Minimizing Total Completion Time in a Flow-shop Scheduling Problems with a Single Server

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Abstract

We consider the problem of two-machine flow-shop scheduling with a single server and equal processing times, we show that this problem is \( NP \) -hard in the strong sense and present a busy schedule for it with worst-case bound \( \frac{7}{6} \).

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1 Introduction

We consider the two-machine flow-shop scheduling problem with minimizing total completion time and equal processing times, that

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is $F^2|p_{i,j} = p|\sum C_j$. Complexity results for $F^2|\sum C_j$ problem obtained by Garey, et al [1], J.A. Hoogereen [2] studied some special cases for two-machine flow-shop problems with minimizing total completion times, and proved that the problem with equal processing times on first machine, that is $F^2|p_{i,j} = p|\sum C_j$, is \textit{NP}-hard in the strong sense, and present an $O(n \log n)$ approximation algorithm for it with worst-case bound $4/3$. Complexity results for flow-shop problems with a single server was obtained by Brucher, et al [3]. In this paper, we derive some new complexity results for two-machine problem with a single server, introduce an improved algorithm, and prove that its worst case is $7/6$, the bound is tight.

2 Complexity of the $F^2, S1|p_{i,j} = p|\sum C_j$ problem

Let $C_{i,j}$ denote the completion times of job $J_j$ on machine $M_i$. If there are no idle times on $M_1$ and $M_2$, we have $C_{1,1} = s_{1,1} + p_{1,1}, C_{2,1} = s_{1,1} + p_{1,1} + s_{2,1} + p_{2,1}$, 

$$C_{i,j} = C_{i,j-1} + s_{i,j} + p_{i,j}, C_{2,j} = \max\{ C_{2,j-1}, C_{1,j} \} + s_{2,j} + p_{2,j}, \text{ for } j = 2, ..., n.$$ 

\textbf{Theorem 1} The problem of deciding whether for a given instance of the $F^2, S1|p_{i,j} = p|\sum C_j$ problem there exists a schedule with cost no more than a given threshold value $y$ is \textit{NP}-hard in the strong sense.

\textit{Proof}. Our proof is based upon a reduction from the problem \textit{Numerical Matching with Target Sums} or, in short, $TS$, which is known to be \textit{NP}-hard in the strong sense.

\textit{TS} Given two multisets $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ of positive integers and an target vector $\{z_1, ..., z_n\}$, where $\sum_{j=1}^{n}(x_j + y_j) = \sum_{j=1}^{n}z_j$, is there a position of the set $X \cup Y$ into $n$ disjoint set $Z_1, ..., Z_n$, each containing exactly one element
from each of $X$ and $Y$, such that the sum of the numbers in $Z_j$ equal $z_j$, for $1, \ldots, n$?

(1) $P$-jobs: $s_{1,i} = b, p_{1,i} = b; s_{2,i} = b + x_i, p_{2,i} = b (i = 1, \ldots, n)$,

(2) $Q$-jobs: $s_{1,i} = 0, p_{1,i} = b; s_{2,i} = b + y_i, p_{2,i} = b (i = 1, \ldots, n)$,

(3) $R$-jobs: $s_{1,i} = 0, p_{1,i} = b; s_{2,i} = b - z_i, p_{2,i} = b (i = 1, \ldots, n)$,

(4) $U$-jobs: $s_{1,i} = 0, p_{1,i} = b; s_{2,i} = 0, p_{2,i} = b (i = 1, \ldots, n)$,

(5) $V$-jobs: $s_{1,i} = 0, p_{1,i} = b; s_{2,i} = 0, p_{2,i} = b (i = 1, \ldots, n)$,

(6) $W$-jobs: $s_{1,i} = 0, p_{1,i} = b; s_{2,i} = 0, p_{2,i} = b (i = 1, \ldots, n)$,

(7) $L$-jobs: $s_{1,i} = 4b, p_{1,i} = b; s_{2,i} = b, p_{2,i} = b (i = 1, \ldots, n)$.

Observe that all processing times are equal to $b$. To prove the theorem we show that in this constructed if the $F2, S1|p_{i,j} = p| \sum C_j$ problem a schedule $S_0$ satisfying $\sum C_j (S_0) \leq y = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (x_i + y_i) + (77n^2 - 13n - 4)b/2$ exists if and only if $T S$ has a solution. Suppose that $T S$ has a solution. The desired schedule $S_0$ exists and can be described as follows. No machine has intermediate idle time. $M_1$ process the jobs in order of the sequence $\sigma$, i.e., in the sequence

$$\sigma = \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20} \}$$

While $M_2$ process the jobs in the sequence

$$\tau = \{ \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20} \}$$

as indicated in Figure 1.

Then we define the sequence $\sigma$ and $\tau$ shown in Figure 1. Obviously, these sequence $\sigma$ and $\tau$ fulfills $C(S) = C(\sigma, \tau) \leq y$. Conversely, assume that the flow-shop scheduling problem has a solution $\sigma$ and $\tau$ with $C(S) \leq y$.

Considering the path composed of $M_1$ operations of jobs $\{P_{1,1}, Q_{1,1}, R_{1,1}, U_{1,1}, V_{1,1}, W_{1,1} \}$, $M_2$ operations of jobs
{R_{2,1}, U_{2,1}, V_{2,1}, W_{2,1}, L_{2,1}, ..., R_{2,n}, U_{2,n}, V_{2,n}, W_{2,n}, L_{2,n}}$, we obtain that

$$C(S) \geq 3b + x_i + 5b + x_i + y_i + 7b + x_i + y_i - z_i + 8b + 9b + 10b + ... + (3 + (n-1)11)b + x_i + y_i + (7 + (n-1)11)b + ... + (11n+1)b$$

$$= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (x_i + y_i) + (77n^2 - 13n - 4)b / 2 = y.$$

So we have $C(S) = y$.

(a) If $S$ has a partition $\mu$, then there is a schedule with finish times $y$. One such schedule is shown in Figure 1.

(b) If $S$ has no partition, then all schedule must have a finish times $> y$. Since $S$ has no partition, then $x_i + y_i \neq z_i (i = 1, ..., n)$. Let $\xi_i = x_i + y_i - z_i (i = 1, ..., n)$, so

$$\sum C_j(S) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (x_i + y_i) + (77n^2 - 13n - 4)b / 2 + 5\sum_{i=1}^{n} \xi_i$$

$$+ 10\sum_{i=1}^{n-1} \xi_i + ... + 5n\xi_1 > y$$

3 **Worst-case for the $F2,S1|p_{i,j} = p|\sum C_j$ problem**

In examining “worst” schedule, we restrict ourselves to busy schedule. A busy schedule is a schedule in which at all times from start to finish at least one server is processing a task.
Theorem 2 The $F2, S1| p_{i,j} = p| \sum C_j$ problem, let $S_o$ be a busy schedule for this problem, $S^*$ be the optimal solution for the $F2, S1| p_{i,j} = p| \sum C_j$ problem, then

$$\sum C_j(S_0) / \sum C_j(S^*) \leq 7/6,$$

the bound is tight.

Proof. For a schedule $S$, let $I_{i,j}(S) (i = 1, 2; j = 1, \ldots, n)$ denote the total idle times of job $J_j$ on $M_i$. Considering the path composed of $M_1$ operations of jobs $1, \ldots, j, M_2$ operation of job $j$, we obtain that

$$C_j = \sum_{i=1}^{j} (s_{i,j} + p_{1,i}) + I_{1,j} + s_{2,j} + p_{2,j} \quad (1)$$

Considering the path composed of $M_1$ operations of jobs $1, M_2$ operation of job $1, 2, \ldots, j$, we obtain that

$$C_j = s_{1,1} + p_{1,1} + \sum_{i=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j} \quad (2)$$

Considering the path composed of $M_1$ operations of jobs $1, \ldots, l, M_2$ operation of job $l, \ldots, j$, we obtain that

$$C_j = \sum_{i=1}^{l} (s_{i,d} + p_{1,i}) + I_{1,l} + \sum_{i=l}^{j} (s_{2,i} + p_{2,i}) + I_{2,j} \quad (3)$$

So we have

$$6 \sum C_j(S_0) = (2(\sum_{i=1}^{j} (s_{1,i} + p_{1,i}) + I_{1,j} )) + (2(\sum_{i=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j} )) +$$

$$2(\sum_{i=1}^{l} (s_{1,i} + p_{1,i}) + I_{1,l} + \sum_{i=l}^{j} (s_{2,i} + p_{2,i}) + (2(s_{1,1} + p_{1,1}) + 2(s_{2,1} + p_{2,1}) +$$

$$\sum C_j(S_0) / \sum C_j(S^*) \leq 7/6.$$

To prove the bound is tight, introduce the following example as follows and show in Figure 2 and Figure 3.

(1) $P$-jobs: $s_{1,i} = 2b, p_{1,i} = b, s_{2,i} = 2b, p_{2,i} = b (i = 1, 2)$;
(2) $Q$-jobs: $s_{1,i} = 0, p_{1,i} = b, s_{2,i} = 0, p_{2,i} = b (i = 3, 4)$. 


So we have \( \sum C_j(S_0) / \sum C_j(S^*) = 35b / 30b = 7/6 \), the bound is tight.

References

