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A Class of an Implicit Stage-two Rational Runge-Kutta Method for Solution of Ordinary Differential Equations

C.E. Abhulimen[1*] and J.A. Uloko[2]

Abstract

In this paper, we derived an implicit stage – two Rational Runge – Kutta method of order Two for solution of ordinary differential equation. The stability analysis of the method shows that our method is A-Stable. The numerical results of the implementation of the scheme on some existing methods which have solved the set of problems.

Mathematics Subject Classification: 65L05, 65L20

Keywords: Implicit Rational Runge-Kutta, Order Two, ordinary differential equation, A-Stable

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[1] School of Mathematical Sciences, National Mathematics Centre, Abuja, Nigeria,
e-mail: cletusabhulimen@yahoo.co.uk
[*] Corresponding author.

[2] Department of Mathematics, Ambrose Alli University, Ekpoma, Nigeria.

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1 Introduction

We consider the numerical integration of first order initials-value problems in differential equation of the form;

$y' = f(x,y), y(x_0) = y_0\,, a \le x \le b$ (1.1)

Where, $y(x_0) = y_0$ is the initial condition, and, $a \le x \le b$ is the interval.

Traditional, Runge-Kutta methods represent important family of explicit iterative methods for approximation initial value problems in ordinary differential equations. However, in order to improve on the weak stability characteristics of these methods, a lot of authors in the past and present times have developed some numerical schemes based on implicit Rational Runge-Kutta methods for solutions of initial value problems of the type (1.1), Several of such authors includes; Lambert [14], Butcher [6][6], King [12], Lambert [15], Hong [11], Fatunla [10], Otunta and Ikhile [17], Ademiluyi [1], Ademiluyi and Babatola [2], Ademiluyi *et al* [4], Otunta and Nwachukwu [18], Butcher and Hojjati [21].

2 Derivatives of the Scheme

The ineffectiveness of some conventional Runge-Kutta scheme to solve some system of initial value problem in ordinary differential equations due to their small region of absolute stability led for the for better numerical methods for solving such initial value problem of the type (1.1).

Rational functions are quotients of polynomials which constitute a much richer class of functions. Their response to problems with singularities make them principal targets for function approximation.

The property of rational functions perhaps motivated Hong Yang Fu [12] and Ademiluyi and Babatola [5] to proposed an R-stage implicit rational R-K scheme of form;

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{2.1}$$

where,

$$K_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{R} a_{ij} K_j\right)$$

$$H_i = hg\left(x_n + d_i h, Z_n + \sum_{j=1}^{R} b_{ij} H_j\right), i = 1(1)R$$

$$g(x_n, Z_n) = -Z_{n^2} f(x_n, y_n), Z_n = \frac{1}{y_n}$$

With the constrains

$$c_i = \sum_{j=1}^{R} a_{ij}$$

$$d_j = \sum_{j=1}^{R} b_{ij}$$

$for\ a_{ij}\ and\ b_{ij} \neq 0 \quad j \geq i$

These parameters $V_i, W_i, c_i, d_i, a_{ij}\ and\ b_{ij}$ are to be determined from the system of (non-linear) equations generated by adopting the following steps recommended by Ademiluyi and Babatola [5].

i. Obtain the Taylor series expansion of $y_{n+1}, Ki's$ and $Hi's$ about point (x_n, y_n) for $i = 1(1)R$ and binomial series expansion of right side of (2.1).
ii. Insert the Taylor series expansion into (1.2)
iii. Compare the final expansion of, $Ki's$ and $Hi's$ to the Taylor series expansion y_{n+1} of about (x_n, y_n) in the powers of h.

Normally the numbers of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of the following conditions are satisfied):

i. Minimum bound of local truncation error exists. [20]
ii. The method has maximized interval of absolute stability.
iii. Minimized computer storage facilities are utilized.

However, to derive the scheme of two-stage of order two by setting R=2 in Equation (2.1) above. We then have,

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{2} W_i K_i}{1 + y_n \sum_{i=1}^{2} V_i H_i} \tag{2.2}$$

where,

$$K_i = hf\left(x_n + c_1 h, y_n + \sum_{j=1}^{2} a_{ij} K_j\right)$$

$$H_i = hg\left(x_n + d_1 h, Z_n + \sum_{j=1}^{2} b_{ij} H_j\right)$$

$$g(x_n, Z_n) = -Zn^2\, f(x_n, y_n)$$

$$Z_n = \frac{1}{y_n},$$

$$g = \frac{-f(x_n, y_n)}{y_{n^2}}$$

with constraints

$$c_i = \sum_{j=1}^{2} a_{ij}$$

$$d_i = \sum_{j=1}^{2} b_{ij}$$

$$y_{n+1} = \frac{y_n + W_1 K_1 + W_2 K_2}{1 + y_n (V_1 H_1 + V_2 H_2)} \tag{2.3}$$

where

$$K_1 = hf(x_n + c_1h, y_n + a_{11}K_1 + a_{12}K_2)$$
$$K_2 = hf(x_n + c_2h, y_n + a_{21}K_1 + a_{22}K_2)$$
$$H_1 = hg(x_n + d_1h, Z_n + b_{11}H_1 + b_{12}H_2)$$
$$H_2 = hg(x_n + d_2h, Z_n + b_{21}H_1 + b_{22}H_2)$$

With the adoption of binomial expansion on the right side of (2.2) yields.

$$y_{n+1} = y_n + W_1K_1 + W_2K_2 - {y_n}^2(V_1H_1 + V_2H_2) + (higher\ order\ terms) \quad (2.4)$$

The Taylor series expansion of y_{n+1} about y_n gives

$$y_{n+1} = y_n + h_1{y_n}^1 + \frac{h_2{y_n}^2}{2!} + \frac{h^3{y_n}^3}{3!} + (0h^4) \quad (2.5)$$

Now we find the partial derivatives of *y* as follows;

$$y'_n = f(x_n, y_n) = f_n$$
$$y''_n = f_x + f_nf_y = Df_n$$
$$y'''_n = f_{xx} + 2f_nf_{xy} + {f_n}^2f_{yy} + f_y(f_x + f_xf_y) = D^2f_n + f_yDf_n$$
$$\begin{aligned} {y_n}^{iv} &= f_{xxx} + 3{f_n}^2f_{xyy} + {f_n}^3f_{yy} + f_y(f_{xx} + 2f_nf_{xy} + {f_n}^2f_{yy}) \\ &\quad + (f_x + f_nf_y)(3f_{xy} + 3f_nf_y + {f_y}^2) \\ &= D^3f_n + f_yD^2f_n + 3Df_nDf_y + {f_y}^2Df_n \end{aligned}$$

$$D^2f_n = f_{xx} + 2f_nf_{xy} + {f_n}^2f_{yy} \quad (2.6)$$
$$Df_y = f_{xy} + f_nf_{yy} + {f_y}^2$$

Substitute (2.6) into (2.5) we obtain,

$$y_{n+1} = y_n + hf_n\frac{h^2Df_{n!}}{2!} + h^3\frac{(D^2f_n + f_yDf_n)}{3!} + 0(h^4) \quad (2.7)$$

$$K_i = hf(x + c_i h, y + a_{i1} hK_i + a_{i2} hK_i)$$

Similarly expanding $K_1$by Taylor series about point (x_n, y_n) for $i = 1,2$ we have the solution of K_i and K_2 being expressed in the form,

$$K_i = hA_i + h^2 B_i + h^3 Di + 0(h^4) \tag{2.8}$$

Equating powers of *h* we have

$$A_i = f_n, B_i = c_i f_x + (a_{i1} A_1 + a_{i2} A_2) f_y = c_i D f_n$$

$$D_i = [(a_{i1} B_1 + a_{i2} B_2) f_y + \frac{1}{2} c_i{}^2 f_{xx} + c_i (a_{i1} A_1 + a_{i2} A_2) f_{xy}$$

$$+ \frac{1}{2} (a_{i1} A_1 + a_{i2} A_2)^2 f_{yy}]$$

$$=(c_1 a_{i1} + c_2 a_{i2}) f_y D f_n + \tfrac{1}{2} c_i{}^2 D^2 f_n \qquad i = 1,2$$

Just as in (2.7), we have

$$H_i = hg(x + d_i h, y + b_{i1} hH_i + b_{i2} hH_i)$$

In the same manner, we expand H_i to obtain,

$$H_i = hN_i + h^2 M_i + h^3 R_i + 0[h^4] \tag{2.9}$$

Equating powers of *h* we have

$$N_i = g_n, M_i = d_i g_x + (b_{i1} N_1 + b_{i2} N_2) g_z$$

$$R_i = (b_{i1} M_1 + b_{i2} M_2) g_z + \frac{1}{2} d_i{}^2 g_{xx} + d_i (b_{i1} N_1 + b_{i2} N_2) g_{xz}$$

$$+ \frac{1}{2} (b_{i1} N_1 + b_{i2} N_2) g_{zz}$$

$$N_i = g_n, \; M_1 = d_i D g_n$$

$$R_i = d_i{}^2 (b_{i1} + b_{i2}) \left(g_z D g_n + \frac{1}{2} D^2 g_n \right)$$

$$= (b_{i1} D g_n + b_{i1} d_2 D g_n) g_z + \frac{1}{2} d_i{}^2 g_{xx} + d_i (b_{i1} + b_{i2}) g g_{zz}$$

$$+ \frac{1}{2} (b_{i1} + b_{i2})^2 g_{zz} \tag{2.10}$$

$$= (d_1 b_{i1} + d_2 b_{i2}) b_z D g_n + \frac{1}{2} d_i{}^2 (g_{xx} + g g_{xz} + g_{zz})$$

$$= (d_1 b_{i1} + d_2 b_{i2}) g_z D g_n + d_i{}^2 D^2 g_n \qquad i = 1,2$$

Thus, let us express g and its partial derivatives in order to facilitate the comparison of coefficients. That is,

$$g_n = \frac{-f_x}{{y_n}^2}, \qquad g_x = \frac{-fx}{{y_n}^2}, \qquad g_{xxx} = \frac{-f_{xxx}}{{y_n}^2},$$

$$g_z = \frac{-2f_x}{y_n} + f_y, \qquad g_{xz} = \frac{-2f_x}{y_n} + f_{xy}, \qquad g_{xxz} = \frac{-2f_{xx}}{y_n} + f_{xxz},$$

$$g_{zz} = -2f_n - {y_n}^2 f_{yy}, \qquad g_{xzz} = -2f_x - 2{y_n}^2 f_{xyy},$$

$$g_{zzz} = 4{y_n}^2 f_y + 6{y_n}^2 f_{xx} + {y_n}^4 f_{yyy}$$

$$Dg_n = g_x + g_n g_z \tag{2.11}$$

$$D^2 g_n = g_{xx} + 2g_n g_{xz} + {g_n}^2$$

$$Dg_z = g_{xz} + g_n g_{zz}$$

Substitution (2.11) and (2.10) into (2.9), we have

$$N_i = \frac{-f_n}{{y_n}^2}, \qquad M_i = \frac{-d_i}{{y_n}^2}\left(Df_n + \frac{2{f_n}^2}{{y_n}^2}\right) \tag{2.12}$$

$$R_i = \frac{1}{{y_n}^2}\left\{(b_{i1}d_1 + b_{i2}d_2)\left(Df_n + \frac{2{f_n}^2}{y_n}\right)(-2f_n \right.$$

$$\left. + f_y)\frac{1}{2}\left[{d_i}^2\left(D^2 f_n + 2f_n\frac{(2f_x}{y_n} + \frac{{f_n}^2)}{{y_n}^2}\right)\right]\right\} \quad i = 1,2$$

Substituting (2.8) and (2.12) into (2.3) to obtain;

$$y_{n+1} = y_n + W_1\left(hA_1 + h^2B_1 + h^3D_1 + 0(h^4)\right) + W_2\left(hA^2 + h^2B_2 + h^3D_2 + 0(h^4)\right) - {y_n}^2\left\{V_1\left(hN_1 + h^2M_1 + h^3R_1 + 0(h^4)\right) + V_2\left(hN_2 + h^2M_2 + h^3R_2 + 0(h^4)\right)\right\} \tag{2.13}$$

$$= y_n + \left(W_1A_1 + W_2A_2 - {y_n}^2(V_1N_1 + V_2N_2)\right)h + \left(W_1B_1 + W_2B_2 - {y_n}^2(V_1M_1 + V_2M_2)\right)h^2 + \left(W_1D_1 + W_2D_2 - {y_n}^2(V_1R_1 + V_2R_2)\right)h^3 + 0(h^4)$$

Comparing the coefficients of the powers of h in the equations (2.13) and (2.7) we obtained

$$W_1A_1 + W_2A_2 - {y_n}^2(V_1N_1 + V_2N_2) = f_n \qquad (2.14)$$

$$W_1 + W_2 + V_1 + V_2 = 1$$

Comparing coefficient of h^2

$$W_1B_1 + W_2B_2 - {y_n}^2(V_1M_1 + V_2M_2) = \frac{1}{2}Df_n \qquad (2.15)$$

$$W_1c_1 + W_2c_2V_1d_1 + V_2d_2 = \frac{1}{2}$$

From equation (2.14) and (2.15) we have the following set constraints

(i) $W_1 + W_2 + V_1 + V_2 = 1$

(ii) $W_1c_1 + W_2c_2 + V_1d_1 + V_2d_2 = \frac{1}{2}$

Where (2.16)

(i) $c_1 = a_{11} + a_{12}, c_2 = a_{21} + a_{22}$

(ii) $d_1 = b_{11} + b_{12}, d_2 = b_{21} + b_{22}$

with constraints taking coefficients of h and h^2 into consideration and imposing condition $T_{n+1} = 0(h^3)$

We obtain the local truncation error of the method

$$T_{n+1} = y_{n+1} - \frac{y_n + W_1K_1 + W_2K_2}{1 + y_nV_1H_1 + V_1H_2}$$

By expansion we have

$$T_{n+1} = \frac{1}{6}\left(D^2 f_n + f_y D f_n\right) - \left[\left(W_1(c_1 a_{11} + c_2 a_{12}) + W_2(c_1 a_{21} + c_2 a_{22})\right)\right]\ f_y D f_n$$
$$+\ \frac{1}{2}({c_n}^2) D^2 f_n [V_1(b_{11} d_1 + b_{12} d_2) + V_2(b_{21} d_1 + b_{22} d_2)]\ D f_n$$
$$+\frac{2{f_n}^2}{y_n}\ (-2f_n + f_y) + \frac{1}{2}({d_i}^2) D^2 f_n + 2f_n \frac{(2f_x}{y_n} + \frac{{f_n}^2)}{y_n} \quad (2.17)$$

Finally, to obtain the family of our method of order two two-stages, we have the following as our constraints;

$$W_1 + W_2 + V_1 + V_2 = 1$$
$$W_1 c_1 + W_1 c_2 + V_1 d_1 + V_2 d_2 = \frac{1}{2}$$

Solving the set of equations (2.18) above for the unknown parameters we have the 2-stage order two scheme.

$$y_{n+1} = \frac{y_n + \frac{1}{4}(K_1 + K_2)}{1 + {}^{y_n}/_4\,(H_1 + H_2)}$$

Where

$$K_1 = hf\left(x_n\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h, y_n + \frac{1}{4}K_1 + \left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)K_2\right)$$
$$K_2 = hf\left(x_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h, y_n + \left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)K_1 + \tfrac{1}{4}K_2\ \right)$$
$$H_1 = hf\left(x_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h, z_n + \left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)H_1 + \tfrac{1}{4}H_2\ \right)$$
$$H_1 = hf\left(x_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h, z_n + \frac{1}{4}H_1 + \left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)H_2\right)$$

3 Stability Analysis

Definition 3.1 A numerical scheme is said to be A-stable is the region of absolute stability includes the entire left half of the complex plane. Lambert [16].

By this definition, our numerical scheme is said to be A-stable if the region of absolute stability includes the entire half of the complex plane. That is, the corresponding region R of absolute stability of the scheme can be defined as

$$R = \{\Omega | \mu(\Omega) \leq 1\}, \;\; where\ \Omega = \lambda h$$

Now to achieve the stability function of our scheme, we introduce the Dalquist [10] stability scalar test function.

$$y' = \lambda y, \qquad y(x_0) = y_0 \tag{3.1}$$

Applying (2.18) to the test equation (3.1), we obtain system of linear equations for K_i's which when written in matrix notation yields;

$$\begin{bmatrix} 1 - \lambda h a_{11} & -a_{12}\lambda h \\ -\lambda h a_{21} & 1 - a_{22}\lambda h \end{bmatrix} \quad \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} \lambda h y_n \\ \lambda h y_n \end{bmatrix} \tag{3.2}$$

This is in compact form because

$$DK = Y$$

and

$$Y = [11]^T \lambda h y_n \tag{3.3}$$

By assuming that D^{-1} exists, equation (1.27) becomes

$$K = D^{-1}\ or\ K = (1 - A\Omega)^{-1} e y_n \tag{3.4}$$

and $Y = [11]^T \lambda h y_n$

$$\Omega = \lambda h$$

Similarly, for H_i's

and $Y = [1,1]^T \lambda h Z_n$

$$H = -1(1 + B\Omega)^{-1} e Z_n \tag{3.5}$$

Our stability function becomes;

$$\mu(\Omega) = \frac{1 + \frac{(W_1 + W_2)\Omega + \left(W_1(a_{12} - a_{22}) + W_2(a_{21} - a_{11})\right)\Omega^2}{1 - (a_{11} + a_{22})\Omega + (a_{11} * a_{22} - a_{12} * a_{21})\Omega^2}}{1 - \frac{(V_1 + V_2)\Omega + \left(V_1(b_{21} + b_{11})\right)\Omega^2}{1 + (b_{11} + b_{22})\Omega + (b_{11} * b_{22} - b_{12} *_{21})\Omega^2}} \tag{3.6}$$

If we simplify (3.6) further, the general stability function for the family of two stage schemes becomes

$$= \frac{1 - \frac{1}{12}\Omega^2}{1 - \Omega + \frac{1}{3}\Omega^2} \leq 1 \tag{3.7}$$

$$= \Omega - \frac{5}{12}\Omega^2 \leq 0$$

$$\Omega \leq 0$$

Region of Stability of our scheme (RAS)

This indicates that our scheme is A-stable with $(-\infty, 0)$ as corresponding interval of absolute stability satisfying

$$\lim_{\Omega \to 0} |(\Omega)| \leq 1$$

These large stability properties encourage us to adopt the use of the proposed scheme for solving initial value problem in ODEs.

4 Numerical Examples and Results

Math lab computer program was used in implementing the 2-stage implicit rational Runge-Kutta scheme of order two with some sample problems with shows a good performance comparing some existing ones.

Example 4.1. Consider initial value problem

$$y' = \lambda(y - x^3) + 3x^2, \quad y_{(0)} = 1 \qquad 0 \leq x \leq 1$$

The theoretical solution is

$$y_{(x)} = x^3 + e^{\lambda x}$$

Result of two stage implicit rational Ruge-Kutta of order two with $\lambda = -10$ are shown in Table 1.

Let our scheme be denoted by ABJ and also Semi implicit of Babatola [5] by BLA.

Table 1: Numerical Result of Problem 1

s/n	H	y exact	Abj	Absolute Abj error	Bla	absolute Bla error
1	1.000000e-001	3.6887944e-001	3.6886264e-001	1.6798089e-005	3.684109e-001	5.4164391e-004
2	0.500000e-001	6.0665565e-001	6.0664098e-001	1.4671388e-005	6.0655736e-001	2.6707970e-005
3	0.500000e-001	7.7881640e-001	7.7881470e-001	1.7031289e-006	7.7880182e-001	1.0359074e-006
4	1.250000e-002	8.8249885e-001	8.8249871e-001	1.4063886e-007	8.8249694e-001	3.7210967e-008
5	6.250000e-003	9.3941330e-001	9.3941329e-001	1.0057223e-008	9.3941306e-001	1.2426234e-009
6	3.125000e-003	9.6923326e-001	9.9623326e-001	6.7245409e-010	9.6923323e-001	2.3452629e-010
7	1.562500e-003	9.8449644e-001	9.8449644e-001	4.3567149e-011	9.8449644e-001	7.5418560e-012
8	7.812500e-004	9.9221793e-001	9.9221793e-001	2.7858826e-012	9.9221794e-001	1.5305313e-010
9	3.906250e-004	9.9610136e-001	9.9610136e-001	1.7785772e-013	9.9610137e-001	9.6331831e-012
10	1.953125e-004	9.9804878e-001	9.9804878e-001	1.1435297e-014	9.9804878e-001	6.0418337e-013
11	9.765625e-005	9.9902391e-001	9.9902391e-001	8.8817841e-016	9.9902391e-001	1.5495538e-010

Remark 4.1: From table 1, our proposed scheme (ABJ) compete favourably with the existing Method of BLA [5].

Example 4.2. Consider the initial value problem

$$y^x = 5e^{5x}(y - x)^2 + 1 \qquad 0 \leq x \leq 1, \; y_{(0)} = -1$$

With exact solution $x - e^{-5x}$,

Table 2: Numerical Examples/Results of Problem 2

s/n	x_n	y exact	Abj	Error	Classical irk order 4	Error
1	0.25	-3.6505e-002	-3.8384e-002	1.8795e-003	4.01435e-001	4.37936e-001
2	0.5	4.1792e-001	3.8069e-001	3.7223e-002	3.434753	3.01956e+000
3	0.75	7.2648e-001	7.5653e-001	-3.0046e-002	1.44639+023	1.44639e+023
4	1.0	9.9326e-001	1.2003e+000	-2.0703e-001	Overflow	

Remark 4.2: From Table 2 above, we observed that our method (ABJ) have better accuracy than that of classical IRK order 4.

5 Conclusion

The new scheme is A-stable and it demonstrates a better accuracy, efficiency when compared with the existing methods that have solved the set of problems.

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