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Improved Approximate Solutions for Nonlinear Evolutions Equations in Mathematical Physics Using the Reduced Differential Transform Method

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Abstract

In this paper, an improved method called the Reduced Differential Transform Method (RDTM) was used to obtain approximate numerical and exact solutions for three different types of nonlinear partial differential equations (NLPDEs), such as; Gardner equation, Variant Nonlinear Water Wave equation (VNWW), and the Fifth-Order Korteweg-de Vries (FKdV) equation. The theoretical analyses of the RDTM are investigated for these equations and are calculated in the form of a series with easily computable terms. The results we obtained are compared with the analytical solutions obtained by other methods used in the past. One can conclude that only few terms of the series expansion are required to obtain approximate solutions using the RDTM with an excellent accuracy. Most of the symbolic and numerical computations were performed using Mathematica software.

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1 Introduction

The Reduced Differential Transform Method [9-11], was first introduced by Keskin to solve linear and nonlinear PDEs that appears in many Mathematical physics and engineering applications. For nonlinear models, the RDTM has shown dependable results and gives analytical approximation that converges very rapidly and in some cases gives exact solutions. Many numerical methods were used to solve nonlinear partial differential equations, such as, the Adomian Decomposition Method (ADM) [1, 2], the Differential Transform Method (DTM) [4], and the Variational Iteration Method (VIM) [6]. In this paper, we solve the following NLPDEs:

First, consider the Gardner equation

$$u_t = 6u^2u_x + 6u + u_{xxx}, \tag{1}$$

subject to the initial condition

$$u(x,0) = \frac{-1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right). \tag{2}$$

Second, Nonlinear Variant Water Wave equation the:

$$u_t + u_x + u_{xxx} + u_{xxxxx} + \left(uu_{xx}\right)_x = 0 \tag{3}$$

subject to the conditions

$$u(x,0) = 2 - 2\tanh^2\left(\frac{\sqrt{10}x}{10}\right), \quad u_t(x,0) = \frac{39}{25}\sqrt{\frac{2}{5}}\sec h^2\left(\frac{x}{10}\right). \tag{4}$$

Third, the FKdV equation:

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0, \tag{5}$$

subject to the condition

$$u(x,0) = e^x \,. \tag{6}$$

The goal of the study is to use the RDTM to solve three different types of nonlinear partial differential equations (NLPDEs). Efficiency and simple applicability of the method for the solution of complicated nonlinear partial differential equations are the main highlights of this study.

Keskin, in his PhD thesis [13], introduced the reduced form of the differential transform method (DTM) as reduced differential transform method (RDTM) and he used the RDTM to solve the Gas Dynamics Equation and linear and nonlinear Klein Gordon Equations and more. Also, Keskin and Oturanc (2010) used the RDTM to solve linear and nonlinear wave equations and they showed the effectiveness, and the accuracy of the method. Moreover, they showed that the number of iterations is less than the one used by the DTM. Finally, Alquran [4] used the DTM to solve the Gardner equation and Kaya and Al-Khaled [9], find a numerical solution to the Kawahara equation.

2 Analysis of the RDTM

In this section, we start with a function of two variables $u(x,t)$ which is analytic and $k-$times continuously differentiable with respect to time t and space x in the domain of our interest. Assume we can represent this function as a product of two single-variable functions, namely $u(x,t) = f(x).g(t)$. From the definitions of the DTM, the function can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x).\, t^k \,, \tag{7}$$

where $U_k(x)$ is the transformed function of $u(x,t)$ which can be defined as:

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k . \tag{8}$$

From equations (7) and (8) we can deduce

$$u\left(x,t\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k . \tag{9}$$

Some basic properties of the reduced differential transformation obtained from equations (7) and (8) are given as follows:

Theorem 2.1 If $f(x,t)=\alpha u(x,t)\pm\beta v(x,t)$, then $F_k(x)=\alpha U_k(x)\pm\beta V_k(x)$, where α and β are constant.

Theorem 2.2 If $f(x,t)=u(x,t).v(x,t)$, then $F_k(x)=\sum_{i=0}^{k}U_i(x)\,V_{k-i}(x)$.

Theorem 2.3 If $f(x,t)=u(x,t).v(x,t).w(x,t)$, then

$$F_k(x)=\sum_{i=0}^{k}\sum_{j=0}^{i}U_j(x)\,V_{i-j}(x)W_{k-i}(x) .$$

Theorem 2.4 If $f(x,t)=\frac{\partial^n}{\partial t^n}u(x,t)$, then $F_k(x)=\frac{\left(k+n\right)!}{K!}U_{k+n}(x)$.

Theorem 2.5 If $f(x,t)=\frac{\partial^n}{\partial x^n}u(x,t)$, then $F_k(x)=\frac{\partial^n}{\partial x^n}U_k(x)$.

Theorem 2.6 If $f(x,t)=x^m t^n u(x,t)$, then $F_k(x)=x^m U_{k-n}\left(x\right)$.

Theorem 2.7 If $f(x,t)=x^m t^n$, then $F_k(x)=x^m\delta\left(k-n\right)$,

where $\delta(k-n)=\begin{Bmatrix}1, & k=n\\ 0, & k\neq n\end{Bmatrix}$.

The proofs of the above theorems and more properties can be found in [13].

To illustrate the RDTM, we write the Gardner equation in standard form

$$L_t\left(u(x,t)\right)-6L\left(u(x,t)\right)-L_{xxx}\left(u(x,t)\right)-N\left(u(x,t)\right)=0, \tag{10}$$

subject to initial conditions

$$u(x,0)=f(x),\ u_t(x,0)=g(x), \tag{11}$$

where $L_t=\frac{\partial}{\partial t}, L_{xxx}=\frac{\partial^3}{\partial x^3}$, and $N\left(u(x,t)\right)$ is the nonlinear term.

Now from equation (10) and (11), we can derive the recursive formulas (according to the theorems mentioned above) as:

$$\left(k+1\right)U_{k+1}(x)=\frac{\partial^3}{\partial x^3}\left(U_k(x)\right)+6U_k(x)+N\left(u(x,t)\right) \tag{12}$$

and

$$U_0(x)=f(x),\ U_1(x)=g(x). \tag{13}$$

To find the rest of the iterations, we first substitute equation (13) into equation (12) and then we find the values of $U_k(x)$'s. Finally, we apply the inverse transformation to all the values $\left\{U_k(x)\right\}_{k=0}^{n}$ to obtain the approximate solution:

$$\hat{u}(x,t)=\sum_{k=0}^{n}U_k(x)t^k, \tag{14}$$

where n is the number of iterations we have used to find the approximate solution. Hence, the exact solution of our problem is given by

$$u(x,t)=\lim_{n\to\infty}\hat{u}(x,t). \tag{15}$$

3 Applications

In this section, we test the RDTM on three numerical examples and then compare our approximate solutions to the exact solutions.

3.1 Examples

Now, we present three examples to show the efficiency of the RDTM.

Example 3.1.1

Consider the Gardner equation

$$u_t = 6u^2 u_x + 6u + u_{xxx}, \tag{16}$$

subject to the initial conditions

$$u(x,0) = \frac{-1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right),\;\; u_t(x,0) = \frac{-1}{4} sech^2\left(\frac{x}{2}\right), \tag{17}$$

where the exact solution is

$$u(x,t) = \frac{-1}{2}\left(1 - \tanh\left(\frac{x-t}{2}\right)\right). \tag{18}$$

Applying the RDTM to (16) and (17), we obtain the recursive relation

$$U_{k+1}(x) = \left(\frac{1}{(k+1)}\right)\left(\frac{\partial^3}{\partial x^3}\left(U_k(x)\right) + 6U_k(x) + 6\sum_{i=0}^{k}\sum_{j=0}^{i} U_{k-i}(x)U_{i-j}(x)\frac{\partial}{\partial x}\left(U_j(x)\right)\right), \tag{19}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = \frac{-1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right),\;\; U_1(x) = \frac{-1}{4} sech^2\left(\frac{x}{2}\right). \tag{20}$$

Now, substitute Eq. (20) into Eq. (19) to obtain the following:

$$\begin{aligned} U_2(x) &= \frac{sech^2\left(\frac{x}{2}\right)}{96}\left(27 sech^4\left(\frac{x}{2}\right) + 27\cosh(x) sech^4\left(\frac{x}{2}\right) - 108\right) \\ &+ \frac{sech^6\left(\frac{x}{2}\right)}{96}\left(6\sinh(2x) - 24\sinh(x)\right), \ldots \end{aligned} \tag{21}$$

And so on. So after the third iteration, the differential inverse transform of $\{U_k(x)\}_{k=0}^{3}$ will provide us with the following approximate solution:

$$\hat{u}\left(x,t\right)=\sum_{k=0}^{3}U_k(x)t^k=U_0(x)+U_1(x)t+U_2(x)t^2+U_3(x)t^3+\ldots$$

$$=\frac{-1}{2}\left(1-\tanh\left(\frac{x}{2}\right)\right)-\frac{1}{4}sech^2\left(\frac{x}{2}\right)t$$

$$+\frac{sech^2\left(\frac{x}{2}\right)}{96}\left(27sech^4\left(\frac{x}{2}\right)+27\cosh(x)sech^4\left(\frac{x}{2}\right)+6\sinh(2x)-24\sinh(x)-108\right)t^2+$$

$$+\frac{sech^2\left(\frac{x}{2}\right)}{96}\left(\begin{array}{c}-388-897sech^4\left(\frac{x}{2}\right)+471\cosh(x)sech^4\left(\frac{x}{2}\right)\\ +\frac{3sech^4\left(\frac{x}{2}\right)(79\sinh(x)-46\sinh(2x)+6(84+\sinh(3x)))}{1+\cosh(x)}\end{array}\right)t^3+\ldots$$

Example 3.1.2

We consider the Nonlinear Variant Water Wave equation

$$u_t+u_x+u_{xxx}+u_{xxxxx}+\left(uu_{xx}\right)_x=0\,, \tag{22}$$

subject to the conditions

$$u(x,0)=2-2\tanh^2\left(\frac{\sqrt{10}x}{10}\right),\ u_t(x,0)=\frac{78}{25}\sqrt{\frac{2}{5}}sech^2\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)\tanh\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right), \tag{23}$$

where the exact solution

$$u(x,t)=2-2\tanh^2\left(\frac{\sqrt{10}}{10}\left(x-\frac{39}{25}t\right)\right) \tag{24}$$

Similar to the previous example, by the theorems above applied to Eq. (23) and Eq. (22) we get

$$U_{k+1}(x)=\frac{-1}{(k+1)}\left(\frac{\partial^3}{\partial x^3}\left(U_k(x)\right)+\frac{\partial}{\partial x}U_k(x)+\frac{\partial^5}{\partial x^5}\left(U_k(x)\right)+\frac{\partial}{\partial x}(\sum_{i=0}^{k}U_i(x)U_{k-i}(x))\right) \tag{25}$$

and

$$U_0(x) = 2 - 2\tanh^2\left(\frac{\sqrt{10}x}{10}\right), U_1(x) = \frac{78}{25}\sqrt{\frac{2}{5}}sech^2\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)\tanh\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right), \tag{26}$$

where, the $U_k(x)$, is the transform function of the $t-$dimensional spectrum.

Now, substitute Eq. (26) into Eq. (25) to obtain the following:

$$U_2(x) = \frac{39sech^5\left(\frac{x}{\sqrt{10}}\right)sech^4\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)\left(88\cosh(\frac{78t-25x}{25\sqrt{10}}) - 5\cosh(\sqrt{\frac{5}{2}}x) - 2\cosh(\frac{x}{\sqrt{10}}) - 41\cosh\left(\frac{3x}{\sqrt{10}}\right) - 88\cosh(\frac{3(-26t+25x)}{25\sqrt{10}}) - 8\cosh\left(\frac{78t+25x}{25\sqrt{10}}\right) + 8\cosh\left(\frac{125x-78t}{25\sqrt{10}}\right)\right)}{10000}$$
$$-\frac{39sech^2\left(\frac{x}{\sqrt{10}}\right)sech^6\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)}{1250}\left(33 - 26\cosh\left(\sqrt{\frac{2}{5}}\left(x-\frac{39t}{25}\right)\right) + \cosh\left(2\sqrt{\frac{2}{5}}\left(x-\frac{39t}{25}\right)\right)\right) - \frac{39}{10000}sech^4\left(\frac{x}{\sqrt{10}}\right)sech^4\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)20\sinh\left(\frac{39}{25}\sqrt{\frac{2}{5}}t\right) + \ldots.$$

So after the third iteration, the differential inverse transform of $\{U_k(x)\}_{k=0}^{3}$ will give the following approximate solution:

$$\begin{aligned}
\hat{u}(x,t) &= \sum_{k=0}^{3} U_k(x)t^k \\
&= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \ldots \\
&= 2 - 2\tanh^2\left(\frac{x}{\sqrt{10}}\right) + \frac{78}{25}\sqrt{\frac{2}{5}}sech^2\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)\tanh\left(\frac{x-\frac{39t}{25}}{\sqrt{10}}\right)t + \ldots
\end{aligned}$$

Example 3.1.3

We consider the FKdV equation

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0, \tag{27}$$

subject to the initial condition

$$u(x,0) = e^x, \tag{28}$$

where the exact solution

$$u(x,t) = e^{x-t}. \tag{29}$$

Applying the RDTM to (28) and (27), we obtain the recursive relation

$$U_{k+1}(x) = \frac{1}{k+1}\left(\sum_{i=0}^{k} U_i(x)\frac{\partial^3}{\partial x^3}U_{k-i}(x) - \sum_{i=0}^{k} U_i(x)\frac{\partial}{\partial x}U_{k-i}(x) - \frac{\partial}{\partial x}U_k(x) \right) \tag{30}$$

So for $k=0$, we obtain $U_1(x) = -\mathrm{e}^x$. Now for $k \geq 1$ we obtain

$U_2(x) = \frac{\mathrm{e}^x}{2}$, $U_3(x) = -\frac{\mathrm{e}^x}{6}$, $U_4(x) = \frac{\mathrm{e}^x}{24}$, $U_5(x) = -\frac{\mathrm{e}^x}{120}$, ... Thus

$$\begin{aligned} u(x,t) &= \mathrm{e}^x - \mathrm{e}^x t + \frac{\mathrm{e}^x t^2}{2} - \frac{\mathrm{e}^x t^3}{6} + \frac{\mathrm{e}^x t^4}{24} - \frac{\mathrm{e}^x t^5}{120} + \ldots \\ &= e^x \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \frac{t^6}{720} - \frac{t^7}{5040} + \frac{t^8}{40320} + O[t]^9 \right) \\ &= e^{x-t}. \end{aligned}$$

This is the exact solution of Eq. (27).

4 Tables and Figures

In this section, we shall illustrate the accuracy and efficiency of the RDTM. For this purpose, we consider the same values for x and t, specifically, $x = \{-0.5, -0.3, 0.3, 0.5\}$ and $t = \{0.0002, 0.0004, 0.0006, 0.001\}$. Also we can do the same for the other example.

Table 1: Comparison of absolute errors of the solution for Gardner equation, by RDTM and the DTM for different values of x and t

x	t	Exact	DTM	RDTM	Error(DTM)(n=8)	Error(RDTM)(n=3)
-0.5	0.0002	−0.62250633	-0.344945	−0.62250635	0.277561	2.16889E-8
	0.0004	−0.62255332	-0.344860	−0.62255341	0.277693	8.6782E-8
	0.0006	−0.62260032	-0.344776	−0.62260051	0.277825	1.95319E-7
	0.001	−0.62269430	-0.344606	−0.62269484	0.278088	5.42886E-7
-0.3	0.0002	−0.57449140	-0.410969	−0.57449142	0.163522	2.18636E-8
	0.0004	−0.57454029	-0.410891	−0.57454038	0.163649	8.74778E-8
	0.0006	−0.57458918	-0.410814	−0.57458938	0.163775	1.96877E-7
	0.001	−0.57468695	-0.410658	−0.57468750	0.164029	5.47172E-7

0.3	0.0002	–0.42560637	-0.578589	–0.42560639	0.152983	2.31256E-8
	0.0004	–0.42565526	-0.578528	–0.42565536	0.152873	9.2525E-8
	0.0006	–0.42570416	-0.578468	–0.42570437	0.152764	2.08231E-7
	0.001	–0.42580195	-0.578346	–0.42580253	0.152544	5.787E-7
0.5	0.0002	–0.37758767	-0.626214	–0.37758769	0.248626	2.31625E-8
	0.0004	–0.37763467	-.626158	–0.37763476	0.248523	9.26768E-8
	0.0006	–0.37768168	-.626101	–0.37768188	0.248420	2.08583E-7
	0.001	–0.37777570	-.625989	–0.37777628	0.248213	5.7973E-7

Table 2: Comparison of absolute errors of the solution for nonlinear variant water wave equation, by RDTM and the DTM for different values of *x* and *t*

x	*t*	*Exact*	*DTM*	*RDTM*	*Error(DTM)(n=11)*	*Error(RDTM)(3)*
-0.5	0.0002	1.95076129	1.84062457	1.95076128	0.11013671	1.43532E -8
	0.0004	1.95070088	1.84059019	1.95070082	0.11011068	5.74234E -8
	0.0006	1.95064043	1.84055580	1.95064030	0.11008462	1.29227E -7
	0.001	1.95051942	1.84048701	1.95051906	0.11003240	3.59096E -7
-0.3	0.0002	1.98207043	1.95835336	1.98207042	0.02371707	1.41368E -8
	0.0004	1.98203338	1.95834003	1.98203333	0.02369335	5.65554E -8
	0.0006	1.98199630	1.95832669	1.98199617	0.02366960	1.27268E -7
	0.001	1.98192201	1.95830001	1.98192165	0.02362200	3.53623E -7
0.3	0.0002	1.98214442	1.98211219	1.98214441	0.00003222	1.41327E -8
	0.0004	1.98218136	1.98211694	1.98218130	0.00006441	5.65227E -8
	0.0006	1.98221826	1.98212169	1.98221813	0.00009657	1.27158E -7
	0.001	1.98229195	1.98213118	1.98229159	0.00016077	3.53113E -7
0.5	0.0002	1.95088202	1.94311128	1.95088201	0.00777074	1.43479E -8

0.0004	1.95094233	1.94312259	1.95094227	0.00781973	5.73808E -8
0.0006	1.95100261	1.94313390	1.95100248	0.00786870	1.29083E-7
0.001	1.95112305	1.94315653	1.95112269	0.00796652	3.58431E -7

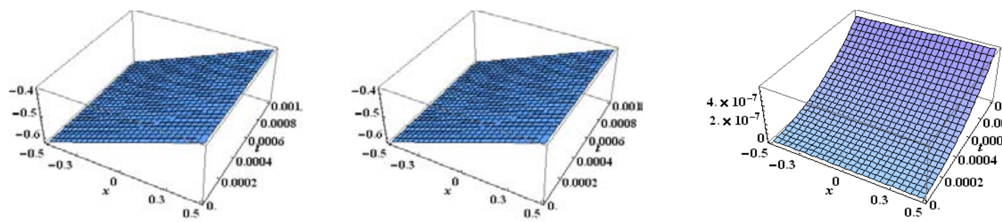

Figure 1: The approximate, exact solutions and absolute error, respectively for example 3.1.1 when -0.5< x <0.5 and 0< t <0.001.

Note that; Figure 1 shows the exact solution, approximate solution and the absolute error, respectively.

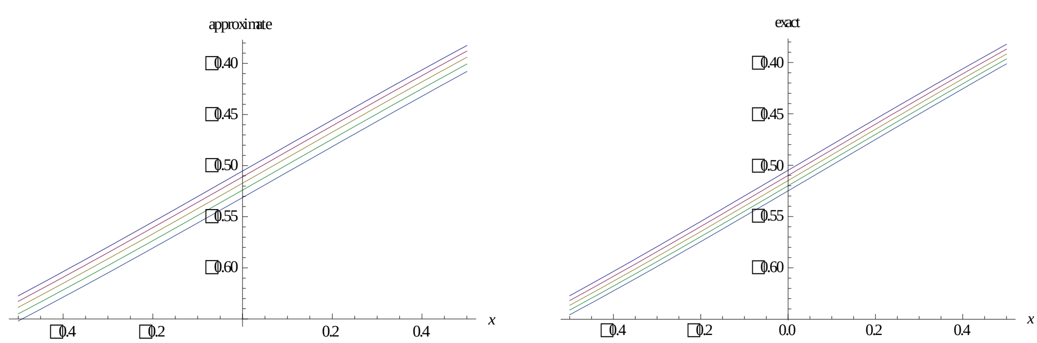

Figure 2: The approximate and exact solutions for example 3.1.1 when -0.5< x <0.5 and t =0.02, 0.04, 0.06, 0.08, 0.1.

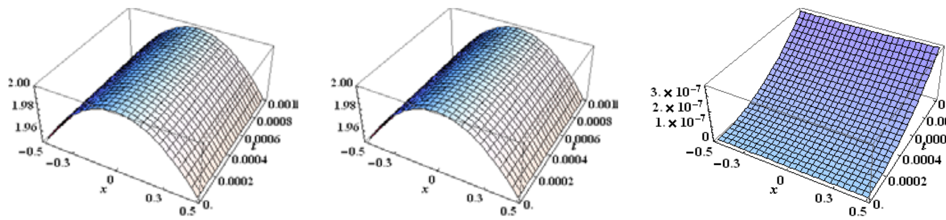

Figure 3: The approximate, exact solutions and absolute error, respectively for example 3.1.2 when -0.5< x <0.5 and 0< t <0.001.

Note that; Figure 3 shows the exact solution, approximate solution and the absolute error, respectively.

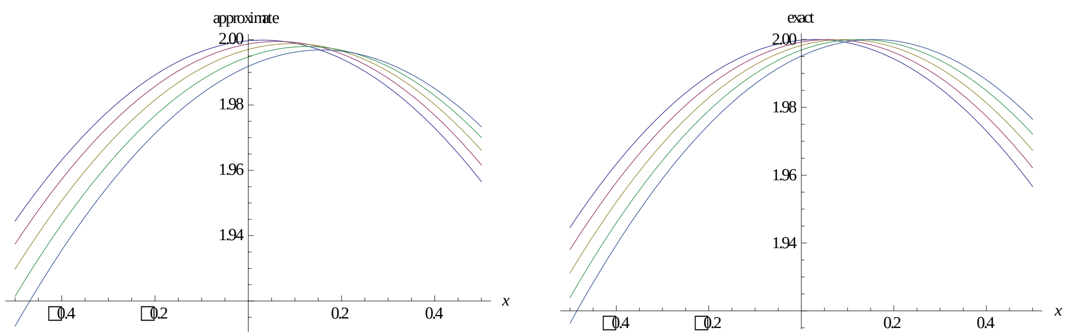

Figure 4: The approximate and exact solutions for example 3.1.1 when -0.5< *x* <0.5 and *t* =0.02, 0.04, 0.06, 0.08, 0.1.

5 Conclusion

In this paper, the Reduced Differential Transform Method (RDTM) was proposed for solving the Gardner equation, Nonlinear Variant Water Wave equation, and the Fifth-Order Korteweg-de Vries (FKdV) equation. We successfully found approximate solutions for the first two nonlinear PDEs by first applying the RDTM to all three physical models. Also I was being able to find exact solution for example (3.1.3). The results we obtained were in excellent agreement with the exact solutions. The RDTM introduces a significant improvement in the fields over existing techniques.

Also a comparative study has been conducted between the DTM and the RDTM. My goal in the future is to apply the RDTM to other nonlinear PDEs which arises in other areas of science. Computations of this paper have been carried out using computer package Mathematica 7.

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