Journal of Applied Mathematics & Bioinformatics, vol.3, no.2, 2013, 15-34
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2013

On Multiobjective Variational Problems with equality and inequality constraints

I. Husain[1] and Vikas K. Jain[2]

Abstract

Fritz John type optimality conditions for a multiobjective variational Problems with equality and inequality constraints are derived. By an application of Karush -Kuhn-Tucker type optimality conditions, a Wolfe type second-order dual to this problem is formulated and various duality results are proved under generalized second-order invexity assumptions. A pair of Wolfe type second-order dual multiobjective with natural boundary values is also presented to investigate duality. Finally, it is pointed out that our duality results established in this research can be viewed as dynamic generalizations of those of static cases already existing in the literature.

Mathematics Subject Classification: 90C30, 90C11, 90C20, 90C26

---

[1] Department of Mathematics, Jaypee University of Engineering and Technology, Guna (M.P), India, e-mail:ihusain11@yahoo.com
[2] Department of Mathematics, Jaypee University of Engineering and Technology, Guna (M.P), India, e-mail:jainvikas13@yahoo.com

Article Info: *Received* : December 21, 2012. *Revised* : February 1, 2013
Published online : June 30, 2013

Keywords: Multiobjective Variational Problems, Wolfe Type Second-Order Duality, Second-Order Generalized invexity, Multiobjective Nonlinear Programming Problems

1 Introduction

Second-order duality in mathematical programming has been widely researched. As second-order dual to a constrained optimization problem gives a tighter bound and hence enjoys computational advantage over the first-order dual to the problem. Mangasarian [1] was the first to study second-order duality in non-linear programming. Motivated with analysis of Mangasarian [1], Chen [2] presented Wolfe type second-order dual to a class of constrained variational problems under an involved invexity like conditions. Later Husain et al [3] introduced second-order invexity and generalized invexity and presented a Mond-Weir type second-order dual to the problem of [2] in order to relax implicit invexity requirements to the generalized second-order invexity.

Multiobjective optimization (also known as multiobjective programming, vector optimization, multicriteria optimization, multiattribute optimization, or Pareto optimization) is an area of multiple criteria decision making. This deals with optimization problems involving more than one objective functions to be optimized simultaneously which appear more often than single objective optimization problem to represent the models of real life problems. Multiobjective optimization has applications in various fields of science that includes engineering, economics and logistics where optimal decisions need to be taken in the presence of trade-off between two or more conflicting objectives. Minimizing weight while maximizing the strength of a particular component, maximizing performance whilst minimizing fuel consumption and emission of pollutants of a vehicle are examples of multiobjective optimization problems involving two or three objectives respectively. In real-life problems, there can be more than three objectives.

Motivated with the above cursory remarks relating to multiobjective optimization problems in this paper we present a multiobjective version of the variational problem considered by Chen [2] with equality and inequality constraints which represent more realistic problems than those variational problem with an inequality constraint only. For this multiobjective variational problem, Fritz John type optimality conditions are obtained. Using Karush-Kuhn-Tucker type optimality conditions, deduced from the Fritz John type optimality conditions requiring a suitable regularity conditions. Wolfe type second-order multiobjective variational to the variational problems is formulated and various duality theorems, viz. weak, strong, strict-converse and converse duality theorems are proved under second-order pseudoinvexity and second-order strict pseudo-invexity. A pair of Wolfe type second-order dual multiobjective variational problems with natural boundary values is also constructed to study duality. Finally, it is indicated that our duality theorems can be regarded as dynamic generalizations of the second-order duality theorems is nonlinear programming, existing in the literature.

2 Pre-Requisites and expression of the problem

Let $I = [a,b]$ be a real interval, $\phi : I \times R^n \times R^n \to R$ and $\psi : I \times R^n \times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $\phi(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x and $\phi_{\dot{x}}$, the first order derivatives of ϕ with respect to $x(t)$ *and* $\dot{x}(t)$, respectively, that is,

$$\phi_x = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \cdots, \frac{\partial \phi}{\partial x^n} \right)^T , \ \phi_{\dot{x}} = \left(\frac{\partial \phi}{\partial \dot{x}^1}, \frac{\partial \phi}{\partial \dot{x}^2}, \cdots, \frac{\partial \phi}{\partial \dot{x}^n} \right)^T$$

Denote by ϕ_{xx} the Hessian matrix of ϕ, and ψ_x the $m\times n$ Jacobian matrix respectively, that is, with respect to $x(t)$, that is, $\phi_{xx} = \left(\dfrac{\partial^2 \phi}{\partial x^i \partial x^j} \right)$, $i, j = 1, 2, \ldots n$, ψ_x the $m\times n$ Jacobian matrix

$$\psi_x = \begin{pmatrix} \dfrac{\partial \psi^1}{\partial x^1} & \dfrac{\partial \psi^1}{\partial x^2} \cdots & \dfrac{\partial \psi^1}{\partial x^n} \\ \dfrac{\partial \psi^2}{\partial x^1} & \dfrac{\partial \psi^2}{\partial x^2} \cdots & \dfrac{\partial \psi^2}{\partial x^n} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial \psi^m}{\partial x^1} & \dfrac{\partial \psi^m}{\partial x^2} \cdots & \dfrac{\partial \psi^m}{\partial x^n} \end{pmatrix}_{m\times n}.$$

The symbols $\phi_{\dot{x}}$, $\phi_{\dot{x}x}$, $\phi_{x\dot{x}}$ and $\psi_{\dot{x}}$ have analogous representations.

Designate by X, the space of piecewise smooth functions $x: I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \int_a^t u(s)\,ds,$$

Thus $\dfrac{d}{dt} = D$ except at discontinuities.

Consider the following constrained multiobjective variational problem:

(VEP): Minimize $\left(\int_I f^1(t,x,\dot{x})dt, \ldots, \int_I f^p(t,x,\dot{x})dt \right)$

subject to $x(a) = \alpha$, $x(b) = \beta$

$$g(t,x,\dot{x}) \leqq 0, t \in I \tag{1}$$

$$h(t,x,\dot{x}) = 0, t \in I \tag{2}$$

where $f^i : I\times R^n \times R^n \to R$, $i \in K = \{1, 2, \ldots, p\}$, $g : I\times R^n \times R^n \to R^m$, $h : I\times R^n \times R^n \to R^l$ are continuously differentiable functions.

The following convention for equality and inequality will be used. If $\alpha, \beta \in R^n$, then

$$\alpha = \beta \Leftrightarrow \alpha^i = \beta^i \qquad i = 1, 2, \ldots, n$$
$$\alpha \geqq \beta \Leftrightarrow \alpha^i \geqq \beta^i \qquad i = 1, 2, \ldots, n$$
$$\alpha \geq \beta \Leftrightarrow \alpha \geqq \beta \quad and \quad \alpha \neq \beta$$
$$\alpha > \beta \Leftrightarrow \alpha^i > \beta^i \qquad i = 1, 2, \ldots, n$$

Definition 2.1 A feasible solution $\bar{x}$ is efficient for (VEP) if there is no feasible $\hat{x}$ for (VEP) such that

$$\int_I f^i(t, \hat{x}, \dot{\hat{x}})dt < \int_I f^i(t, \bar{x}, \dot{\bar{x}})dt \ , \quad \text{for some} \quad i \in \{1, 2, \ldots, p\}$$

$$\int_I f^j(t, \hat{x}, \dot{\hat{x}})dt \leqq \int_I f^j(t, \bar{x}, \dot{\bar{x}})dt \ , \quad \text{for all} \quad j \in \{1, 2, \ldots, p\}$$

In the case of maximization, the signs of above inequalities are reversed. We need the following lemma due to Chankong and Haimes [4] for the proof of strong duality result.

Lemma 2.1 If $\bar{x}$ is efficient solution of (VEP) if and only if $\bar{x}$ solves $P_k(\bar{x})$, for all K, defined as

$P_k(\bar{x})$: Minimize $\int_I f^k(t, x, \dot{x})dt$

subject to

$$x(a) = \alpha \ , x(b) = \beta$$

$$g(t, x, \dot{x}) \leqq 0, \qquad t \in I$$

$$h(t, x, \dot{x}) = 0, \qquad t \in I$$

$$f^j(t, \hat{x}, \dot{\hat{x}}) \leqq f^j(t, \bar{x}, \dot{\bar{x}}) \ , t \in I \ \ for\ all \ \ j \in \{1, 2, \ldots, p\}, \ j \neq k$$

We incorporate the following definitions which are required in the subsequent analysis.

Definitions 2.2 The function $\int_I \phi(t,x,\dot{x})dt$ where $\phi : I \times R^n \times R^n \to R$, is said to be second order-pseudo invex with respect to $\eta = \eta(t,x,u)$ if

$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt \geqq 0$$
$$\Rightarrow \int_I \phi(t,x,\dot{x})dt \geqq \int_I \left(\phi(t,u,\dot{u}) - \frac{1}{2}\beta^T(t)A\beta(t) \right) dt$$

where $A = \phi_{xx}(t,x,\dot{x}) - 2D\phi_{x\dot{x}}(t,x,\dot{x}) + D^2\phi_{\dot{x}\dot{x}}(t,x,\dot{x}) - D^3\phi_{\dot{x}\ddot{x}}(t,x,\dot{x})$, $t \in I$.

The functional $\int_I \phi(t,x,\dot{x})dt$ is said to strictly pseudoinvex with respect to η if

$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt \geqq 0$$
$$\Rightarrow \int_I \phi(t,x,\dot{x})dt > \int_I \phi(t,u,\dot{u})dt$$

Or equivalently

$$\int_I \phi(t,x,\dot{x})dt \leqq \int_I \phi(t,u,\dot{u})dt$$
$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt < 0$$

Remark 2.1 If ϕ does not depend explicitly on t then the above definitions reduce to those given in [5].

3 Necessary optimality conditions

In order to obtain the Fritz John type necessary optimality conditions, we state the following variational problem treated by Chandra et al [6]:

(P): Minimize $\int_I f(t,x,\dot{x})dt$

subject to

$$x(a)=\alpha \ , x(b)=\beta$$

$$g(t,x,\dot{x}) \leqq 0, \quad t \in I$$

$$h(t,x,\dot{x}) = 0, \quad t \in I$$

where $f : I \times R^n \times R^n \to R$ and g and h are the same as earlier.

The problem (P) may be written as

Minimize F(x)

subject to

$$G(x) \in S \ ,$$
$$H(x) = 0 \ ,$$

where $G : X \to C(I, R^m)$ giving $G(x)(t) = g(t, x, \dot{x})$ for all $x \in X$ and $t \in I$ and $C(I, R^m)$ denotes the space of continuous function from an interval I into R^m, S is the convex cone of function in $C(I, R^m)$ where components are non-negative. The function $H : X \to C(I, R^l)$ is defined by

$$H(x)(t) = h(t, x, \dot{x}) \ \ for \ \ x \in X \ \ , t \in I \ .$$

The following theorem gives the Fritz John type necessary optimality conditions for the single objective variational problems derived in [6].

Theorem 3.1 (Fritz John optimality conditions) If $\bar{x}$ is an optimal solution of (P) and $h_x(., x(.), \dot{x}(.))$ maps onto a closed subspace of $C(I, R^l)$, then there exist Lagrange multipliers $\tau \in R$ and piecewise smooth $\bar{y} : I \to R^m$, $\bar{z} : I \to R^l$ such that

$$\tau f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{y}(t)^T g_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)^T h_x(t,\bar{x}(t),\dot{\bar{x}}(t))$$
$$= D\Big(\tau f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))) + z(t)^T h_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\Big), \qquad t \in I$$

$$\bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0, \qquad t \in I$$
$$\left(\tau, \bar{y}(t)\right) \geqq 0, \qquad t \in I$$
$$\left(\tau, \bar{y}(t), \bar{z}(t)\right) \neq 0, \qquad t \in I$$

Remark 3.1 The above Fritz John necessary optimality conditions become Karush-Kuhn-Tucker John optimality condition if $\tau = 1$ (then $\bar{x}$ may be called normal). It is sufficient for $\tau = 1$ that the Zowe's form of Slater condition [6] is assumed, that is, there exist $q \in X$,

$$G(\bar{x}) + G'(\bar{x})q \in \operatorname{int} S$$
$$H'(\bar{x})q = 0$$

The Karush-Kuhn-Tucker type necessary optimality conditions for (P) can explicitly be given in the following theorem:

Theorem 3.2 (Karush-Kuhn-Tucker type necessary optimality conditions) If $\bar{x}$ is an normal and optimal solution of (VEP) and $h_x(., x(.), \dot{x}(.))$ maps onto a closed subspace of $C(I, R^l)$, then there exist piecewise smooth $\bar{y}: I \to R^m$, $\bar{z}: I \to R^l$ such that

$$\begin{aligned} &f_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{y}(t)^T g_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{z}(t)^T h_x(t, \bar{x}(t), \dot{\bar{x}}(t)) \\ &\quad = D\Big(f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{z}(t)^T h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\Big), \quad t \in I \end{aligned}$$

$$\bar{y}(t)^T g\left(t, \bar{x}\left(t\right), \dot{\bar{x}}\left(t\right)\right) = 0, \quad t \in I$$
$$\bar{y}(t) \geqq 0, \qquad t \in I$$

In this section, we will establish the following theorem giving the Fritz John type necessary optimality conditions for (VEP):

Theorem 3.3 (Fritz John type necessary optimality conditions) Let $\bar{x}$ be an efficient solution of (VEP). Then there exist $\bar{\lambda}^i \in R,\ i \in K$ and piecewise smooth functions $\bar{y} : I \to R^m$, $\bar{z} : I \to R^l$ such that

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i - Df_{\dot{x}}^i \right) + \left(\bar{y}(t)^T g_x + \bar{z}(t)^T h_x \right) - D\left(\bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}} \right) = 0,\ t \in I$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}\right) = 0,\ t \in I$$

$$\left(\bar{\lambda}, \bar{y}(t)\right) \geqq 0,\ t \in I$$

$$\left(\bar{\lambda}, \bar{y}(t), \bar{z}(t)\right) \neq 0,\ t \in I$$

Proof: Since $\bar{x}$ is an efficient solution of (VEP), by Lemma 2.1, $\bar{x}$ is an optimal $\left(P_k\right)$ for each $p \in K$ and in particular of $\left(P_1\right)$. Therefore, by Theorem 3.1, there exist $\bar{\lambda}^i \in R,\ i \in K$

$$\left(\bar{\lambda}^T f_x - D\bar{\lambda}^T f_{\dot{x}} \right) + \left(\bar{y}(t)^T g_x + \bar{z}(t)^T h_x \right) - D\left(\bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}} \right) = 0,\ t \in I$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}\right) = 0,\ t \in I$$

$$\left(\bar{\lambda}, \bar{y}(t)\right) \geqq 0,\ t \in I$$

$$\left(\bar{\lambda}, \bar{y}(t), \bar{z}(t)\right) \neq 0,\ t \in I$$

which validates the theorem.

4 Wolfe type Duality

In this section, we present the following Wolfe type second- order dual to the problem (VEP) and prove various duality theorems

(WVED):

$$\text{Maximize} \left(\int_I \left(f^1(t, x, \dot{x}) + y(t)^T g(t, x, \dot{x}) + z(t)^T h(t, x, \dot{x}) - \frac{1}{2} \beta(t)^T H^1 \beta(t) \right) dt, \right.$$

$$\left. \ldots, \int_I \left(f^p(t, x, \dot{x}) + y(t)^T g(t, x, \dot{x}) + z(t)^T h(t, x, \dot{x}) - \frac{1}{2} \beta(t)^T H^p \beta(t) \right) dt \right)$$

subject to

$$x\left(a\right) = \alpha\ ,\ x\left(b\right) = \beta \tag{3}$$

$$\lambda^T f_x + y(t)^T g_x + z(t)^T h_x - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right) + H\beta(t) = 0 \tag{4}$$

$$y(t) \geqq 0 \tag{5}$$

$$\lambda > 0,\ \lambda^T e = 1,\ \text{where}\ \ e = (1,1,\ldots,1) \in R^n \tag{6}$$

where

$$H^i = f^i_{xx} - 2Df^i_{x\dot{x}} + D^2 f^i_{\dot{x}\dot{x}} - D^3 f^i_{\dot{x}\ddot{x}} + \left(y(t)^T g_x + z(t)^T h_x\right)_x - 2D\left(y(t)^T g_x + z(t)^T h_x\right)_{\dot{x}}$$
$$+ D^2\left(y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right)_{\dot{x}} - D^3\left(y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right)_{\ddot{x}}$$

$$H = \lambda^T f_{xx} - 2D\lambda^T f_{x\dot{x}} + D^2\lambda^T f_{\dot{x}\dot{x}} - D^3\lambda^T f_{\dot{x}\ddot{x}} + \left(y(t)^T g_x + z(t)^T h_x\right)_x$$
$$- 2D\left(y(t)^T g_x + z(t)^T h_x\right)_{\dot{x}} + D^2\left(y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right)_{\dot{x}} - D^3\left(y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right)_{\ddot{x}},$$

and $\qquad A^i = f^i_{xx} - 2Df^i_{x\dot{x}} + D^2 f^i_{\dot{x}\dot{x}} - D^3 f^i_{\dot{x}\ddot{x}}$.

Theorem 4.1 (Weak Duality) Assume that for all feasible x for (VEP) and all feasible $(u, y, z, \lambda, \beta)$ for (WVED).

$$\int_I \left\{ f^i(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x})\right\} dt$$

is second-order pseudoinvex with respect to η . Then the following cannot hold:

$$\int_I f^i(t,x,\dot{x})dt < \int_I \left\{ f^i(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) + z(t)^T h(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^i \beta(t)\right\} dt$$

for some $i \in \{1,2,\ldots,p\}$. (7)

$$\int_I f^j(t,x,\dot{x})dt \leqq \int_I \left\{ f^j(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) + z(t)^T h(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^j \beta(t)\right\} dt$$

for all $j \in \{1,2,\ldots,p\}$. (8)

Proof: Suppose contrary to the result that (7) and (8) hold. Since x is feasible for (VEP) and $(u, y, z, \lambda, \beta)$ is feasible for (WVED), it follows from $g(t,x,\dot{x}) \leqq 0$, $y(t) \geqq 0$, (7) and (8) that

$$\int_I \left\{ f^i(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) \right\} dt$$

$$< \int_I \left\{ f^i(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) + z(t)^T h(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} dt$$

for some $i \in \{1, 2, ..., p\}$. (9)

$$\int_I \left\{ f^j(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) \right\} dt$$

$$\leqq \int_I \left\{ f^j(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) + z(t)^T h(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^j \beta(t) \right\} dt$$

for all $j \in \{1, 2, ..., p\}$, with $j \neq i$. (10)

Using the second-order pseudoinvexity of

$\int_I \left\{ f^i(t,.,.) + y(t)^T g(t,.,.) + z(t)^T h(t,.,.) \right\} dt$ from the relations (9) and (10), we get

$$\int_I \{\eta^T(t,x,u)\left(f_u^i(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u}) \right)$$

$$+\left(\frac{d\eta}{dt}\right)^T \left(f_{\dot{u}}^i(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u}) \right) - \frac{1}{2}\eta^T H^i \beta(t)\} dt < 0$$

for all $i \in \{1, 2, ..., p\}$ (11)

multiplying each inequality of (11) by $\lambda^i > 0,\ i = 1, 2, ..., p$ and adding , we have

$$0 > \int_I \{\eta^T(t,x,u)\left(\lambda^T f_u^i(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u}) \right)$$

$$+\left(\frac{d\eta}{dt}\right)^T \left(\lambda^T f_{\dot{u}}^i(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u}) \right) + \eta^T H \beta(t)\} dt$$

$$= \int_I \eta^T(t,x,u)\{\left(\lambda^T f_u^i(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u}) \right)$$

$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u}) \right) + H\beta(t)\} dt$$

$$+\eta^T \left(\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u}) \right)\Big|_{t=a}^{t=b}$$

$$\int_I \eta^T(t,x,u)\{\left(\lambda^T f_u^i(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u})\right)$$
$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u})\right) + H\beta(t)\}dt < 0$$

This contradicts the feasibility of $(u, y, z, \lambda, \beta)$ for (WVED). Hence the result follows.

Corollary 4.1 Assume that the weak duality (Theorem 4.1) holds between (VEP) and (WVED). If $\bar{x}$ is feasible for (VEP) and $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \beta)$ is feasible for (WVED) with $\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}\right) = 0, \beta(t) = 0, t \in I$. Then $\bar{x}$ is efficient for (VEP) and $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \beta)$ is efficient for (WVED).

Proof: Suppose $\bar{x}$ is not efficient for (VEP). Then there exist some feasible $\hat{x}$ for (VEP) such that

$$\int_I f^i(t, \hat{x}, \dot{\hat{x}})dt < \int_I f^i(t, \bar{x}, \dot{\bar{x}})dt, \text{ for some } i \in \{1, 2, ..., p\}$$

$$\int_I f^j(t, \hat{x}, \dot{\hat{x}})dt \leqq \int_I f^j(t, \bar{x}, \dot{\bar{x}})dt, \text{ for all } j \in \{1, 2, ..., p\}$$

Since $\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}\right) = 0, \bar{z}(t)^T h\left(t, \bar{x}, \dot{\bar{x}}\right) = 0, t \in I$ and $\beta(t) = 0, t \in I$

$$\int_I f^i(t, \bar{x}, \dot{\bar{x}})dt < \int_I \left\{ f^i(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t)^T h(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} dt$$

for some $i \in \{1, 2, ..., p\}$.

$$\int_I f^j(t, \bar{x}, \dot{\bar{x}})dt \leqq \int_I \left\{ f^j(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t)^T h(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T H^j \beta(t) \right\} dt$$

for all $j \in \{1, 2, ..., p\}$.

This contradicts weak duality. Hence $\bar{x}$ is efficient for (VEP).

Now suppose that $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{z}, \beta)$ is not efficient for (WVED) such that

$$\int_I \left\{ f^i(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} dt$$
$$< \int_I \left\{ f^i(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} dt$$

for some $i \in \{1,2,...,p\}$

$$\int_I \left\{ f^j(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^j \beta(t) \right\} dt$$
$$\leqq \int_I \left\{ f^j(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T H^j \beta(t) \right\} dt$$

for all $j \in \{1,2,...,p\}$, $j \neq i$.

Since $\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right) = 0$, $\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}}\right) = 0$, $\beta(t) = 0$, $t \in I$, we have

$$\int_I \left\{ f^i(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} dt < \int_I f^i(t,\bar{x},\dot{\bar{x}}) dt$$

for some $i \in \{1,2,...,p\}$.

$$\int_I \left\{ f^j(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^j \beta(t) \right\} dt \leqq \int_I f^j(t,\bar{x},\dot{\bar{x}}) dt$$

for all $j \in \{1,2,...,p\}$.

This contradicts weak duality. Hence $\left(\bar{x},\bar{\lambda},\bar{y},\bar{z},\beta(t)\right)$ is efficient for (WVED).

Theorem 4.2 (Strong Duality) Let $\bar{x}$ be an efficient solution of (VEP) and $\bar{x}$ satisfy the constraint qualification of Theorem 3.1 for $P_k(\bar{x})$, for at least one $k \in \{1,2,...,p\}$. Then there exist $\bar{\lambda} \in R^k$ and piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^l$ such that $\left(\bar{x},\bar{\lambda},\bar{y},\bar{z},\beta = 0\right)$ is feasible for (WVED) and $\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right) = 0$, $t \in I$. If the weak duality also holds between (VEP) and (WVED) then $\left(\bar{x},\bar{\lambda},\bar{y},\bar{z},\beta = 0\right)$ is efficient for (WVED).

Proof: As $\bar{x}$ satisfies the constraint qualification of Theorem 3.2 for $P_k(\bar{x})$, for at least one $k \in \{1,2,...,p\}$, it follows from Theorem 3.2 that there exist $\lambda' \in R^{k-1}$

and piecewise smooth $y' : I \to R^m$ and $z' : I \to R^l$ such that for all $t \in I$

$$f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i \left(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right)$$

$$+y'(t)^T g_x(t,\bar{x},\dot{\bar{x}}) - Dy'(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + z'(t)^T h_x(t,\bar{x},\dot{\bar{x}}) - Dz'(t)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) = 0, t \in I$$

$$y'(t)^T g_x(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I$$

$$y'(t)^T \geqq 0, t \in I$$

$$\lambda'^i \geqq 0, i = 1,2,\ldots,p, i \neq k$$

Setting

$$\bar{\lambda} = \lambda'^i \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i\right), \bar{\lambda}^k = 1 \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i\right)$$

$$\bar{y}(t) = y'(t) \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i\right),\ \bar{z}(t) = z'(t) \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i\right)$$

and dividing by $\left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \lambda'^i\right)$, we finally get

$$\begin{aligned} &\bar{\lambda}^T f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h_x(t,\bar{x},\dot{\bar{x}}) \\ &-D\left(\bar{\lambda}^T f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\right) = 0, t \in I \end{aligned}$$

Also we get $\sum_{i=1}^{p} \bar{\lambda}^i = 1$ and $\bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}}) = 0, t \in I$. It implies that

$\left(\bar{x}, \bar{\lambda}, \bar{y}, \bar{z}, \beta = 0\right)$ is feasible for (WVED).

The weak duality holds along $\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0$ and $\beta(t) = 0, t \in I$. This by corollary 4.1 implies that $\left(\bar{x}, \bar{\lambda}, \bar{y}, \bar{z}, \beta = 0\right)$ is efficient for (WVED).

Theorem 4.3 (Strict- converse Duality) Let $\bar{x}$ be efficient and normal solution of (VEP) and $\left(u, y, z, \lambda, \beta\right)$ be efficient for (WVED) such that

$$\int_I \sum_{i=1}^{p} \lambda^i f^i(t,\bar{x},\dot{\bar{x}})dt$$
$$= \int_I \{\sum_{i=1}^{p} \lambda^i f(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^T g(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^T h(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\beta(t)^T H \beta(t)\}dt \tag{12}$$

If $\int_I \left(\lambda^T f_x + y(t)^T g + z(t)^T h\right) dt$ is second-order strictly pseudoinvex with respect to η, then

$$\bar{x}(t) = \bar{u}(t)\ , \quad t \in I$$

Proof: By second-order strict pseudoinvexity of $\int_I \left(\lambda^T f(t,.,.) + y(t)^T g(t,.,.) + z(t)^T h(t,.,.)\right) dt$, (12) implies

$$0 > \int_I \Big\{ \eta^T \left(\lambda^T f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u})\right)$$
$$+ \left(\frac{d\eta}{dt}\right)^T \left(\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u})\right) + \eta^T H \beta \Big\} dt$$
$$= \int_I \eta^T \{ \left(\lambda^T f_u(t,\bar{u},\dot{\bar{u}}) + y(t)^T g_u(t,\bar{u},\dot{\bar{u}}) + z(t)^T h_u(t,\bar{u},\dot{\bar{u}})\right)$$
$$- D\left(\lambda^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + y(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + z(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}})\right) + H\beta(t)\}dt$$
$$+ \eta^T \left(\lambda^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + y(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + z(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}})\right)\Big|_{t=a}^{t=b}$$

Using $\eta = 0,$ at $t = a,\ t = b$, we have

$$\int_I \eta^T \{ \left(\lambda^T f_u^i(t,\bar{u},\dot{\bar{u}}) + y(t)^T g_u(t,\bar{u},\dot{\bar{u}}) + z(t)^T h_u(t,\bar{u},\dot{\bar{u}})\right)$$
$$- D\left(\lambda^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + y(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + z(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}})\right) + H\beta(t)\}dt < 0$$

This contradicts the equality constraint of (WVED).Hence $\bar{x}(t) = \bar{u}(t)\ ,\ \ t \in I$.

Theorem 4.4 (Converse duality) Let $\left(\bar{x}(t), y(t), z(t), \lambda, \beta(t)\right)$ be an efficient solution of (WVED) for which

(C_1): H is non-singular

$$
\begin{aligned}
(\mathbf{C_2}):\ & \Big[\left(\sigma(t)^T H^i \sigma(t)\right)_x - D\left(\sigma(t)^T H^i \sigma(t)\right)_{\dot{x}} \\
& + D^2\left(\sigma(t)^T H^i \sigma(t)\right)_{\ddot{x}} - D^3\left(\sigma(t)^T H^i \sigma(t)\right)_{\dddot{x}} + D^4\left(\sigma(t)^T H^i \sigma(t)\right)_{\ddddot{x}}\Big] \\
& + 2\Big[\sigma(t)^T\left(H\sigma(t)\right)_x - \sigma(t)^T D\left(H\sigma(t)\right)_{\dot{x}} + \sigma(t)^T D^2\left(H\sigma(t)\right)_{\ddot{x}} \\
& - \sigma(t)^T D^3\left(H\sigma(t)\right)_{\dddot{x}} + \sigma(t)^T D^4\left(H\sigma(t)\right)_{\ddddot{x}}\Big] = 0 \Rightarrow \sigma(t) = 0,
\end{aligned}
$$

where $\sigma(t)$ is a vector function.

Then $\bar{x}(t)$ is feasible for (VEP) and the two objective functionals have the same value. Also, if the weak duality theorem holds for all feasible solutions of (VEP) and (WVED), then $\bar{x}(t)$ is efficient for (VEP).

Proof: Since $\left(\bar{x}, y, z, \lambda, \beta(t)\right)$ is an efficient solution of (WVED), there exists $\alpha, \xi \in R^p, \delta \in R$ and piecewise smooth $\theta: I \to R^n$, $\eta: I \to R^m$ such that following Fritz John conditions (Theorem (3.3)) are satisfied at $\left(\bar{x}, y, z, \lambda, \beta(t)\right)$:

$$
\begin{aligned}
& \sum_{i=1}^{p} \alpha^i \Big[\left(f^i + y^T g_x + z^T h_x\right) - D\left(f^i + y^T g_{\dot{x}} + z^T h_{\dot{x}}\right) - \frac{1}{2}\left(\beta(t)^T H^i \beta(t)\right)_x \\
& \quad + \frac{1}{2} D\left(\beta(t)^T H^i \beta(t)\right)_{\dot{x}} - \frac{1}{2} D^2\left(\beta(t)^T H^i \beta(t)\right)_{\ddot{x}} + \\
& \quad + \frac{1}{2} D^3\left(\beta(t)^T H^i \beta(t)\right)_{\dddot{x}} - \frac{1}{2} D^4\left(\beta(t)^T H^i \beta(t)\right)_{\ddddot{x}}\Big] \\
& + \theta(t)^T\left[H + \left(H\beta\right)_x - D\left(H\beta\right)_{\dot{x}} + D^2\left(H\beta\right)_{\ddot{x}} - D^3\left(H\beta\right)_{\dddot{x}} + D^4\left(H\beta\right)_{\ddddot{x}}\right] = 0 \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\alpha^T e\right)\left[g^j - \frac{1}{2}\beta(t)^T g^j_{xx} \beta(t)\right] \\
& + \theta(t)^T\left[g^j_x - D g^j_{\dot{x}} + \left(g^i_{xx} - 2D g^i_{x\dot{x}} + D^2 g^i_{\dot{x}\dot{x}} - D^3 g^i_{\dot{x}\ddot{x}}\right)\beta(t)\right] + \eta^i(t) = 0 \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\alpha^T e\right)\left[h^j - \frac{1}{2}\beta(t)^T h^j_{xx} \beta(t)\right] \\
& + \theta(t)^T\left[h^j_x - D h^j_{\dot{x}} + \left(h^i_{xx} - 2D h^i_{x\dot{x}} + D^2 h^i_{\dot{x}\dot{x}} - D^3 h^i_{\dot{x}\ddot{x}}\right)\beta(t)\right] = 0 \qquad (15)
\end{aligned}
$$

$$
\left(\theta(t) - \left(\alpha^T e\right)\beta\right)^T H = 0 \qquad (16)
$$

$$
\theta(t)^T\left(f^i_x - D f^i_{\dot{x}} - A^i \beta(t)\right) + \xi^i + \delta^i = 0 \qquad (17)
$$

$$\eta(t)^T y(t) = 0, \quad t \in I \qquad (18)$$

$$\xi^T \lambda = 0 \qquad (19)$$

$$\delta\left(\sum_{i=1}^{p} \lambda^i - 1\right) = 0 \qquad (20)$$

$$(\alpha, \eta, \xi, \delta) \geqq 0 \qquad (21)$$

$$(\alpha, \theta, \eta, \xi, \delta) \neq 0 \qquad (22)$$

Since $\lambda > 0$, (19) implies $\xi = 0$.

Since H is non singular, (16) implies

$$\theta(t) = \left(\alpha^T e\right)\beta(t) \;, t \in I \qquad (23)$$

Using the equality constraint of the dual and (23) , we

$$-\sum \alpha^i \begin{bmatrix} H\beta(t)^T - \frac{1}{2}\left(\beta(t)^T H^i \beta(t)\right)_x + \frac{1}{2} D\left(\beta(t)^T H^i \beta(t)\right)_{\dot{x}} - \frac{1}{2} D^2\left(\beta(t)^T H^i \beta(t)\right)_{\ddot{x}} \\ + \frac{1}{2} D^3\left(\beta(t)^T H^i \beta(t)\right)_{\dddot{x}} - \frac{1}{2} D^4\left(\beta(t)^T H^i \beta(t)\right)_{\ddddot{x}} \end{bmatrix}$$
$$+\left(\alpha^T e\right)\begin{bmatrix} H\beta(t)^T + \beta(t)^T\left(H\beta\right)_x - \beta(t)^T D\left(H\beta\right)_{\dot{x}} + \beta(t)^T D^2\left(H\beta\right)_{\ddot{x}} \\ -\beta(t)^T D^3\left(H\beta\right)_{\dddot{x}} + \beta(t)^T D^4\left(H\beta\right)_{\ddddot{x}} \end{bmatrix} = 0$$

Consequently, we obtain

$$\begin{bmatrix} \left(\beta(t)^T H^i \beta(t)\right)_x - D\left(\beta(t)^T H^i \beta(t)\right)_{\dot{x}} + D^2\left(\beta(t)^T H^i \beta(t)\right)_{\ddot{x}} \\ -D^3\left(\beta(t)^T H^i \beta(t)\right)_{\dddot{x}} + D^4\left(\beta(t)^T H^i \beta(t)\right)_{\ddddot{x}} \end{bmatrix}$$
$$+2\begin{bmatrix} \beta(t)^T\left(H\beta\right)_x - \beta(t)^T D\left(H\beta\right)_{\dot{x}} + \beta(t)^T D^2\left(H\beta\right)_{\ddot{x}} \\ -\beta(t)^T D^3\left(H\beta\right)_{\dddot{x}} + \beta(t)^T D^4\left(H\beta\right)_{\ddddot{x}} \end{bmatrix} = 0$$

This, because of the hypothesis (C_2) implies $\beta(t) = 0$, $t \in I$,

Using $\beta(t) = 0$, $t \in I$ in (23), we have

$$\theta(t) = 0 \;, \quad t \in I.$$

Using $\theta(t) = 0$ and $\beta(t) = 0$, $t \in I$. From (14), we have

$$\left(\alpha^T e\right) g^j + \eta^i(t) = 0 \qquad (24)$$

Let $\alpha = 0$, then (14) and (17) respectively give $\delta^i = 0$ and $\eta^i = 0$. Consequently $(\alpha, \theta, \eta, \xi, \delta) = 0$ leading to a contradiction to (22). Hence $\alpha > 0$. From (24), we have

$$g^j = -\frac{\eta^i(t)}{(\alpha^T e)} \leqq 0, t \in I \tag{25}$$

or $$g(t, \bar{x}, \dot{\bar{x}}) \leqq 0, t \in I$$

the relations (24) and (15) respectively imply that

$$y(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0, t \in I \quad \text{and} \quad h(t, x, \dot{x}) = 0$$

Also $g(t, \bar{x}, \dot{\bar{x}}) \leqq 0$ *and* $h(t, \bar{x}, \dot{\bar{x}}) = 0$, $t \in I$ imply that $\bar{x}$ is feasible for (VEP). Now, we have

$$\int_I \left(f^i(t, \bar{x}, \dot{\bar{x}}) + y(t)^T g(t, \bar{x}, \dot{\bar{x}}) + z(t)^T h(t, \bar{x}, \dot{\bar{x}}) \right) dt = \int_I f^i(t, \bar{x}, \dot{\bar{x}})\, dt,\ i \in K,$$

implying the equality of objective values. By Corollary 4.1 the efficiency of $\bar{x}$ for (VEP) follows.

5 Multiobjective variational problems with natural boundary values

The following is a pair of Wolfe type second- order Multiobjective variational problems with natural values:

(VEP)N: Minimize $\left(\int_I f^1(t, x, \dot{x}) dt, \ldots, \int_I f^p(t, x, \dot{x}) dt \right)$

subject to

$$g(t, x, \dot{x}) \leqq 0, \quad t \in I$$

$$h(t, x, \dot{x}) = 0, \quad t \in I$$

(WVED)N: Maximize

$$\begin{pmatrix} \int_I \left(f^1(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^1 \beta(t) \right) dt, \ldots, \\ \int_I \left(f^p(t,x,\dot{x}) + y(t)^T g(t,x,\dot{x}) + z(t)^T h(t,x,\dot{x}) - \frac{1}{2}\beta(t)^T H^p \beta(t) \right) dt \end{pmatrix}$$

subject to

$$\lambda^T f_x + y(t)^T g_x + z(t)^T h_x - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right) + H\beta(t) = 0, \quad t \in I$$

$$y(t) \geqq 0\ , \quad t \in I$$

$$\lambda > 0,\ \lambda^T e = 1, \text{ where } e = (1,1,\ldots,1) \in R^k$$

$$\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u}) = 0, \quad \text{at} \quad t = a \text{ and } t = b$$

The theorems established in the proceeding sections can easily be proved for the above pair of problems.

6 Wolfe type second–order Multiobjective nonlinear programming problems

If the problems (VEP) and WVED) are independent of t, i.e. if f, g and h do not depend explicitly on t, then these problems essentially reduce to the static cases of nonlinear programming studied in [7], namely

(VEP)$_0$: Minimize $\left(f^1(x), f^2(x), \ldots, f^p(x)\right)$

subject to $g(x) \leqq 0\ ,\ h(x) = 0$

(WVED)$_0$: Maximize

$$\begin{pmatrix} f^1(u) + y^T g(u) + z^T h(u) - \frac{1}{2}\beta^T \nabla^2 \left(f^1(u) + y^T g(u) + z^T h(u) \right) \beta, \ldots, \\ f^p(u) + y^T g(u) + z^T h(u) - \frac{1}{2}\beta^T \nabla^2 \left(f^p(u) + y^T g(u) + z^T h(u) \right) \beta \end{pmatrix}$$

subject to

$$\lambda^T f_u(u) + y^T g_u(u) + z^T h_u(u) + \nabla^2 \left(\lambda^T f_{\dot{u}}(u) + y^T g_{\dot{u}}(u) + z^T h_{\dot{u}}(u) \right) \beta = 0$$

$$y \geqq 0$$

$$\lambda > 0, \ \lambda^T e = 1$$

Acknowledgements. The authors acknowledge the valuable comments of the anonymous referees which have substantially improved the presentation of this research work.

References

[1] O.L. Mangasarian, Second- and higher-order duality in nonlinear programming, *J. Math. Anal. Appl.*, **51**, (1979), 607-620.

[2] X. Chen, Second order duality for the variational problems, *J. Math. Anal. Appl.*, **286**, (2003), 261-270.

[3] I. Husain, A. Ahmed and Mashoob Masoodi, Second-order duality for variational problems, *European Journal of Pure and Applied Mathematics*, **2**(2), (2009), 278- 295.

[4] V. Chankong and Y.Y. Haimes, *Multiobjective Decision Making Theory and Methodology*, North- Holland, New York, 1983.

[5] J. Zhang and B. Mond, Duality for a nondifferentiable programming problem, *Bull. Austral. Math. Soc.*, **55**, (1997), 29-44.

[6] S. Chandra, B.D. Craven and I. Husain, A class of nondifferentiable continuous programming problem, J. *Math. Anal. Appl.*, **107**, (1985), 122-131.

[7] Mond and J. Zhang, Duality for multiobjective programming involving second-order V-invex functions, *Proceedings of the Optimization Miniconference*, B.M. Glover and V. JeyaKumar, Editors, University of South Wales, Sydney, Australia, (1985) , 89-100.