\dot{x}}) + \left(y(t)^T g_x\right)_x - D\left(y(t)^T g_{\dot{x}}\right)_x - D\left(rf_{\dot{x}x} + \left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right) \\ +\left((rF + H)\beta(t)\right)_x - D\left((rF + H)\beta(t)\right)_{\dot{x}} \\ +D^2(rF + H)\beta(t)_{\ddot{x}} - D^3\left((rF + H)\beta(t)\right)_{\dddot{x}} + D^4\left((rF + H)\beta(t)\right)_{\ddddot{x}} \end{bmatrix} +$$

$$+\omega\left[\begin{array}{l}\left(\left(y(t)^T g_x\right)-D\left(y(t)^T g_{\dot{x}}\right)\right)-\frac{1}{2}\left(\beta(t)^T H\beta(t)\right)_x+\frac{1}{2}D\left(\beta(t)^T H\beta(t)\right)_{\dot{x}} \\ -\frac{1}{2}D^2\left(\beta(t)^T H\beta(t)\right)_{\ddot{x}}+\frac{1}{2}D^3\left(\beta(t)^T H\beta(t)\right)_{\dddot{x}}-\frac{1}{2}D^4\left(\beta(t)^T H\beta(t)\right)_{\ddddot{x}}\end{array}\right]=0 \quad (17)$$

$$\theta(t)^T\left(g_x+g_{xx}\beta(t)\right)+\omega\{g-\frac{1}{2}\left(\beta(t)^T g_{xx}\beta(t)\right)\}+\eta(t)=0, \quad t\in I \qquad (18)$$

$$\left(r\theta-\tau\beta\right)F+\left(\theta(t)-\omega\beta(t)\right)H=0, \ t\in I \qquad (19)$$

$$\theta(t)^T\left(f_x-D f_{\dot{x}}+F\beta\right)+\xi=0, \quad t\in I \qquad (20)$$

$$\omega\int_I\{y(t)^T g-\frac{1}{2}\left(\beta(t)^T H\beta(t)\right)\}\,dt=0, \ t\in I \qquad (21)$$

$$\eta(t)^T y(t)=0, \ t\in I \qquad (22)$$

$$\xi r=0, \qquad (23)$$

$$\left(\tau,\xi,\omega,\eta(t)\right)\geq 0, \ t\in I \qquad (24)$$

$$\left(\tau,\xi,\theta(t),\omega,\eta(t)\right)\neq 0, \ t\in I \qquad (25)$$

By the hypothesis (A_1), (19) yields

$$r\theta(t)-\tau\beta(t)=0, \ t\in I \qquad (26)$$

$$\theta(t)-\omega\beta(t)=0, \ t\in I \qquad (27)$$

Using equality constraint of (SF$_r$CD) along with (26), (27) in (17), we have

$$\begin{array}{l}(\tau-r\omega)\left[y(t)^T g_x-D\,y(t)^T g_x+H\beta(t)\right] \\ +r\tau\{-\frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_x+\frac{1}{2}D\left(\beta(t)^T F\beta(t)\right)_{\dot{x}}-\frac{1}{2}D^2\left(\beta(t)^T F\beta(t)\right)_{\ddot{x}} \\ \qquad +\frac{1}{2}D^3\left(\beta(t)^T F\beta(t)\right)_{\dddot{x}}-\frac{1}{2}D^4\left(\beta(t)^T F\beta(t)\right)_{\ddddot{x}}\} \\ +r\theta(t)^T\{\left(\left(rF+H\right)\beta(t)\right)_x-D\left(\left(rF+H\right)\beta(t)\right)_{\dot{x}}+D^2\left(rF+H\right)\beta(t)_{\ddot{x}} \\ \qquad -D^3\left(\left(rF+H\right)\beta(t)\right)_{\dddot{x}}+D^4\left(\left(rF+H\right)\beta(t)\right)_{\ddddot{x}}\} \\ +r\omega\{-\frac{1}{2}\left(\beta(t)^T H\beta(t)\right)_x+\frac{1}{2}D\left(\beta(t)^T H\beta(t)\right)_{\dot{x}}-\frac{1}{2}D^2\left(\beta(t)^T H\beta(t)\right)_{\ddot{x}} \\ \qquad +\frac{1}{2}D^3\left(\beta(t)^T H\beta(t)\right)_{\dddot{x}}-\frac{1}{2}D^4\left(\beta(t)^T H\beta(t)\right)_{\ddddot{x}}\}=0\end{array} \qquad (28)$$

In view of (A_2), the equality constraint of the problem (SF_rCD), implies that $r \neq 0$. Hence $r > 0$.

Let $\omega = 0$, (27) implies $\theta(t) = 0, t \in I$ and from (26) implies $\frac{\tau}{r}\beta = 0$, i.e., $\tau\beta = 0$, for $r > 0$.

Consequently from (28), we have

$$\tau\left[y(t)^T g_x - D\, y(t)^T g_{\dot{x}} + H\beta(t)\right] = 0,\ t \in I,$$

which because of hypothesis (A_2), yields $\tau = 0$. From (20), we have $\xi = 0$. Hence $(\tau, \theta(t), \xi, \eta(t), \omega) = 0,\ t \in I,$ ensuing a contradiction to Fritz John condition to (25), thus $\omega > 0$.

Multiplying (18) by y (t), then (26), (27) and (21) along with $\omega > 0$, we get

$$\int_I \{2\beta(t)^T (y(t)^T g_x) + \beta(t)^T (H + (yg_x)_x)\beta(t)\} dt = 0$$

$$\int_I \left\{\beta(t)^T (H + (yg_x)_x)\beta(t)\right\} dt = -2\int_I \beta(t)^T (y(t)^T g_x) dt \leq 0$$

But $\int_I \beta(t)^T \left(H + (y(t)^T g_x)_x\right)\beta(t) dt > 0.$

Hence $\beta(t) = 0,\ t \in I$. From equation (27), $\theta(t) = 0,\ t \in I$ and from equation (18) $g = -\frac{\eta(t)}{\omega} \leq 0,\ t \in I,$ implying the feasibility of x for (CP) and the objective functions of (CP) and (SF_rCD) are equal there. Under the stated second-order generalized invexity condition, by Theorem 4.1, $\bar{x}$ is an optimal solution for (CP).

6 Natural Boundary Values

In this section, we formulate a pair of dual variational problems with natural boundary values rather than fixed end points.

(CP_0) Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$g(t,x,\dot{x}) \leq 0, \qquad t \in I. \tag{29}$$

(SF_rCD_0) Maximize $\int_I \{ f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F(t)\beta(t)\}\,dt$

Subject to

$$\bar{r}\ f_u + y(t)^T g_u - D(\bar{r}\ f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (\bar{r}F + H)\,\beta(t) = 0,\ \ t \in I \tag{30}$$

$$\int_I \{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t)\}dt \geq 0,\ \ t \in I \tag{31}$$

$$(\bar{r}, y(t)) \geq 0,\ \ t \in I \tag{32}$$

$$(\bar{r}, y(t)) \neq 0,\ \ t \in I \tag{33}$$

$$r\ f_{\dot{u}}(t,u,\dot{u}) + \bar{y}(t)^T\ g_{\dot{u}}(t,u,\dot{u})\Big|_{t=a} = 0, t \in I \tag{34}$$

$$r\ f_{\dot{u}}(t,u,\dot{u}) + \bar{y}(t)^T\ g_{\dot{u}}(t,u,\dot{u})\Big|_{t=b} = 0,\ t \in I \tag{35}$$

We shall not repeat the proofs of Theorems 3.1-5.1, as these follow exactly on the lines of the analysis of the preceding section with slight modifications.

7 Nonlinear Programming Problems

If all functions in the problems (CP_0) and (SF_rCD_0) are independent of t and $b-a=1$, then these problems will reduce to following pair of Fritz John dual nonlinear programming problems, studied by Husain et.al [7].

(NP): Minimize $f(x)$

Subject to

$$g(x) \leq 0.$$

(SF_rD_0) Maximize $f(u)-\frac{1}{2}\beta^T\hat{F}\beta$

Subject to

$$r\,f_u+y^Tg_u+(r\,\hat{F}+\hat{H})\,\beta=0$$
$$y^Tg-\frac{1}{2}\beta^T\hat{H}\,\beta\geq 0$$
$$(r,y)\geq 0,(r,y)\neq 0,$$

where

$\hat{F}=f_{uu}\left(u\right)=\nabla^2 f\left(u\right)$, and $\hat{H}=\left(y(t)^Tg_x\right)_x=\nabla^2\left(y^Tg\right)$.

$\left(SGF_rCD_0\right)$: Maximize $\left(f(u)-\frac{1}{2}\beta^T\hat{F}\beta\right)$

Subject to

$$rf_u(u)+y^Tg_u(u)-\nabla^2\left(rf(u)+y^Tg(u)\right)\beta=0$$
$$\sum_{i\in I_\alpha}\left(y^ig^i(u)-\frac{1}{2}\beta\hat{H}^i\beta\right)\geq 0,\ \alpha=0,1,2,...,k$$
$$\left(r,y^i\right)\geq 0$$
$$\left(r,y^i,i\in I_\alpha,\alpha\in N\right)\neq 0.$$

It is pointed out that the duality results between (CP_0) and (SF_rD_0), and between (CP_0) and (SGF_rCD_0) are immediate consequences of the preceding extensive analysis of this research. If $\beta=0$, the dual problems (SF_rCD) and (SGF_rCD) reduce to the problem to the following nonlinear programming problems whose duality is extensively discussed by Weir and Mond [11]:

(NP): Minimize $f\left(x\right)$

Subject to

$$g\left(x\right)\leq 0.$$

(SF_rD) Maximize $f(u)$

Subject to

$$r\ f_u + y^T g_u = 0,$$
$$y^T g \geq 0,$$
$$(r, y) \geq 0,$$
$$(r, y) \neq 0.$$

$\left(SGF_rCD\right)$: Maximize $\left(f(u)\right)$

Subject to

$$rf_u(u) + y^T g_u(u) = 0,$$
$$\sum_{i\in I_\alpha}\left(y^i g^i(u)\right) \geq 0, \quad \alpha = 0,1,2,\ldots,k,$$
$$\left(r, y^i\right) \geq 0,$$
$$\left(r, y^i, i \in I_\alpha, \alpha \in N\right) \neq 0.$$

References

[1] O.L. Mangasarian, Second and higher order duality in nonlinear programming, *J. Math. Anal. Appl.*, **51**, (1979), 605-620.

[2] B. Mond, Second order duality in non-linear programming, *Opsearch*, **11**, (1974), 90-99.

[3] Mond and M.A. Hanson, Duality for variational problems, *J. Math. Anal. Appl*., **18**, (1965), 355-364.

[4] X.Chen, Second order duality for the variational problems, *J. Math. Anal. Appl*., **28**, (2003), 261-270.

[5] I. Husain, A. Ahmed and Mashoob Masoodi, Second order duality for variational problems, *European Journal of Pure and Applied Mathematics*, **2**(2), (2009), 278- 295.

[6] B. Mond and T. Weir, Generalized convexity and higher order duality, *J. Math. Anal. App.*, **46**, (1974), 169-174.

[7] I. Husain, N.G. Rueda and Z. Jabeen, Fritz John Second-Order Duality for Nonlinear Programming, *Applied Mathematics Letters*, **14**, (2001), 513-518.

[8] J. Zhang and B. Mond, Duality for a nondifferentiable programming problem, *Bulletin Australian Mathematical society*, **55**, (1997), 20-44.

[9] S. Chandra, B.D. Craven and I. Husain, A class of nondifferentiable continuous programming problems, *J. Math. Anal. Appl.*, **107**, (1985), 122-131.

[10] O.L Mangasarian, *Nonlinear Programming*, McGraw-Hill Book Company, New York, 1969.

[11] T. Weir and B. Mond, Sufficient Fritz John optimality conditions and duality for Nonlinear programming Problems, *Opsearch*, **23**(3), (1986), 129-141.