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On Nondifferentiable Nonlinear Programming

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Abstract

In this paper, we obtain optimality conditions for a class of nondifferentiable nonlinear programming problems with equality and inequality constraints in which the objective contains the square root of a positive semidefinite quadratic function and is ,therefore, not differentiable. Using Karush-Kuhn-Tucker optimality, Mond-Weir dual to this problem is constructed and various duality results are validated under suitable generalized invexity hypotheses. A mixed type dual to the problem is also formulated and duality results are similarly derived.

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1 Introduction

In mathematical programming, there are a large number of researchers who have discussed duality for a problem involving the square root of a positive semidefinite. Mond studied Wolfe type duality while Chandra et. al. [1] investigated Mond-Weir type duality for a class of nondifferentiable mathematical programming problem containing square root term. Subsequently several research papers have appeared in the literature, namely Chandra and Husain [2] and references exist there. The popularity of this kind of problem seems to originate from the fact that, even though the objective functions and or constraint functions are non-smooth, a simple representation for the dual problem may be found. The nonsmooth mathematical programming deals with much more general kind of functions by means of generalized subdifferential, or quasidifferentials. However, the square root of a positive semidefinite quadratic form is one of the four cases of a nondifferentiable function for which subdifferential can explicitly be written.

In this paper, we obtain optimality conditions for a class of nondifferentiable nonlinear programming problem having square root term in the objective function with equality and inequality constraints. It is to be remarked here, a constrained optimization problem with equality and inequality constraints represents a majority of real life problems, and hence it is important. For this class of problem, Mond-Weir duality is investigated using generalized invexity assumptions. A mixed type dual problem to this problem is also constructed to obtain various duality results.

2 Statement of the Problem and Related Pre-requisites

We consider the following nondifferentiable nonlinear programming problems:

(EP): Minimize $f(x)+\left(x^T B x\right)^{1/2}$

subject to

$$g(x) \leq 0, \tag{1}$$

$$h(x) = 0, \tag{2}$$

Where

(i) $f : R^n \to R,\ g : R^n \to R^m$ and $h : R^n \to R^p$ are continuously differentiable functions.

(ii) B is an $n \times n$ symmetric positive semi-definite matrix.

We recall the following definitions of generalized invexity which will be used to derive various duality results.

Definitions 2.1: (i) A function $\phi : R^n \to R$ is said to be quasi-invex with respect to a vector function $\eta = \eta(x,u)$, if

$$\phi(x) \leq \phi(u) \Rightarrow \eta^T(x,u)\nabla\phi(u) \leq 0 .$$

(ii) A function ϕ is said to be pseudo-invex with respect to a vector function $\eta = \eta(x,u)$, if

$$\eta^T(x,u)\nabla\phi(u) \geq 0 \Rightarrow \phi(x) \geq \phi(u).$$

(iii) ϕ is said to be the strictly pseudoinvex with respect to η if $x \neq u$,

$\eta^T(x,u)\nabla\phi(u) \geq 0 \Rightarrow \phi(x) > \phi(u)$

Equivalently, if

$$\phi(x) \leq \phi(u) \Rightarrow \eta^T(x,u)\nabla\phi(u) < 0.$$

We shall make use of the generalized Schwartz inequality [3]

$$\left(x^T B\omega\right) \leq \left(x^T B x\right)^{1/2}\left(\omega^T B\omega\right)^{1/2}$$

The equality holds if, for $\lambda \geq 0$, $Bx = \lambda B\omega$. The function $\phi(x) = \left(x^T B x\right)^{1/2}$, being convex and everywhere finite, has a subdifferential in the sense of convex analysis.

The subdifferential of $\left(x^T B x\right)^{1/2}$ is given by

$$\partial\left(x^T B x\right)^{1/2} = \left\{Bw \middle| x^T Bw = \left(x^T B x\right)^{1/2}, \text{ where } w \in R^n, \text{ and } w^T B w \leq 1\right\}.$$

We also require the Mangasarian-Fromovitz constraint qualification which is described as the following:

Let $\bar{x} \in \Omega$ be the set of feasible solution of the problem (EP), that is, $\Omega = \left\{x \in R^n \middle| g(x) \leq 0, h(x) = 0\right\}$, and by $A(\bar{x})$, the set of inequality active constraint indices, that is, $A(\bar{x}) = \left\{j, \middle| g_j(\bar{x}) = 0\right\}$, where $\bar{x} \in \Omega$.

We say the Mangasarian-Fromovitz constraint qualification holds at $\bar{x} \in \Omega$ when the equality constraint gradients $\nabla h_1(\bar{x}), \nabla h_2(\bar{x}), ..., \nabla h_p(\bar{x})$ are linearly independent and there exist a vector $d \in R^n$ such that $\nabla h(\bar{x}) d = 0$ and $\nabla g_j(\bar{x}) d < 0$, for all $j \in A(\bar{x})$.

3 Optimality Conditions

In this section, optimality conditions for the problem (EP) are obtained.

Theorem 3.1 (Fritz John Necessary Optimality Condition) If $\bar{x}$ is an optimal solution of (EP), then there exist $r \in R$, $y \in R^m$, $z \in R^p$ and $w \in R^n$ such that

$$r\nabla f\left(\bar{x}\right)+B\omega+\nabla y^{T}g\left(\bar{x}\right)+\nabla z^{T}h\left(\bar{x}\right)=0,$$
$$y^{T}g\left(\bar{x}\right)=0,$$
$$\bar{x}^{T}Bw=\left(\bar{x}^{T}B\bar{x}\right)^{1/2},$$
$$\omega^{T}B\omega\leq 1,$$
$$\left(r,y\right)\geq 0$$
$$\left(r,y,z\right)\neq 0.$$

Proof: The problem (EP) may be written as

Minimize $\phi\left(x\right)=f\left(x\right)+\psi\left(x\right),$

Subject to

$$-g\left(x\right)\in R_{+}^{m},$$
$$h\left(x\right)=0.$$

Where R_{+}^{m} is the non-negative orthant of R^{m} and the nondifferentiable convex function $\psi=\left(x^{T}B\,x\right)^{1/2}$.

A Fritz John theorem [4] shown that the necessary condition for a minimum of (EP) at $\bar{x}$ are the existence of Lagrange multipliers $r\in R,\ y\in R^{m}$, $z\in R^{p}$ and $\omega\in R^{n}$ such that

$$0\in r\,\partial\phi\left(\bar{x}\right)+r\partial\psi\left(\bar{x}\right)+\partial y^{T}g\left(\bar{x}\right)+\partial z^{T}h\left(\bar{x}\right)$$

This implies

$$0\in r\left\{\nabla f\left(\bar{x}\right)\right\}+r\{B\omega\,\big|\,\omega\in R^{n},\bar{x}^{T}Bw=\left(\bar{x}^{T}B\bar{x}\right)^{1/2},\ \omega^{T}B\omega\leq 1\}+\left\{\nabla y^{T}g\left(\bar{x}\right)+\nabla z^{T}h\left(\bar{x}\right)\right\}$$

which implies

$$r\left(\nabla f\left(\bar{x}\right)+B\omega\right)+\nabla y^{T}g\left(\bar{x}\right)+\nabla z^{T}h\left(\bar{x}\right)=0,$$
$$y^{T}g\left(\bar{x}\right)=0,$$
$$\bar{x}^{T}Bw=\left(\bar{x}^{T}B\bar{x}\right)^{1/2},$$
$$\omega^{T}B\omega\leq 1,$$
$$\left(r,y\right)\geq 0,$$
$$\left(r,y,z\right)\neq 0.$$

Thus the theorem follows.

Karush-Kuhn-Tucker type optimality conditions can be deduced from the above Fritz John optimality condition if Mangasarian-Fromovitz Constraint Qualification holds at $\bar{x} \in \Omega$.

The following theorem gives the Karush-Kuhn-Tucker types optimality conditions.

Theorem 3.2 (K-K-T optimality condition) If $\bar{x}$ is an optimal solution of (EP) and satisfies Mangasarian-Fromovitz constraint qualification, then there exist $r \in R, y \in R^m, z \in R^p$ and $\omega \in R^n$ such that

$$\nabla f(\bar{x}) + B\omega + \nabla y^T g(\bar{x}) + \nabla z^T h(\bar{x}) = 0,$$

$$y^T g(\bar{x}) = 0,$$

$$\bar{x}^T B w = \left(\bar{x}^T B \bar{x}\right)^{1/2},$$

$$\omega^T B \omega \leq 1,$$

$$y \geq 0.$$

4 Mond-Weir type Duality

In this section, we present the Mond-Weir type dual to (EP). Using Karush-Kuhn-Tucker necessary optimality conditions, Wolfe type dual to the following problem was formulated in [5].

Problem: (P) Minimize $f(x) + \left(x^T B x\right)^{1/2}$

Subject to

$$g(x) \leq 0.$$

Dual (WD):

Maximize $f(u) + y^T g(u) - u^T \left(\nabla y^T g(u) + \nabla f(u)\right)$

subject to

$$\nabla f(u) + B\,\omega + \nabla y^T g(u) = 0,$$
$$\omega^T B\,\omega \le 1,$$
$$y \ge 0.$$

The problem (WD) is a dual to (P) assuming that f and g are convex:

Further, Chandra et. al. [1] in order to weaken the convexity requirements in [5] for duality to hold, formulated the following Mond-Weir type dual to the problem (P).

(M-WD): Maximize $f(u) + u^T B\omega$

subject to

$$\nabla f(u) + B\,\omega + \nabla y^T g(u) = 0,$$
$$y^T g(u) \ge 0,$$
$$\omega^T B\omega \le 1,$$
$$y \ge 0.$$

and established duality results assuming that $f(.) + (.)^T B\omega$ is pseudoconvex for all $\omega \in R^n$ and $y^T g(.)$ is quasiconvex. Later Mond and Smart [6] generalized the results by Zhang and Mond [7] and Chandra et. al. [1] to invexity conditions.

Here, we propose the following Mond-Weir type dual to the problem (EP) to study duality:

(M-WED): Maximize $f(u) + u^T B\omega$

Subject to

$$\nabla f(u) + B\omega + \nabla y^T g(u) + \nabla z^T h(u) = 0, \quad (3)$$
$$\omega^T B\omega \le 1, \quad (4)$$
$$y^T g(u) \ge 0, \quad (5)$$
$$z^T h(u) \ge 0, \quad (6)$$
$$y \ge 0. \quad (7)$$

Theorem 4.1 (Weak Duality) Let x be feasible for (EP) and (u, y, ω) feasible for (M-WED). If for all feasible (x, u, y, ω), $f(.) + (.)^T B\omega$ is pseudoinvex, $y^T g(.)$ and $z^T h(.)$ are quasi-invex with respect to the same η, then

infimum (EP) $\geq$ supremum (M-WED)

Proof: Since x is feasible for (EP) and (u, y, ω) is feasible for (M-WED), we have

$$y^T g(x) \leq y^T g(u) \tag{8}$$

$$z^T h(x) \leq z^T h(u) \tag{9}$$

By quasi-invexity of $y^T g(.)$ and $z^T h(.)$ with respect to the same η, (8) and (9) respectively yield

$$\eta^T (x,u) \nabla y^T g(u) \leq 0 \tag{10}$$

$$\eta^T (x,u) \nabla z^T h(u) \leq 0 \tag{11}$$

Combining (10) and (11), we have

$$\eta^T (x,u)\left(\nabla y^T g(u) + \nabla z^T h(u)\right) \leq 0$$

which because of (3) gives

$$\eta^T (x,u)\left(\nabla f(x) + B\omega\right) \geq 0$$

By pseudoinvexity of $f(.) + (.)^T B\omega$, this implies

$$f(x) + x^T B\omega \geq f(u) + u^T B\omega$$

Since $\omega^T B\omega \leq 1$, by the generalized Schwartz inequality, it follows that

$$f(x) + \left(x^T B x\right)^{1/2} \geq f(u) + u^T B\omega$$

giving

infimum (EP) $\geq$ supremum (M-WED).

Theorem 4.2 (Strong Duality) If $\bar{x}$ is an optimal solution of (EP) and MFCQ holds at $\bar{x}$, then there exist $y \in R^m$, $z \in R^p$ and $\omega \in R^n$ such that $(\bar{x}, y, \omega, z)$ is feasible for (M-WED) and the corresponding values of (EP) and (M-WED) are equal. If, also $f(.)+(.)^T B\omega$ is pseudoinvex for all $\omega \in R^n$, $y^T g(.)$ and $z^T h(.)$ are quasi-invex with respect to the same η. Then $(\bar{x}, y, \omega, z)$ is an optimal solution of (M-WED).

Proof: Since $\bar{x}$ is an optimal solution of (EP) and MFCQ holds at $\bar{x}$, then from Theorem 3.2, then there exist $y \in R^m$, $z \in R^p$ and $\omega \in R^n$ such that

$$\nabla f(\bar{x}) + B\omega + \nabla y^T g(\bar{x}) + \nabla z^T h(\bar{x}) = 0,$$

$$y^T g(\bar{x}) = 0,$$

$$\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2},$$

$$\omega^T B\omega \leq 1,$$

$$y \geq 0.$$

Since $\bar{x}$ is feasible for (EP) and (M-WED), $h(\bar{x}) = 0$, which implies

$$z^T h(\bar{x}) = 0,$$

$$\nabla f(\bar{x}) + B\omega + \nabla y^T g(\bar{x}) + \nabla z^T h(\bar{x}) = 0,$$

$$y^T g(\bar{x}) = 0,$$

$$\omega^T B\omega \leq 1,$$

and $$y \geq 0.$$

If $f(.)+(.)^T B\omega$ is pseudoinvex for all $\omega \in R^n$, $y^T g(.)$ and $z^T h(.)$ are quasi-invex with respect to the same η. Then from Theorem 4.1, $(\bar{x}, y, \omega, z)$ is an optimal solution for (M-WED).

The following is another dual to the following (EP):

(M-WED1): Maximize $f(u) + u^T B\omega$

subject to

$$\nabla f(u) + B\omega + \nabla y^T g(u) + \nabla z^T h(u) = 0,$$
$$y_i g_i(u) \geq 0, \ i = 1, 2, ..., m$$
$$z^T h(u) \geq 0,$$
$$\omega^T B\omega \leq 1,$$
$$y \geq 0.$$

This is a dual of (EP) if $f(.) + (.)^T B\omega$ is pseudo-invex and each $y_i g_i$, $i = 1, 2, ..., m$ is quasi-invex and $z^T h(u)$ is quasi-invex with respect to the same η. It is remarked here that if $g_i(.)$ is quasi-invex with respect to same η, $y_i \geq 0$, then $y_i g_i$ is quasi-invex with respect to η. The problem (M-WED1) is a dual to (EP) if $f(.) + (.)^T B\omega$ is pseudoinvex and each $g_i, i = 1,2,..m$ is quasi-invex and $z^T h(.)$ is quasi-invex with respect to the same η.

Theorem 4.3 (Strict Converse duality) Assume that $f(.) + (.)^T B\omega$ for all $\omega \in R^n$ is strictly pseudoinvex, $y^T g(.)$ is quasi-invex and $z^T h(.)$ is quasi-invex with respect to the same η. Assume also that (EP) has an optimal solution $\bar{x}$ which satisfies Mangasarian-Fromovitz constraint qualification. If $(\bar{u}, y, z, \omega)$ is an optimal solution. Then $\bar{u}$ is an optimal solution of (EP) with $\bar{u} = \bar{x}$.

Proof: We assume that $\bar{u} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution of (EP), it implies from Theorem 4.2 that there exists $y \in R^m, z \in R^p$ and $\omega \in R^n$ such that $(\bar{x}, y, z, \omega)$ is an optimal solution of (M-WED1). Since $(\bar{u}, y, z, \omega)$ is also an optimal solution of (M-WED1), it follows that

$$f(\bar{x}) + \bar{x}^T B\omega = f(\bar{u}) + \bar{u} B\omega$$

This, by strict pseudoinvexity of $f(.) + (.)^T B\omega$ yields

$$\eta^T(x,u)\left(\nabla f(\bar{u}) + B\omega\right) < 0 \tag{12}$$

From the constraints of (EP) and (M-WED), we have

$$y^T g(\bar{x}) \leq y^T g(\bar{u}),$$

$$z^T h(\bar{x}) \leq z^T h(\bar{u}),$$

which by quasi-invexity of $y^T g(.)$ and $z^T h(.)$ with respect to the same η, yield

$$\eta^T(\bar{x},\bar{u})\nabla y^T g(\bar{u}) \leq 0 \tag{13}$$

$$\eta^T(\bar{x},\bar{u})\nabla z^T h(\bar{u}) \leq 0 \tag{14}$$

From (12), (13) and (14), we have

$$\eta^T\left[\left(\nabla f(x) + B\bar{\omega}\right) + \nabla y^T g(u) + \nabla z^T h(u)\right] < 0,$$

contradicting the feasibility of $(\bar{u}, y, z, \omega)$ for (M-WED).

Hence

$$\bar{u} = \bar{x}.$$

Theorem 4.4 (Converse duality) Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\omega})$ be optimal to (M-WED) at which

(A_1): the matrix $\nabla^2\left(f(\bar{x}) + \bar{y}^T g(\bar{x}) + \bar{z}^T h(\bar{x})\right)$, is positive or negative definite and

(A_2): the vectors $\nabla \bar{y}^T g(\bar{x})$ and $\nabla \bar{z}^T h(\bar{x})$ are linearly independent.

If, for all feasible (x, u, y, z, ω), $f(.) + (.)^T B\omega$ is pseudoinvex, $y^T g(.)$ and $z^T h(.)$ are quasi-invex with respect to the same η, then $\bar{x}$ is an optimal solution of (EP).

Proof: By Theorem 3.1, there exist $\tau \in R$, $\theta \in R^n$, $\alpha \in R$, $\beta \in R$, $\eta \in R^m$ and $\gamma \in R$ such that

$$\tau\left(\nabla f+B\omega\right)+\theta^{T}\left(\nabla^{2}f\left(u\right)+\nabla^{2}y^{T}g\left(u\right)+\nabla^{2}z^{T}h\left(u\right)\right)$$
$$+\ \alpha\nabla y^{T}g+\beta\nabla z^{T}h\left(u\right)=0, \tag{15}$$

$$\theta\nabla g\left(u\right)+\alpha g+\eta=0, \tag{16}$$

Implies

$$\theta^{T}\nabla y^{T}g\left(u\right)=0, \tag{17}$$

$$\theta^{T}\nabla z^{T}h\left(u\right)=0, \tag{18}$$

$$\tau\left(x^{T}B\right)+\theta B-2\gamma B\omega=0, \tag{19}$$

$$\alpha\, y^{T}g=0, \tag{20}$$

$$\beta\, z^{T}g=0, \tag{21}$$

$$\gamma\left(1-\omega^{T}B\omega\right)=0, \tag{22}$$

$$\eta^{T}\ y=0, \tag{23}$$

$$\left(\tau,\alpha,\beta,\gamma,\eta\right)\geq 0, \tag{24}$$

$$\left(\tau,\alpha,\beta,r,\eta,\theta\right)\neq 0. \tag{25}$$

Using (3) in (15), we have

$$\tau\left(\nabla y^{T}g\left(u\right)+\nabla z^{T}h\left(u\right)\right)+\theta\nabla^{2}\left(f\left(\bar{u}\right)+y^{T}g\left(u\right)+z^{T}h\left(u\right)\right)$$
$$+\alpha\nabla y^{T}g\left(u\right)+\beta\nabla^{2}\left(z^{T}h\left(u\right)\right)=0,$$

$$\left(\alpha-\tau\right)\nabla y^{T}g\left(u\right)+\left(\beta-\tau\right)\nabla^{2}z\,h\left(u\right)+\theta^{T}\nabla^{2}\left(f\left(u\right)+y^{T}g\left(u\right)+z^{T}h\left(u\right)\right)=0,$$

$$\Rightarrow\left(\alpha-\tau\right)\theta^{T}\nabla y^{T}g\left(u\right)+\left(\beta-\tau\right)\theta^{T}\nabla^{2}z^{T}h\left(u\right)+\theta^{T}\nabla^{2}\left(f\left(u\right)+y^{T}g\left(u\right)+z^{T}h\left(u\right)\right)=0,$$

Which because of (17) and (18), yields

$$\theta^{T}\ \nabla^{2}\left(f\left(u\right)+y^{T}g\left(u\right)+z^{T}h\left(u\right)\right)=0,$$

By$\left(\mathrm{A}_1\right)$, it follows that $\theta=0.$

Then (15) implies

$$\left(\alpha-\tau\right)\nabla\ y^{T}g\left(u\right)+\left(\beta-\tau\right)\nabla\ z^{T}h\left(u\right)=0.$$

By the linear independent of $\nabla \bar{y}^T g(\bar{x})$ and $\nabla \bar{z}^T h(\bar{x})$, this gives $(\alpha - \tau) = 0$, and $(\beta - \tau) = 0$.

Let $\theta = 0$, $\tau > 0$. Then $\alpha = 0, \beta = 0$. Consequently (16) implies $\eta = 0$.
The relation (19) and (22) together imply that $\gamma = 0$. This leading to a contradiction to (25). Hence
$\tau > 0$, $\alpha > 0$ and $\beta > 0$. Also $\theta = 0$, $\tau > 0$ and (19) give
$x^T B = \frac{2\gamma}{\tau} B\omega$. Hence $\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2} \left(\omega^T B\omega\right)^{1/2}$.

If $\gamma > 0$, then (22) give $\omega^T B\omega = 1$ and so $\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2}$.

If $\gamma = 0$, (19) gives $B\bar{x} = 0$. so we still get

$$\left(\bar{x}^T B\omega\right) = \left(\bar{x}^T B\bar{x}\right)^{1/2}. \tag{26}$$

Thus, in earlier case, we obtain $\left(\bar{x}^T B\omega\right) = \left(\bar{x}^T B\bar{x}\right)^{1/2}$.

Therefore, from (26), we have

$$f(\bar{x}) + \left(\bar{x}^T B\bar{x}\right)^{1/2} = f(\bar{x}) + \left(\bar{x}^T B\omega\right).$$

The equality of objective values follows.

If, for all feasible $(\bar{x}, u, y, z, \omega)$, $f(.) + (.)^T B\omega$ is pseudoinvex, $y^T g(.)$ and $z^T h(.)$ is quasi-invex with respect to the same η, then by Theorem 4.1 , it implies $\bar{x}$ is an optimal solution of (EP).

5 Mixed type duality

Let $M = \{1, 2, ..., m\}$, $L = \{1, 2, ...l\}$, $I_\alpha \subseteq M$, $\alpha = 0, 1, 2, ..., r$

with $I_\alpha \cap I_\beta = \phi, \alpha \neq \beta$ and $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $J_\alpha \subseteq L, \alpha = 0,1,2,3...,r$ with $J_\alpha \cap J_\beta = \phi, (\alpha \neq \beta)$ and $\bigcup_{\alpha=0}^{r} J_\alpha = L$.

In relation to (EP), consider the problem.

Mix (ED): Maximize $f(u) + u^T B\omega + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j h_j(u)$

subject to

$$\nabla f(u) + B\omega + \nabla\ y^T g(u) + \nabla z^T h(u) = 0 \tag{27}$$

$$\sum_{i \in I_\alpha} y_i\, g_i(u) \geq 0, \quad \alpha = 1,2,...,r. \tag{28}$$

$$\sum_{j \in J_\alpha} z_j h_j(u) \geq 0, \quad \alpha = 1,2,...,r. \tag{29}$$

$$\omega^T B\omega \leq 1 \tag{30}$$

$$y \geq 0 \tag{31}$$

Theorem 5.1 (Weak Duality) Let x be feasible and (u, y, z, ω) feasible for (ED). If for all feasible (x, u, y, z, ω), $f(.) + (.)^T B\omega + \sum_{i \in I_0} y_i g_i(.) + \sum_{j \in J_0} z_j h_j(.)$ is pseudoinvex, $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1,2,...,r$ is quasi-invex and $\sum_{j \in J_\alpha} z_j h_j(.)$, $\alpha = 1,2,...,r$ is quasi-invex with respect to the same η, then

$$f(x) + (x^T Bx)^{1/2} \geq f(u) + u^T B\omega + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j h_j(u)$$

That is,

infimum (EP) $\geq$ supremum Mix (ED).

Proof: Since x is a feasible for (ED) and (u, y, z, ω) is feasible for Mix (ED), we have

$$\sum_{i\in I_\alpha} y_i g_i(x) \le \sum_{i\in I_\alpha} y_i g_i(u) \ , \quad \alpha = 1,2...r$$

By quasi-invexity of $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1,2,...,r$, this implies

$$\eta^T(x,u)\, \nabla \left(\sum_{i\in I_\alpha} y_i g_i(u) \right) \le 0, \quad \alpha = 1,2,...,r$$

$$\eta^T(x,u)\, \nabla \left(\sum_{i\in M-I_0} y_i g_i(u) \right) \le 0, \quad \alpha = 1,2,...,r \tag{32}$$

$f(.)+(.)^T B\omega + \sum_{i\in I_0} y_i(.) g_i(.) + \sum_{j\in J_0} z_j(.) h_j(.)$ is pseudo-invex.

Also $\sum_{j\in J_\alpha} z_j h_j(x) \le \sum_{j\in J_\alpha} z_j h_j(u)$, $\alpha = 1,2,...r$.

By quasi-invexity of $\sum_{j\in J_\alpha} z_j h_j(.)$, $\alpha = 1,2,...r$

This gives $\eta^T(x,u)\, \nabla \left(\sum_{j\in I_\alpha} z_j h_j(u) \right) \le 0$, $\alpha = 1,2,...,r$

$$\eta^T(x,u)\, \nabla \, (\sum_{j\in L-J_0} z_j h_j(u)) \le 0, \quad \alpha = 1,2,...,r \tag{33}$$

From the (27), it follows that

$$\eta^T(x,u)[\nabla f(u) + B\omega + \nabla \sum_{i\in I_0} y_i g_i(u) + \nabla \sum_{j\in J_0} z_j h_j(u)] \ge 0$$

The pseudoinvexity of $f(.)+(.)B\omega + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j h_j(u)$, this yield

$$f(x) + \left(x^T B\omega\right) \ge f(u) + u^T B\omega + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j h_j(u)$$

Since $\omega^T B\, \omega \le 1$, by the generalized Schwartz inequality, this implies

$$f(x) + \left(x^T B x\right)^{1/2} \ge f(u) + u^T B\omega + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j h_j(u)$$

implying,

infimum (EP) ≥ supremum Mix (ED).

Theorem 5.2 (Strong Duality) If $\overline{x}$ is an optimal solution of (EP) and Mangasarian-Fromovitz Constraint Qualification is satisfied at $\overline{x}$, then there exist $y \in R^m$, $z \in R^l$ and $\omega \in R^n$ such that (x, u, z, ω) is feasible for Mix (ED) and the corresponding value of (EP) and Mix(ED) are equal. If, also, $f(.)+(.)B\omega+\sum_{i\in I_0} y_i g_i(.)+\sum_{j\in J_0} z_j h_j(.)$ is pseudoinvex for all $\omega \in R^n$, $\sum_{i\in I_\alpha} y_i g_i(.)$ and $\sum_{j\in J_\alpha} z_j h_j(.)$ are quasi-invex with respect to the same η, then (x, y, z, ω) is optimal for Mix (ED).

Proof: Since $\overline{x}$ is optimal solution of (EP) and MFCQ is satisfied at x, then from Theorem 3.2, there exist $y \in R^m, z \in R^l$ and $\omega \in R^n$ such that

$$f(\overline{x})+B\omega+\nabla y^T g(\overline{x})+\nabla z^T h(\overline{x})=0,$$

$$y^T g(\overline{x}) \geq 0,$$

$$\omega^T B\ \omega \leq 1,$$

$$\overline{x}\, B\omega = \left(\overline{x}^{\,T} B \overline{x}\right)^{1/2},$$

$$y \geq 0.$$

The relation $y^T g(\overline{x}) \geq 0$, and $z^T h(\overline{x}) \geq 0$, are obvious. From the above it implies that $(\overline{x}, y, z, \omega)$ is feasible and the corresponding value of (EP) and (Mix ED) are equal. If $f(.)+(.)^T B\omega+\sum_{i\in I_0} y_i g_i(.)+\sum_{j\in J_0} z_j h_j(.)$ is quasi-invex, $y^T g(.)$ and $z^T h(.)$ are quasi-invex with respect to the same η, then from Theorem 5.1, (x, y, z, ω) must be an optimal solution of (Mix ED).

We now consider some special cases of (Mix ED).

If $I_0 = \phi,\ I_\alpha = M$ for some $\alpha \in \{1, 2, ..., r\}$ and $J_\alpha = \phi$, $\alpha = 0, 1, 2, ..., r$. Then (Mix ED) be (WD) and (WD) is dual to (EP) if $f(.)+(.)^T B\omega + y^T g(.)$ is pseudoinvex with respect to the same η.

If $J_\alpha = \phi$, $\alpha = 0,1,2,...,r$. Then (Mix ED) become the Mixed dual to the problem (P) considered by Zhang and Mond [7] and generalized dual to (P) if $f(.)+(.)^T B\omega + y^T g(.)$ is pseudoinvex and $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1,2,...,r$ is quasi-invex with respect to the same η.

Theorem5.3 (Converse duality) Let $(\bar{x},y,z,\omega)$ be an optimal solution of Mix (ED) at which

(A_1): the matrix $\nabla^2\left(f(\bar{x})+y^T g(\bar{x})+z^T h(\bar{x})\right)$ is positive or negative definite and

(A_2): the vectors $\left\{\sum_{i\in I_\alpha}\nabla y_i g_i(\bar{x}),\ \sum_{j\in J_\alpha}\nabla z_j h_j(\bar{x}),\ \alpha = 1,2,...,r\right\}$ are linearly independent. If $f(.)+(.)^T B\omega$ is pseudoinvex, $\sum_{i\in I_\alpha} y_i g_i(.)$ and $\sum_{j\in J_\alpha} z_j h_j(.)$ are quasi-invex with respect to the same η. Then $\bar{x}$ is an optimal solution of (EP).

Proof:

$$\tau(\nabla f(x)+B\omega+\sum_{i\in I_0}\nabla y_i^T g_i(\bar{x})+\sum_{j\in J_0}\nabla z_j h_j(\bar{x})) + \theta^T\left(\nabla^2\left(f(x)+y^T g(\bar{x})+z^T h(\bar{x})\right)\right)$$

$$+\sum_{\alpha=1}^{r} p_\alpha\left(\sum_{i\in I_\alpha}\nabla y_i g_i(\bar{x})\right)+\sum_{\alpha=1}^{r} q_\alpha\left(\sum_{j\in J_\alpha}\nabla z_j h_j(\bar{x})\right)=0 \tag{34}$$

$$\tau B\bar{x}+\theta^T B-2\gamma B\omega=0 \tag{35}$$

$$\tau g_i(\bar{x}) + \theta^T\nabla g_i(\bar{x})+\eta=0,\ i\in I_\alpha,\ \alpha = 1,2,...,r \tag{36}$$

$$\tau h_j(\bar{x})+\theta^T\nabla h_j(\bar{x})=0,\ j\in J_\alpha,\ \alpha = 1,2,...,r \tag{37}$$

$$p_\alpha(\sum_{i\in I_\alpha} y_i g_i(x))=0,\ \alpha = 1,2,...,r \tag{38}$$

$$q_\alpha(\sum_{j\in J_\alpha} z_j h_j(x))=0,\ \alpha = 1,2,...,r \tag{39}$$

$$\eta^T y=0 \tag{40}$$

$$\gamma\left(1-\omega^T B\omega\right)=0 \tag{41}$$

$$\left(\tau, p_1, p_2, ..., p_r, \gamma, \eta, q_1, q_2, ..., q_r\right) \geq 0 \tag{42}$$

$$\left(\tau, p_1, p_2, ..., p_r, \gamma, \theta, \eta, q_1, q_2, ..., q_r\right) \neq 0 \tag{43}$$

From (36), it follows that

$$\theta^T \sum_{i \in I_\alpha} \nabla y_i g_i\left(\bar{x}\right) = 0, \quad \alpha = 1, 2, ..., r \tag{44}$$

From (37), it follows that

$$\theta^T \sum_{j \in j_\alpha} \nabla Z_J h_j\left(\bar{x}\right) = 0, \quad \alpha = 1, 2, ..., r \tag{45}$$

Using equality constraint of the problem Mix(ED) in (34), we have

$$\sum_{\alpha=1}^{r}\left(p_\alpha - \tau\right)\left(\sum_{i \in I_\alpha} \nabla y_i g_i\left(\bar{x}\right)\right) + \sum_{\alpha=1}^{r}\left(q_\alpha - \tau\right)\left(\sum_{j \in J_\alpha} \nabla z_j h_j\left(\bar{x}\right)\right)$$
$$+\theta^T \nabla^2 \left(f\left(\bar{x}\right) + y^T g\left(\bar{x}\right) + z^T h\left(\bar{x}\right)\right) = 0 \tag{46}$$

Multiplying (46) by θ, and then using (44) and (45), we have

$$\theta^T \nabla^2 \left(f\left(\bar{x}\right) + y^T g\left(\bar{x}\right) + z^T h\left(\bar{x}\right)\right)\theta = 0.$$

By $\left(A_1\right)$, from this we obtain

$$\theta = 0. \tag{47}$$

Then (46) implies

$$\sum_{\alpha=1}^{r}\left(p_\alpha - \tau\right)\left(\sum_{i \in I_\alpha} \nabla y_i g_i\left(\bar{x}\right)\right) + \sum_{\alpha=1}^{r}\left(q_\alpha - \tau\right)\left(\sum_{j \in J_\alpha} \nabla z_j h_j\left(\bar{x}\right)\right) \tag{48}$$

Since the vector $\{\sum_{i \in I_\alpha} \nabla y_i g_i\left(\bar{x}\right), \sum_{j \in J_\alpha} \nabla z_j h_j\left(\bar{x}\right), \ \alpha = 1, 2, ..., r\}$ are linearly independent, (48) yields

$$p_\alpha - \tau = 0, \quad q_\alpha - \tau = 0, \ \alpha = 1, 2, ..., r \tag{49}$$

If $\tau = 0$, (49) implies $p_\alpha = 0 = q_\alpha$, $\alpha = 1, 2, ..., r$.

From (36) implies $\eta = 0$ and from (35) together with (41), we have $\gamma = 0.$

Hence $\left(\tau, \theta, p_1, p_2, ..., p_r, \gamma, \eta, q_1, q_2, ..., q_r\right) = 0$ contradicting to (43), hence $\tau > 0.$

Consequently $p_\alpha > 0$ and $q_\alpha > 0$, $\alpha = 1, 2, ..., r.$

Multiplying (36) by y_i, $i \in I_0$ and using (40), we have

$$\tau y_i g_i(\bar{x}) = 0, \ i \in I_0 \tag{50}$$

Multiplying (37) by $z_j, j \in I_0$, we have

$$\tau z_j h_j(\bar{x}) = 0, \ j \in J_0 \tag{51}$$

Then from $\tau > 0$, (47) and (48) implies that

$$y_i g_i(\bar{x}) = 0, \ i \in I_0 \tag{52}$$

$$z_j h_j(\bar{x}) = 0, \ j \in J_0 \tag{53}$$

Also $\theta = 0, \tau > 0$ and (35) implies

$$B\bar{x} = \frac{2r}{\tau} B\omega \tag{54}$$

Hence

$$\bar{x}^T B\omega = \left(\bar{x}^T Bx\right)^{1/2} \left(\bar{\omega}^T B\omega\right)^{1/2} \tag{55}$$

If $\gamma > 0$, then (41) implies $\omega^T B\omega = 1$.

consequently, (55) yields $\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2}$

If $\gamma = 0$, then (35), yields $B\bar{x} = 0$. So we obtain $\left(\bar{x}^T B\bar{x}\right)^{1/2} = \bar{x}^T B\omega$.

Thus in either case, we obtain $\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2}$.

Therefore from (49) and (50), with $\bar{x}^T B\omega = \left(\bar{x}^T B\bar{x}\right)^{1/2}$, we have

$$f(\bar{x}) + \left(\bar{x}^T B\bar{x}\right)^{1/2} = f(\bar{x}) + \bar{x}^T B\omega + \sum_{i \in I_0} y_i g_i(\bar{x}) + \sum_{j \in J_0} z_j h_j(\bar{x}) \ .$$

If the hypotheses of Theorem 5.1 hold then $\bar{x}$ is an optimal solution of (EP).

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