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Harmonic multivalent meromorphic functions defined by an integral operator

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Abstract

The object of this article is to study a class $M_H(p)$ of harmonic multivalent meromorphic functions of the form $f(z) = h(z) + \overline{g(z)}$, $0 < |z| < 1$, where h and g are meromorphic functions. An integral operator is considered and is used to define a subclass $M_H(p, \alpha, m, c)$ of $M_H(p)$. Some properties of $M_H(p)$ are studied with the properties like coefficient condition, bounds, extreme points, convolution condition and convex combination for functions belongs to $M_H(p, \alpha, m, c)$ class.

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1 Introduction and Prelimnaries

A function $f = u + iv$, which is continuos complex-valued harmonic in a domain D $\subset \mathbb{C}$, if both u and v are real harmonic in D. Cluine and Sheil-

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small [3] investigated the family of all complex valued harmonic mappings f defined on the open unit disk U, which admits the representation $f(z) = h(z) + \overline{g(z)}$ where h and g are analytic univalent in U. Hengartner and Schober [4] considered the class of functions which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{\mathrm{U}} = \{z : |z| > 1\}$ so that $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| , \tag{1}$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$$

are analytic in $\widetilde{\mathrm{U}} = \{z = |z| > 1\}$, $\alpha, \beta, A \in \mathbb{C}$ with $0 \leq |\beta| \leq |\alpha|$ and $w(z) = \overline{f_{\overline{z}}}/f_z$ is analytic with $|w(z)| < 1$ for $z \in \widetilde{\mathrm{U}}$. $\sum_H'$ denotes a class of functions of the form (1) with $\alpha = 1, \beta = 0$. The class $\sum_H'$ has been studied in various research papers such as [5], [6] and [7].

Theorem 1.1. *[4] If $f \in \sum_H'$, then the diameter D_f of $\mathbb{C} \setminus f(U)$, satisfies*

$$D_f \geq 2 |1 + b_1| .$$

This estimate is sharp for

$$f(z) = z + b_1/\overline{z} + A \log |z|$$

whenever $|b_1| < 1$ and $|A| \leq \left(1 - |b_1|^2\right) / |1 + b_1|$, $|b_1| = 1$ and $A = 0$, or $b_1 = -1$ and $|A| \leq 2$.

A function is said to be meromorphic if poles are its only singularities in the complex plane $\mathbb{C}$.

Let M_p $(p \in \mathbb{N}_0 = \{1, 2, ...\})$ be a class of multivalent meromorphic functions of the form:

$$h(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1}, \; a_{-p} \geq 0, a_{n+p-1} \in \mathbb{C}, z \in \mathrm{U}^* = \mathrm{U} \setminus \{0\}. \tag{2}$$

Definition 1.2. *A Bernardi type integral operator* $I_{p,c}^{m}\,(m \geq 0, c > p)$ *for meromorphic multivalent function* $h \in M_p$ *is defined as :*

$$\begin{aligned} I_{p,c}^{0}h(z) &= h(z) \\ I_{p,c}^{1}h(z) &= \frac{c-p+1}{z^{c+1}} \int_0^z t^c I_{p,c}^{0}h(t)dt \\ I_{p,c}^{m}h(z) &= \frac{c-p+1}{z^{c+1}} \int_0^z t^c I_{p,c}^{m-1}h(t)dt,\ m \geq 1. \end{aligned}$$

The Series expansion of $I_{p,c}^{m}h(z)$ for $h(z)$ of the form (2) is given by

$$I_{p,c}^{m}h(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \theta^m(n)\ a_{n+p-1}\ z^{n+p-1}\,(c > p, m \geq 0)\,, \tag{3}$$

where

$$\theta^m(n) = \left(\frac{c-p+1}{n+p+c}\right)^m. \tag{4}$$

Note that $0 < \theta^m(n) < \theta^m(1) = \left(\frac{c-p+1}{1+p+c}\right)^m$.

For fixed integer $p \geq 1$, denote by $M_H(p)$, a family of harmonic multivalent meromorphic functions of the form

$$f(z) = h(z) + \overline{g(z)},\ z \in \mathrm{U}^*, \tag{5}$$

where $h \in M_p$ with $a_{-p} > 0$ of the form (2) and $g \in M_p$ of the form

$$g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}, b_{n+p-1} \in \mathbb{C},\ z \in \mathrm{U}. \tag{6}$$

In the expression (5), h is called a meromorphic part and g co-meromorphic part of f. The class $M_H(p)$ with its subclass has been studied in [1] and [8] for $a_{-p} = 1$. A quite similar to the above mentioned integral operator was used for harmonic analytic functions in [2].

In terms of the operator defined in Defininition 1.2, an operator $f \in M_H(p)$ is defined as follows:

Definition 1.3. *Let* $f = h + \overline{g}$ *be of the form* (5)*, the integral operator* $I_{p,c}^{m}f(z)$ *is defined as*

$$I_{p,c}^{m}f(z) = I_{p,c}^{m}h(z) + (-1)^m \overline{I_{p,c}^{m}g(z)}, z \in U^* \tag{7}$$

the series expansions for $I_{p,c}^{m}h(z)$ *is given by* (3) *with* $a_{-p}>0$ *and for* $I_{p,c}^{m}g(z)$ *with* $g(z)$ *of the form* (6) *is given as:*

$$I_{p,c}^{m}g(z)=\sum_{n=1}^{\infty}\theta^{m}(n)b_{n+p-1}z^{n+p-1}\,(c>p,m\geq 0)\,. \tag{8}$$

Involving operator $I_{p,c}^{m}$ define by (7), a class $M_H(p,\alpha,m,c)$ of functions $f\in M_H(p)$ is defined as follows:

Definition 1.4. *A function* $f\in M_H(p)$ *is said to be in* $M_H(p,\alpha,m,c)$ *if and only if it satisfies*

$$Re\left\{\frac{I_{p,c}^{m}f(z)}{I_{p,c}^{m+1}f(z)}\right\}>\alpha,\ \ z\in U^{*},0\leq\alpha<1,\ \ c>p,\ \ m\in\mathbb{N}. \tag{9}$$

Let $M_{\overline{H}}(p,\alpha,m,c)$ *be a subclass of* $M_H(p,\alpha,m,c)$*, consists of harmonic multivalent meromorphic functions* $f_m=h_m+\overline{g_m}$*, where* h_m *and* g_m *are of the form*

$$\begin{aligned} h_m(z)&=\frac{a_{-p}}{z^{p}}-\sum_{n=1}^{\infty}\left|a_{n+p-1}\right|z^{n+p-1},\ \ a_{-p}>0\\ &\text{and}\\ g_m(z)&=(-1)^{m+1}\sum_{n=1}^{\infty}\left|b_{n+p-1}\right|z^{n+p-1}. \end{aligned} \tag{10}$$

2 Some results for the class $M_H(p)$

In this section, some results for $M_H(p)$ class are derived.

Theorem 2.1. *Let* $f\in M_H(p)$ *be of the form (5), then the diameter* D_f *of* $clf(U^{*})$ *satisfies*

$$D_f\geq 2\left|a_{-p}\right|.$$

This estimate is sharp for $f(z)=a_{-p}z^{-p}$.

Proof. Let $D_f(r)$ denotes the diameter of $clf(\mathrm{U}_r^*)(r)$, where $\mathrm{U}_r^*(r) = \{z : 0 < |z| < r,\ 0 < r < 1\}$ and

$$D_f^*(r) := \max_{|z|=r} |f(z) - f(-z)|.$$

Since $D_f^*(r) \geq D_f(r)$, it follows that

$$\begin{aligned}
\left[D_f^*(r)\right]^2 &\geq \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta}) - f(-re^{i\theta})\right|^2 d\theta \\
&= 4\left[\frac{|a_{-p}|^2}{r^{2p}} + \sum_{n=1}^{\infty}\left(|a_{2n+p-2}|^2 + |b_{2n+p-2}|^2\right) r^{2(2n+p-2)}\right],\ p \text{ is odd} \\
&= 4\left[\frac{|a_{-p}|^2}{r^{2p}} + \sum_{n=1}^{\infty}\left(|a_{2n+p-1}|^2 + |b_{2n+p-1}|^2\right) r^{2(2n+p-1)}\right],\ p \text{ is even} \\
&\geq 4\frac{|a_{-p}|^2}{r^{2p}},
\end{aligned}$$

noted that as $r \longrightarrow 1$, $D_f(r)$ decreases to D_f, which concludes the result. □

Remark 2.2. *[4]Taking* $p = 1, a_{-p} = 1$ *and* $w(z) := f(1/z)$ *in Theorem 2.1, same result as Theorem 1.1 has been obtained for* $w(z)$*, which is of the form* (1) *with* $A = 0$.

Theorem 2.3. *If* $f \in M_H(p)$ *be of the form* (5)*, then*

$$\sum_{n=1}^{\infty}(n+p-1)\left(|a_{n+p-1}|^2 - |b_{n+p-1}|^2\right) \geq p\,|a_{-p}|^2.$$

Equality occurs if and only if $C\backslash f(U^*)$ *has zero area.*

Proof. The area of the ommited set is

$$\begin{aligned}
\lim_{r\to 1}\frac{1}{2i}\int_{|z|=r} \overline{f}df &= \lim_{r\to 1}\left[\frac{1}{2i}\int_{|z|=r} \overline{h}h'dz + \frac{1}{2i}\int_{|z|=r} g\overline{g'}d\overline{z}\right] \\
&= \pi\left[-p\,|a_{-p}|^2 + \sum_{n=1}^{\infty}(n+p-1)\left(|a_{n+p-1}|^2 - |b_{n+p-1}|^2\right)\right] \\
&\geq 0.
\end{aligned}$$

□

Remark 2.4. *[4]Taking* $p = 1, a_{-p} = 1$ *and* $w(z) := f(1/z)$ *in Theorem 2.3, then for* $w \in \sum'_H$*,with* $A = 0$*, it follows that*

$$\sum_{n=1}^{\infty} n\left(|a_n|^2 - |b_n|^2\right) \leq 1 + 2\Re b_1.$$

3 Coefficient Conditions

In this section, sufficient coefficient condition for a functiom $f \in M_H(p)$ to be in $M_H(p, \alpha, m, c)$ class is established and then it is shown that this coefficient condition is necessary for its subclass $M_{\overline{H}}(p, \alpha, m, c)$.

Theorem 3.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *be the form* (5) *and* $\theta^m(n)$ *is given by (4) if*

$$\sum_{n=1}^{\infty} \theta^m(n) \left[\left(1 - \alpha\theta^1(n)\right)|a_{n+p-1}| + \left(1 + \alpha\theta^1(n)\right)|b_{n+p-1}|\right] \leq (1 - \alpha)a_{-p}, \quad (11)$$

holds for $0 \leq \alpha < 1$ *and* $m \in \mathbb{N}$*, then* $f(z)$ *is harmonic in* U^* *and* $f \in M_H(p, \alpha, m, c)$.

Proof. Let the function $f(z) = h(z) + \overline{g(z)}$ be the form (5) satisfying (11). In order to show $f \in M_H(p, \alpha, m, c)$, it suffices to show that

$$Re\left\{\frac{I^m_{p,c} f(z)}{I^{m+1}_{p,c} f(z)}\right\} > \alpha \quad (12)$$

or,

$$Re\left\{\frac{I^m_{p,c} h(z) + (-1)^m \overline{I^m_{p,c} g(z)}}{I^{m+1}_{p,c} h(z) + (-1)^{m+1} \overline{I^{m+1}_{p,c} g(z)}}\right\} > \alpha$$

where $z = re^{i\theta}$, $0 < r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \alpha < 1$.
Let

$$A(z) := I^m_{p,c} h(z) + (-1)^m \overline{I^m_{p,c} g(z)} \quad (13)$$

and

$$B(z) := I^{m+1}_{p,c} h(z) + (-1)^{m+1} \overline{I^{m+1}_{p,c} g(z)}. \quad (14)$$

It is observed that (12) holds if

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\geq 0. \tag{15}$$

From (13) and (14), it follows that

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)| \\
&= \left|I_{p,c}^{m}h(z)+(-1)^{m}\overline{I_{p,c}^{m}g(z)}+(1-\alpha)\left(I_{p,c}^{m+1}h(z)+(-1)^{m+1}\overline{I_{p,c}^{m+1}g(z)}\right)\right| \\
&= \left|(2-\alpha)\frac{a_{-p}}{z^{p}}+\sum_{n=1}^{\infty}\left[\theta^{m}(n)+(1-\alpha)\theta^{m+1}(n)\right]a_{n+p-1}z^{n+p-1}\right. \\
&\qquad \left.+(-1)^{m}\sum_{n=1}^{\infty}\left[\theta^{m}(n)-(1-\alpha)\theta^{m+1}(n)\right]\overline{b_{n+p-1}z^{n+p-1}}\right| \\
&\geq \frac{(2-\alpha)a_{-p}}{|z|^{p}}-\sum_{n=1}^{\infty}\theta^{m}(n)\left[1+(1-\alpha)\theta^{1}(n)\right]|a_{n+p-1}|\,|z|^{n+p-1} \\
&\qquad -\sum_{n=1}^{\infty}\theta^{m}(n)\left[1-(1-\alpha)\theta^{1}(n)\right]|b_{n+p-1}|\,|z|^{n+p-1}
\end{aligned}$$

and

$$\begin{aligned}
&|A(z)-(1+\alpha)B(z)| \\
&= \left|I_{p,c}^{m}h(z)+(-1)^{m}\overline{I_{p,c}^{m}g(z)}-(1+\alpha)\left(I_{p,c}^{m+1}h(z)+(-1)^{m+1}\overline{I_{p,c}^{m+1}g(z)}\right)\right| \\
&= \left|(-\alpha)\frac{a_{-p}}{z^{p}}+\sum_{n=1}^{\infty}\left[\theta^{m}(n)-(1+\alpha)\theta^{m+1}(n)\right]a_{n+p-1}z^{n+p-1}\right. \\
&\qquad \left.+(-1)^{m}\sum_{n=1}^{\infty}\left[\theta^{m}(n)+(1+\alpha)\theta^{m+1}(n)\right]\overline{b_{n+p-1}z^{n+p-1}}\right| \\
&= \left|\alpha\frac{a_{-p}}{z^{p}}-\sum_{n=1}^{\infty}\left[\theta^{m}(n)-(1+\alpha)\theta^{m+1}(n)\right]a_{n+p-1}z^{n+p-1}\right. \\
&\qquad \left.-(-1)^{m}\sum_{n=1}^{\infty}\left[\theta^{m}(n)+(1+\alpha)\theta^{m+1}(n)\right]\overline{b_{n+p-1}z^{n+p-1}}\right| \\
&\leq \alpha\frac{a_{-p}}{|z|^{p}}+\sum_{n=1}^{\infty}\theta^{m}(n)\left[1-(1+\alpha)\theta^{1}(n)\right]|a_{n+p-1}|\,|z|^{n+p-1} \\
&\qquad +\sum_{n=1}^{\infty}\theta^{m}(n)\left[1+(1+\alpha)\theta^{1}(n)\right]|b_{n+p-1}|\,|z|^{n+p-1}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&\geq \frac{2(1-\alpha)a_{-p}}{|z|^p}-2\sum_{n=1}^{\infty}\theta^m(n)\left[1-\alpha\theta^1(n)\right]|a_{n+p-1}|\,|z|^{n+p-1} \\
&\qquad -2\sum_{n=1}^{\infty}\theta^m(n)\left[1+\alpha\theta^1(n)\right]|b_{n+p-1}|\,|z|^{n+p-1} \\
&\geq \frac{2}{|z|^p}\left\{a_{-p}(1-\alpha)-\sum_{n=1}^{\infty}\theta^m(n)\left[1-\alpha\theta^1(n)\right]|a_{n+p-1}|\,|z|^{n-1}\right. \\
&\qquad \left.-\sum_{n=1}^{\infty}\theta^m(n)\left[1+\alpha\theta^1(n)\right]|b_{n+p-1}|\,|z|^{n-1}\right\} \\
&\geq 2\left\{(1-\alpha)a_{-p}-\sum_{n=1}^{\infty}\theta^m(n)\left[\left(1-\alpha\theta^1(n)\right)|a_{n+p-1}|+\left(1+\alpha\theta^1(n)\right)|b_{n+p-1}|\right]\right\} \\
&\geq 0,
\end{aligned}
$$

if (11) holds. This proves the Theorem. □

Theorem 3.2. *Let $f_m = h_m + \overline{g_m}$ where h_m and g_m are of the form* (10) *then $f_m \in M_{\overline{H}}(p,\alpha,m,c)$ under the same parametric conditions taken in Theorem 3.1, if and only if the inequality* (11) *holds.*

Proof. Since $M_{\overline{H}}(p,\alpha,m,c) \subset M_H(p,\alpha,m,c)$, if part is proved in Theorem 3.1. It only needs to prove the "only if" part of the Theorem. For this, it suffices to show that $f_m \notin M_{\overline{H}}(p,\alpha,m,c)$ if the condition (11) does not hold. If $f_m \in M_{\overline{H}}(p,\alpha,m,c)$, then writing corresponding series expansions in (9) , it follows that $Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0$ for all values of z in U* where

$$
\begin{aligned}
\xi(z) &= (1-\alpha)\frac{a_{-p}}{z^p}+\sum_{n=1}^{\infty}\theta^m(n)\left[1-\alpha\theta^1(n)\right]|a_{n+p-1}|\,z^{n+p-1} \\
&\qquad -\sum_{n=1}^{\infty}\theta^m(n)\left[1+\alpha\theta^1(n)\right]|b_{n+p-1}|\,z^{n+p-1}
\end{aligned}
$$

and

$$
\eta(z)=\frac{a_{-p}}{z^p}+\sum_{n=1}^{\infty}\theta^{m+1}(n)\,|a_{n+p-1}|\,z^{n+p-1}-\sum_{n=1}^{\infty}\theta^{m+1}(n)\,|b_{n+p-1}|\,z^{n+p-1}.
$$

Since

$$\left|\frac{\xi(z)}{\eta(z)}\right| \geq Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0,$$

hence the condition $Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0$ holds if

$$\frac{(1-\alpha)a_{-p} - \sum_{n=1}^{\infty}\theta^m(n)\left[(1-\alpha\theta^1(n))\left|a_{n+p-1}\right| + (1+\alpha\theta^1(n))\left|b_{n+p-1}\right|\right] r^{n+2p-1}}{a_{-p} + \sum_{n=1}^{\infty}\theta^{m+1}(n)\left|a_{n+p-1}\right| r^{n+2p-1} + \sum_{n=1}^{\infty}\theta^{m+1}(n)\left|b_{n+p-1}\right| r^{n+2p-1}} \geq 0. \tag{16}$$

Now if the condition (11) does not holds then the numerator of above equation is non-positive for r sufficiently close to 1, which contradicts that $f_m \in M_{\overline{H}}(p,\alpha,m,c)$ and this proves the required result. □

4 Bounds and Extreme Points

In this section, bounds for functions belonging to the class $M_{\overline{H}}(p,\alpha,m,c)$ are obtained and also provide extreme points for the same class.

Theorem 4.1. *If $f_m = h_m + \overline{g_m} \in M_{\overline{H}}(p,\alpha,m,c)$ for $0 \leq \alpha < 1$, $0 < |z| = r < 1$, and $\theta^m(n)$ is given by (4), under the same parametric conditions taken in Theorem 3.1 then*

$$\frac{a_{-p}}{r^p} - r^p\frac{(1-\alpha)a_{-p}}{[1-\alpha\theta^1(1)]} \leq \left|I^m_{p,c}f_k(z)\right| \leq \frac{a_{-p}}{r^p} + r^p\frac{(1-\alpha)a_{-p}}{[1-\alpha\theta^1(1)]}.$$

Proof. Let $f_m = h_m + \overline{g_m} \in M_{\overline{H}}(p,\alpha,m,c)$. Taking the absolute value of $I^m_{p,c}f_m$ from (7), it follows that

$$\begin{aligned}\left|I^m_{p,c}f_m(z)\right| &= \left|\frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty}\theta^m(n)\left|a_{n+p-1}\right| z^{n+p-1} + (-1)^{2m+1}\overline{\sum_{n=1}^{\infty}\theta^m(n)\left|b_{n+p-1}\right| z^{n+p-1}}\right| \\ &= \left|\frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty}\theta^m(n)\left(\left|a_{n+p-1}\right| - \left|b_{n+p-1}\right|\right) z^{n+p-1}\right|\end{aligned}$$

$$\begin{aligned}
\left|I^m_{p,c}f_m(z)\right| &\leq \frac{a_{-p}}{r^p} + r^p \sum_{n=1}^{\infty} \frac{[\theta^m(n) - \alpha\theta^{m+1}(n)]}{[\theta^m(n) - \alpha\theta^{m+1}(n)]} \theta^m(n) \left(|a_{n+p-1}| + |b_{n+p-1}|\right) \\
&\leq \frac{a_{-p}}{r^p} + r^p \sum_{n=1}^{\infty} \frac{[\theta^m(n) - \alpha\theta^{m+1}(n)]}{\theta^m(n)\left[1 - \alpha\theta^1(n)\right]} \theta^m(n) \left(|a_{n+p-1}| + |b_{n+p-1}|\right) \\
&\leq \frac{a_{-p}}{r^p} + \frac{r^p}{[1 - \alpha\theta^1(1)]} \sum_{n=1}^{\infty} \left[\theta^m(n) - \alpha\theta^{m+1}(n)\right] \left(|a_{n+p-1}| + |b_{n+p-1}|\right) \\
&\leq \frac{a_{-p}}{r^p} + \frac{r^p}{[1 - \alpha\theta^1(1)]} \left[\sum_{n=1}^{\infty} \theta^m(n) \left[\left(1 - \alpha\theta^1(n)\right) |a_{n+p-1}|\right] \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \theta^m(n) \left[\left(1 + \alpha\theta^1(n)\right) |b_{n+p-1}|\right] \right] \\
&\leq \frac{a_{-p}}{r^p} + r^p \frac{(1-\alpha)a_{-p}}{[1 - \alpha\theta^1(1)]}
\end{aligned}$$

and

$$\begin{aligned}
\left|I^m_{p,c}f_m(z)\right| &= \left| \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \theta^m(n) |a_{n+p-1}| z^{n+p-1} + (-1)^{2m+1} \overline{\sum_{n=1}^{\infty} \theta^m(n) |b_{n+p-1}| z^{n+p-1}} \right| \\
&= \left| \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \theta^m(n) \left(|a_{n+p-1}| - |b_{n+p-1}|\right) z^{n+p-1} \right|
\end{aligned}$$

$$\begin{aligned}
\left|I^m_{p,c}f_m(z)\right| &\geq \frac{a_{-p}}{r^p} - r^p \sum_{n=1}^{\infty} \frac{[\theta^m(n) - \alpha\theta^{m+1}(n)]}{[\theta^m(n) - \alpha\theta^{m+1}(n)]} \theta^m(n) \left(|a_{n+p-1}| + |b_{n+p-1}|\right) \\
&\geq \frac{a_{-p}}{r^p} - r^p \sum_{n=1}^{\infty} \frac{[\theta^m(n) - \alpha\theta^{m+1}(n)]}{\theta^m(n)\left[1 - \alpha\theta^1(n)\right]} \theta^m(n) \left(|a_{n+p-1}| + |b_{n+p-1}|\right) \\
&\geq \frac{a_{-p}}{r^p} - \frac{r^p}{[1 - \alpha\theta^1(1)]} \sum_{n=1}^{\infty} \left[\theta^m(n) - \alpha\theta^{m+1}(n)\right] \left(|a_{n+p-1}| + |b_{n+p-1}|\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{a_{-p}}{r^p} - \frac{r^p}{[1 - \alpha\theta^1(1)]} \left[\sum_{n=1}^{\infty} \theta^m(n) \left[\left(1 - \alpha\theta^1(n)\right) |a_{n+p-1}|\right] \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \theta^m(n) \left[\left(1 + \alpha\theta^1(n)\right) |b_{n+p-1}|\right] \right] \\
&\geq \frac{a_{-p}}{r^p} - r^p \frac{(1-\alpha)a_{-p}}{[1 - \alpha\theta^1(1)]}
\end{aligned}$$

This proves the required result. □

Corollary 4.2. *Let $f_m = h_m + \overline{g_m} \in M_{\overline{H}}(p, \alpha, m, c)$, for $z \in U^*$ and $\theta^m(n)$ is given by (4), under the same parametric conditions taken in Theorem 3.1 then*

$$\left\{ w : |w| < a_{-p} - \frac{(1-\alpha)a_{-p}}{[1 - \alpha\theta^1(1)]} \right\} \nsubseteq f(U^*).$$

Theorem 4.3. *Let $f_m = h_m + \overline{g_m}$, where h_m and g_m are of the form* (10) *then $f_m \in M_{\overline{H}}(p, \alpha, m, c)$ and $\theta^m(n)$ is given by (4), under the same parametric conditions taken in Theorem 3.1, if and only if f_m can be expressed as:*

$$f_m(z) = \sum_{n=0}^{\infty} \left(x_{n+p-1} h_{m_{n+p-1}}(z) + y_{n+p-1} g_{m_{n+p-1}}(z) \right), \tag{17}$$

where $z \in U^*$ and

$$h_{m_{p-1}}(z) = \frac{a_{-p}}{z^p}, \quad h_{m_{n+p-1}}(z) = \frac{a_{-p}}{z^p} + \frac{(1-\alpha)a_{-p}}{\theta^m(n) - \alpha\theta^{m+1}(n)} z^{n+p-1}, \tag{18}$$

$$g_{m_{p-1}}(z) = \frac{a_{-p}}{z^p}, \quad g_{m_{n+p-1}}(z) = \frac{a_{-p}}{z^p} + (-1)^{m+1} \frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} \overline{z^{n+p-1}} \tag{19}$$

for $n = 1, 2, 3, \ldots\ldots$, and

$$\sum_{n=1}^{\infty} (x_{n+p-1} + y_{n+p-1}) = 1, \qquad x_{n+p-1},\, y_{n+p-1} \geq 0. \tag{20}$$

Proof. Let

$$\begin{aligned}
f_m(z) &= \sum_{n=0}^{\infty} \left(x_{n+p-1} h_{m_{n+p-1}}(z) + y_{n+p-1} g_{m_{n+p-1}}(z) \right) \\
&= x_{p-1} h_{m_{p-1}} + y_{p-1} g_{m_{p-1}} + \sum_{n=1}^{\infty} x_{n+p-1} \left(\frac{a_{-p}}{z^p} + \frac{(1-\alpha)a_{-p}}{\theta^m(n) - \alpha\theta^{m+1}(n)} z^{n+p-1} \right) \\
&\quad + \sum_{n=1}^{\infty} y_{n+p-1} \left(\frac{a_{-p}}{z^p} + (-1)^{m+1} \frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} \overline{z^{n+p-1}} \right) \\
f_m(z) &= \sum_{n=0}^{\infty} (x_{n+p-1} + y_{n+p-1}) \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \left\{ \left(\frac{(1-\alpha)a_{-p}}{\theta^m(n) - \alpha\theta^{m+1}(n)} x_{n+p-1} \right) \right. \\
&\quad \left. + (-1)^{m+1} \frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} y_{n+p-1} \right\} z^{n+p-1}.
\end{aligned}$$

Thus by Theorem 3.2, $f_m \in M_{\overline{H}}(p, \alpha, m, c)$, since,

$$\sum_{n=1}^{\infty} \left\{ \frac{\theta^m(n) - \alpha\theta^{m+1}(n)}{(1-\alpha)a_{-p}} \left(\frac{(1-\alpha)a_{-p}}{\theta^m(n) - \alpha\theta^{m+1}(n)} x_{n+p-1} \right) - \frac{\theta^m(n) + \alpha\theta^{m+1}(n)}{(1-\alpha)a_{-p}} \left(\frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} y_{n+p-1} \right) \right\}$$

$$= \sum_{n=1}^{\infty} (x_{n+p-1} + y_{n+p-1}) = (1 - x_{p-1} - y_{p-1}) \leq 1.$$

Conversly, suppose that $f_m \in M_{\overline{H}}(p, \alpha, m, c)$, then (11) holds.
Setting

$$x_{n+p-1} = \frac{\theta^m(n) - \alpha\theta^{m+1}(n)}{(1-\alpha)a_{-p}} |a_{n+p-1}|$$

$$y_{n+p-1} = \frac{\theta^m(n) + \alpha\theta^{m+1}(n)}{(1-\alpha)a_{-p}} |b_{n+p-1}|$$

which satisfy (20), thus

$$\begin{aligned} f_m(z) &= \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{n+p-1} + (-1)^{m+1} \sum_{n=1}^{\infty} |b_{n+p-1}| z^{n+p-1} \\ &= \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} x_{n+p-1} z^{n+p-1} \\ &\quad + (-1)^{m+1} \sum_{n=1}^{\infty} \frac{(1-\alpha)a_{-p}}{\theta^m(n) + \alpha\theta^{m+1}(n)} y_{n+p-1} z^{n+p-1} \end{aligned}$$

$$\begin{aligned} &= \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \left[h_{m_{n+p-1}} - \frac{a_{-p}}{z^p} \right] x_{n+p-1} + \sum_{n=1}^{\infty} \left[g_{m_{n+p-1}} - \frac{a_{-p}}{z^p} \right] y_{n+p-1} \\ &= \frac{a_{-p}}{z^p} \left[1 - \sum_{n=1}^{\infty} x_{n+p-1} - \sum_{n=1}^{\infty} y_{n+p-1} \right] + \sum_{n=1}^{\infty} h_{m_{n+p-1}} x_{n+p-1} + g_{m_{n+p-1}} y_{n+p-1} \\ &= x_{p-1} h_{m_{p-1}} + y_{p-1} g_{m_{p-1}} + \sum_{n=1}^{\infty} h_{m_{n+p-1}} x_{n+p-1} + \sum_{n=1}^{\infty} g_{m_{n+p-1}} y_{n+p-1} \\ &= \sum_{n=1}^{\infty} \left(x_{n+p-1} h_{m_{n+p-1}}(z) + y_{n+p-1} g_{m_{n+p-1}}(z) \right) \end{aligned}$$

This proves the Theorem. □

Remark 4.4. *The extreme points for the class* $M_{\overline{H}}(p, \alpha, m, c)$ *are given by* (18) *and* (19).

5 Convolution and Integral Convolution

In this section, convolution and integral convolution properties of the class $M_{\overline{H}}(p, \alpha, m, c)$ are established.

Let $f_m, F_m \in M_{\overline{H}}(p)$ be defined as follows:

$$f_m(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} |a_{n+p-1}|\, z^{n+p-1} + (-1)^{m+1} \sum_{n=1}^{\infty} |b_{n+p-1}|\, z^{n+p-1},\ z \in \mathrm{U}^*,$$

$$F_m(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} |A_{n+p-1}|\, z^{n+p-1} + (-1)^{m+1} \sum_{n=1}^{\infty} |B_{n+p-1}|\, z^{n+p-1},\ z \in \mathrm{U}^*.$$

The convolution of f_m and F_m for $m \in \mathbb{N}$, $z \in \mathrm{U}^*$ is defined as:

$$\begin{aligned} (f_m \star F_m)(z) &= f_m(z) \star F_m(z) \\ &= \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} |a_{n+p-1} A_{n+p-1}|\, z^{n+p-1} + \\ &\quad +(-1)^{m+1} \sum_{n=1}^{\infty} |b_{n+p-1} B_{n+p-1}|\, z^{n+p-1}. \end{aligned} \tag{21}$$

The integral convolution of f_m and F_m for $m \in \mathbb{N}$, $z \in \mathrm{U}^*$ is defined as:

$$\begin{aligned} (f_m \diamondsuit F_m)(z) &= f_m(z) \diamondsuit F_m(z) \\ &= \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \frac{p\, |a_{n+p-1} A_{n+p-1}|}{n+p-1} z^{n+p-1} \\ &\quad +(-1)^{m+1} \sum_{n=1}^{\infty} \frac{p\, |b_{n+p-1} B_{n+p-1}|}{n+p-1} \overline{z}^{n+p-1}. \end{aligned} \tag{22}$$

Theorem 5.1. *For* $0 \le \alpha < 1$, $m \in \mathbb{N}$. *If* $f_m, F_m \in M_{\overline{H}}(p, \alpha, m, c)$ *and* $\theta^m(n)$ *is given by (4) then* $(f_m \star F_m) \in M_{\overline{H}}(p, \alpha, m, c)$.

Proof. Since $F_m \in M_{\overline{H}}(p,\alpha,m,c)$, then by Theorem 3.2, $|A_{n+p-1}| \leq 1$ and $|B_{n+p-1}| \leq 1$, hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\theta^m(n)\left[\left\{\frac{1-\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|A_{n+p-1}a_{n+p-1}| \right.\\
&\qquad\qquad \left. +\left\{\frac{1+\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|B_{n+p-1}b_{n+p-1}|\right]\\
\leq\ &\sum_{n=1}^{\infty}\theta^m(n)\left[\left\{\frac{1-\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|a_{n+p-1}| \right.\\
&\qquad\qquad \left. +\left\{\frac{1+\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|b_{n+p-1}|\right]\\
\leq\ & 1
\end{aligned}$$

as $f_m \in M_{\overline{H}}(p,\alpha,m,c)$. Thus by the Theorem 3.2, $(f_m \star F_m) \in M_{\overline{H}}(p,\alpha,m,c)$. □

Theorem 5.2. *For* $0 \leq \alpha < 1$, $m \in \mathbb{N}$. *If* $f_m, F_m \in M_{\overline{H}}(p,\alpha,m,c)$ *and* $\theta^m(n)$ *is given by (4) then* $(f_m \diamondsuit F_m) \in M_{\overline{H}}(p,\alpha,m,c)$.

Proof. Since $F_m \in M_{\overline{H}}(p,\alpha,m,c)$, then by Theorem 3.2, $|A_{n+p-1}| \leq 1$, $|B_{n+p-1}| \leq 1$ and $\frac{p}{n+p-1} \leq 1$ hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\theta^m(n)\left[\left\{\frac{1-\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}\frac{p\,|a_{n+p-1}A_{n+p-1}|}{n+p-1} \right.\\
&\qquad\qquad \left. +\left\{\frac{1+\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}\frac{p\,|b_{n+p-1}B_{n+p-1}|}{n+p-1}\right]\\
\leq\ &\sum_{n=1}^{\infty}\theta^m(n)\left[\left\{\frac{1-\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|a_{n+p-1}| \right.\\
&\qquad\qquad \left. +\left\{\frac{1+\alpha\theta^1(n)}{(1-\alpha)a_{-p}}\right\}|b_{n+p-1}|\right]\\
\leq\ & 1
\end{aligned}$$

as $f_m \in M_{\overline{H}}(p,\alpha,m,c)$. Thus by the Theorem 3.2, $(f_m \diamondsuit F_m) \in M_{\overline{H}}(p,\alpha,m,c)$. □

6 Convex Combination

In this section, it is proved that the class $M_{\overline{H}}(p, \alpha, m, c)$ is closed under convex linear combination of its members.

Theorem 6.1. *Let the functions $f_{m_j}(z)$ defined as:*

$$f_{m_j}(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} |a_{n+p-1,j}|\, z^{n+p-1} + (-1)^{m+1} \sum_{n=1}^{\infty} |b_{n+p-1,j}|\, z^{n+p-1},\ z \in \mathbb{U}^*. \tag{23}$$

be in the class $M_{\overline{H}}(p, \alpha, m, c)$ for every $j = 1, 2, 3....$, then the function

$$\psi(z) = \sum_{j=1}^{\infty} c_j f_{m_j}(z)$$

is also in the class $M_{\overline{H}}(p, \alpha, m, c)$, where $\sum_{j=1}^{\infty} c_j = 1$ for $c_j \geq 0$ $(j = 1, 2, 3....)$.

Proof. From the definition

$$\psi(z) = \frac{a_{-p}}{z^p} + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} c_j |a_{n+p-1,j}| \right) z^{n+p-1} + (-1)^{m+1} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} c_j |b_{n+p-1,j}| \right) z^{n+p-1}.$$

Since $f_{m_j}(z) \in M_{\overline{H}}(p, \alpha, m, c)$ for every $j = 1, 2, 3...$, then by Theorem 3.2, it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \theta^m(n) \left[\left\{ \frac{1 - \alpha\theta^1(n)}{(1-\alpha)a_{-p}} \right\} \left(\sum_{j=1}^{\infty} c_j |a_{n+p-1,j}| \right) \right. \\ &\qquad \left. + \left\{ \frac{1 + \alpha\theta^1(n)}{(1-\alpha)a_{-p}} \right\} \left(\sum_{j=1}^{\infty} c_j |b_{n+p-1,j}| \right) \right] \\ &= \sum_{j=1}^{\infty} c_j \left(\sum_{n=1}^{\infty} \left\{ \frac{1 - \alpha\theta^1(n)}{(1-\alpha)a_{-p}} \right\} |a_{n+p-1,j}| \right. \\ &\qquad \left. + \left\{ \frac{1 + \alpha\theta^1(n)}{(1-\alpha)a_{-p}} \right\} |b_{n+p-1,j}| \right) \\ &\leq \sum_{j=1}^{\infty} c_j .1 \leq 1. \end{aligned}$$

Hence $\psi(z) \in M_{\overline{H}}(p, \alpha, m, c)$, which is the desired result. □

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