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Stepanov-like Pseudo Almost Automorphic Solutions to Nonautonomous Neutral Partial Evolution Equations

Mohamed Zitane[1] **and Charaf Bensouda**[2]

Abstract

In this paper, upon making some suitable assumptions such as the Acquistapace-Terreni condition and exponential dichotomy, we obtain new existence and uniqueness theorems of pseudo almost automorphic solutions to some nonautonomous neutral partial evolution equations. Some examples are presented to illustrate the main findings.

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Keywords: Evolution family, Exponential dichotomy, Fixed point theorem, Stepanov-like pseudo almost automorphiy

[1] Université Ibn Tofaïl, Faculté des Sciences, Département de Mathématiques, Laboratoire d'An. Maths et GNC, B.P. 133, Kénitra 1400, Maroc, e-mail: zitanem@gmail.com

[2] Université Ibn Tofaïl, Faculté des Sciences, Département de Mathématiques, Laboratoire d'An. Maths et GNC, B.P. 133, Kénitra 1400, Maroc, e-mail: bensoudach@yahoo.fr

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1 Introduction

In this paper, we continue the investigation of [6, 7, 12, 13, 14]. That is, we aim to establish the existence and uniqueness theorem of pseudo almost automorphic mild solutions to nonautonomous neutral partial evolution equations with Stepanov-like pseudo almost automorphic term

$$\frac{d}{dt}[u(t)+f(t,u(t))]=A(t)[u(t)+f(t,u(t))]+h(t) \quad , t\in\mathbb{R}, \tag{1}$$

$$\frac{d}{dt}[u(t)+f(t,u(t))]=A(t)[u(t)+f(t,u(t))]+g(t,u(t)) \quad , t\in\mathbb{R}, \tag{2}$$

where $A(.)$ is a given family of closed linear operators on $D(A(t))$ satisfying the well-known Acquistapace-Terreni conditions, $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is pseudo almost automorphic, $h:\mathbb{R}\to\mathbb{X}$ and $g:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ are Stepanov-like pseudo almost automorphic for $p>1$ and continuous, where $\mathbb{X}$ is a Banach space.

Another aim in this paper is to investigate pseudo almost automorphic mild solutions to perturbed nonautonomous neutral partial evolution equation

$$\frac{d}{dt}[u(t)+f(t,Bu(t))]=A(t)[u(t)+f(t,Bu(t))]+g(t,Cu(t)) \quad , t\in\mathbb{R}, \tag{3}$$

where B, C are bounded linear operators.

In recent years, the theory of almost automorphy and its various extensions have attracted a great deal of attention due to their significance and applications in areas such as physics, mathematical biology, control theory, and others.

The concept of pseudo almost automorphy, which is the central issue in this work, was introduced in the literature by Xiao, Liang and Zhang [21] as a natural generalization of both the classical concept of almost automorphy and that of pseudo almost periodicity.

In [17], N'Guérékata and Pankov introduced the notion of Stepanov-like almost automorphic function. Using this new concept, the autors in [9, 10] stydied the composition of Stepanov-like almost automorphic functions and obtained the existence and uniqueness of almost automorphic solutions to some abstract differential equations.

Recently, Diagana [8] studied the basic properties of a new class of functions, called Stepanov-like pseudo almost automorphic functions, which generalizes both the Stepanov-like almost automorphy and pseudo almost automorphy.

We now turn to a summary of this work. In Section 2, we review the concept of almost automorphy and its generalisations such as pseudo almost automorphy and Stepanov-like almost automorphy. Section 3 and 4 are devoted to the existence and uniqueness of pseudo almost automorphic solutions to nonautonomous neutral partial evolution equations and perturbed nonautonomous neutral partial evolution equations, respectively. In Section 5, we provide some examples to illustrate our main results.

2 Preliminaries

In this section, we fix notation and collect some preliminary facts that will be used in the sequel. throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of positive integer, real and complex numbers, $(\mathbb{X}, \|.\|)$, $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ stand for Banach spaces, $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with natural topology. If $\mathbb{Y} = \mathbb{X}$ it is simply denotes by $B(\mathbb{X})$.

2.1 pseudo almost automorphy

Let $C(\mathbb{R}, \mathbb{X})$ denote the collection of continuous functions from $\mathbb{R}$ into $\mathbb{X}$. Let $BC(\mathbb{R}, \mathbb{X})$ denote the Banach space of all $\mathbb{X}$-valued bounded continuous functions equipped with the sup norm $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|$ for each $u \in BC(\mathbb{R}, \mathbb{X})$.

Similarly, $C(\mathbb{R}\times\mathbb{Y}, \mathbb{X}))$ denotes the collection of all jointly continuous functions from $\mathbb{R} \times \mathbb{Y}$ into $\mathbb{X}$, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ denotes the collection of all jointly bounded continuous functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$.

Definition 2.1. [5, 16, 25] *A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that*

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_n - s_m) = f(t) \text{ for each } t \in \mathbb{R}.$$

this limit means that

$$g(t) := \lim_{n\to\infty} f(t + s_n) \quad \text{is well defined for each } t \in \mathbb{R}$$

and

$$f(t) = \lim_{n\to\infty} g(t - s_n) \quad \textit{for each } t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{X})$*.*

Theorem 2.2. [16] *Assume* $f, g : \mathbb{R} \to \mathbb{X}$ *are almost automorphic and* λ *is any scalar. Then the following holds true:*
(1) $f + g, \lambda f, f_\tau(t) := f(t + \tau)$ *and* $\widehat{f}(t) := f(-t)$ *are almost automorphic.*
(2) *The range* R_f *of* f *is precompact, so* f *is bounded.*
(3) *If* $\{f_n\}$ *is a sequence of almost automorphic functions and* $f_n \to f$ *uniformly on* $\mathbb{R}$*, then* f *is almost automorphic.*

Definition 2.3. [12] *A function* $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ *is called bi-almost automorphic if for every sequence of real numbers* $(s'_n)_n$ *we can extract a subsequence* $(s_n)_n$ *such that* $g(t, s) := \lim_{n\to\infty} f(t + s_n, t + s_n)$ *is well defined for each* $t, s \in \mathbb{R}$ *and* $f(t, s) = \lim_{n\to\infty} g(t - s_n, t - s_n)$ *for each* $t, s \in \mathbb{R}$*. The collection of all such functions will be denoted by* $bAA(C(\mathbb{R} \times \mathbb{R}, \mathbb{X}))$*.*

We set

$$AA_0(\mathbb{X}) := \{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(\sigma)\| d\sigma = 0\}.$$

Definition 2.4. [5] *A function* $f \in BC(\mathbb{R}, \mathbb{X})$ *is said to be pseudo almost automorphic if it can be decomposed as* $f = g + \varphi$ *where* $g \in AA(\mathbb{X})$ *and* $\varphi \in AA_0(\mathbb{X})$*. The set of all such functions will be denoted by* $PAA(\mathbb{X})$*.*

Theorem 2.5. [21] *If one equips* $PAA(\mathbb{X})$ *with the sup norm, then* $PAA(\mathbb{X})$*. turns out to be a Banach space.*

Theorem 2.6. [12] *Let* $u \in PAA(\mathbb{Y}), B \in B(\mathbb{Y}, \mathbb{X})$*. If for each* $t \in \mathbb{R}, Bu(t) = v(t)$*, then* $v \in PAA(\mathbb{X})$*.*

Definition 2.7. [15] *A function* $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ *is said to be almost automorphic if* $f(t, u)$ *is almost automorphic in* $t \in \mathbb{R}$ *uniformly for all* $u \in K$*, where* K *is any bounded subset of* $\mathbb{Y}$*. The collection of all such functions will be denoted by* $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$*.*

We set

$$AA_0(\mathbb{R}\times\mathbb{Y},\mathbb{X}) := \{f\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|f(\sigma,u)\|d\sigma = 0$$
$$\text{uniformly for } u \text{ in any bounded subset of } \mathbb{Y}\}.$$

Definition 2.8. [15] *A function $f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to be pseudo almost automorphic if it can be decomposed as $f=g+\varphi$ where $g\in AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\varphi\in AA_0(\mathbb{R}\times\mathbb{Y},\mathbb{X})$. The set of all such functions will be denoted by $PAA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.*

Theorem 2.9. [8, 12] *Assume $F\in PAA(\mathbb{R}\times\mathbb{X})$. Suppose that $F(t,u)$ is Lipschitz in $u\in\mathbb{X}$ uniformly in $t\in\mathbb{R}$, in the sense that there exists $L>0$ such that*

$$\|F(t,u)-F(t,v)\|\le L\|u-v\| \quad \text{for all } t\in\mathbb{R}, u,v\in\mathbb{X}$$

If $\Phi(.)\in PAA(\mathbb{X})$, then $F(.,\Phi(.))\in PAA(\mathbb{X})$.

2.2 Stepanov-like almost automorphy

Let $L^p(\mathbb{R},\mathbb{X})$ denote the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $\|f\|\in L^p(\mathbb{R})$. Let $L^p_{loc}(\mathbb{R},\mathbb{X})$ denote the space of all classes of equivalence of measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that the restriction of every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R},\mathbb{X})$.

Definition 2.10. [8, 17, 19] *The Bochner transform $f^b(t,s), t\in\mathbb{R},\ s\in[0,1]$, of a function $f:\mathbb{R}\to\mathbb{X}$ is defined by $f^b(t,s):=f(t+s)$.*

Remark 2.11. [19] *A function $\varphi(t,s), t\in\mathbb{R}, s\in[0,1]$, is the Bochner transform of a certain function $f, \varphi(t,s)=f^b(t,s)$, if and only if $\varphi(t+\tau,s-\tau)=\varphi(s,t)$ for all $t\in\mathbb{R}, s\in[0,1]$ and $\tau\in[s-1,s]$.*

Definition 2.12. [19] *The Bochner transform $F^b(t,s,u), t\in\mathbb{R}, s\in[0,1]$, $u\in\mathbb{X}$, of a function $F:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is defined by $F^b(t,s,u):=F(t+s,u)$ for each $u\in\mathbb{X}$.*

Definition 2.13. [19] *Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X}))$. This is a Banach space with the norm*

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p d\tau\Big)^{\frac{1}{p}}$$

Definition 2.14. [17] *The space $S^pAA(\mathbb{X})$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic functions) , consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0, 1; \mathbb{X}))$. That is, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ and a function $g \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ such that*

$$\Big(\int_0^1 \|f(s + s_n) - g(s)\|^p ds\Big)^{\frac{1}{p}} \to 0 \quad \text{and} \quad \Big(\int_0^1 \|g(s - s_n) - f(s)\|^p ds\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

2.3 Stepanov-like pseudo almost automorphy

Definition 2.15. [8] *A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like pseudo almost automorphic (or S^p-pseudo almost automorphic) if it can be decomposed as $f = g + \varphi$ where $g^b \in AA(L^p(0, 1; \mathbb{X}))$ and $\varphi^b \in AA_0(L^p(0, 1; \mathbb{X}))$. The set of all such functions will be denoted by $S^pPAA(\mathbb{X})$.*

Lemma 2.16. [8] *If $f \in PAA(\mathbb{X})$, then $f \in S^pPAA(\mathbb{X})$ for each $1 \le p < \infty$. In other words, $PAA(\mathbb{X}) \subseteq S^pPAA(\mathbb{X})$.*

Lemma 2.17. [8] *The space $S^pPAA(\mathbb{X})$ equipped with the norm $\|.\|_{S^p}$ is a Banach space.*

Definition 2.18. [8] *A function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ with $F(., u) \in L^p(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be Stepanov-like pseudo almost automorphic (or S^p-pseudo almost automorphic) if it can be decomposed as $F = G + \Phi$ where $G^b \in AA(\mathbb{R} \times L^p(0, 1; \mathbb{Y}))$ and $\Phi^b \in AA_0(\mathbb{R} \times L^p(0, 1; \mathbb{Y}))$. The collection of such functions will be denoted by $S^pPAA(\mathbb{R} \times \mathbb{Y})$.*

Theorem 2.19. [8] *Assume $F \in S^pPAA(\mathbb{R} \times \mathbb{X})$. Suppose that $F(t, u)$ is Lipschitz in $u \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$, in the sense that there exists $L > 0$ such that*

$$\|F(t, u) - F(t, v)\| \leq L\|u - v\| \quad \text{for all } t \in \mathbb{R}, u, v \in \mathbb{X}$$

If $\Phi(.) \in S^pPAA(\mathbb{X})$ and $K := \overline{\{\Phi(t) : t \in \mathbb{R}\}}$ is compact in $\mathbb{X}$, then $F(., \Phi(.)) \in S^pPAA(\mathbb{X})$.

2.4 Evolution family and exponential dichotomy

Definition 2.20. [18] *A family of bounded linear operators $(U(t, s))_{t \geq s}$ on a Banach space X is called a strongly continuous evolution family if*
(1) $U(t, s) = U(t, r)U(r, s)$ and $U(t, t) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$
(2) the map $(t, s) \mapsto U(t, s)x$ is continuous for all $x \in \mathbb{X}, t \geq s$ and $t, s \in \mathbb{R}$,

Definition 2.21. [18, 11] *An evolution family $(U(t, s))_{t \geq s}$ on a Banach space X is called hyperbolic (or has exponential dichotomy) if there exist projections $P(t), t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, and constants $M > 0, \delta > 0$ such that*
(1) $U(t, s)P(s) = p(t)U(t, s)$ for $t \geq s$ and $t, s \in \mathbb{R}$
(2) the restriction $U_Q(t, s) : Q(s)\mathbb{X} \mapsto Q(t)\mathbb{X}$ of $U(t, s)$ is invertible for $t \geq s$ (and we set $U_Q(s, t) := U(t, s)^{-1}$)
(3) $\|U(t, s)P(s)\| \leq Me^{-\delta(t-s)}, \|U_Q(s, t)Q(t)\| \leq Me^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.
Here and below we set $Q := I - P$.

Definition 2.22. [18, 3] *Given an hyperbolic evolution family, we define its so-called Green's function by*

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & \text{for } t \geq s, \quad t,s \in \mathbb{R} \\ U_Q(t,s)Q(s), & \text{for } t < s, \quad t,s \in \mathbb{R} \end{cases} \tag{4}$$

3 Neutral partial evolution equations

Throughout this paper, we make the following basic assumptions

(H_1) the family of closed linear operators $A(t)$, for $t \in \mathbb{R}$, on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions; namely, there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $M_1, M_2 \geq 0$, and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0) \ni \lambda, \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{M_1}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq M_2|t-s|^\alpha |\lambda|^{-\beta} \tag{5}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |arg\lambda| \leq \theta\}$

(H_2) The evolution family $(U(t,s))_{t \geq s}$ generated by $A(t)$ has an exponential dichotomy with constants $M > 0, \delta > 0$, dichotomy projections $P(t), t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$.

(H_3) $\Gamma(t,s) \in bAA(\mathbb{R} \times \mathbb{R}, B(\mathbb{X}))$

(H_4) $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is pseudo almost automorphic and Lipschitz with respect to the second argument uniformly in the first argument in the sence that there exist K_f such that

$$\|f(t,u) - f(t,v)\| \leq K_f \|u - v\|$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$

(H_5) $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is Stepanov-like pseudo almost automorphic for $p > 1$, jointly continuous and Lipschitz with respect to the second argument uniformly in the first argument in the sence that there exist K_g such that

$$\|g(t,u) - g(t,v)\| \leq K_g \|u - v\|$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$.

Let us mention that in the case when $A(t)$ has a constant domain $D(A(t)) = D$, it is well-known [4, 20] that equation (5) can be replaced with the following

(H_1') There exist constant M_2 and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(s))R(\lambda_0, A(r))\| \leq M_2 |t-s|^{\alpha}, \quad \text{for } s, t, r \in \mathbb{R} \tag{6}$$

Let us also mention that (H_1) was introduced in the literature by Acquistapace and Terreni in [1, 2]. Among other things, from [3, Theorem 2.3]; see also [1, 23, 24]; it ensure that the family $(A(t))_t$ generates a unique strongly continuous evolution family $(U(t,s))_{t\geq s}$ on $\mathbb{X}$.

Definition 3.1. *A mild solution to Equation (1) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying*

$$u(t) = -f(t, u(t)) + U(t,s)[u(s) + f(s, u(s))] + \int_s^t U(t,\sigma)h(\sigma)d\sigma \tag{7}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Now, we state our first result.

Theorem 3.2. *Assume that $(H_1) - (H_4)$ hold and $h \in S^pAA(\mathbb{X}) \cap C(\mathbb{R}, \mathbb{X})$ for $p > 1$. if $K_f < 1$, then there exists a unique solution $u \in PAA(\mathbb{X})$ of equation (1) such that*

$$u(t) = -f(t, u(t)) + \int_{-\infty}^t U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_t^{+\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma, \quad t \in \mathbb{R} \tag{8}$$

Lemma 3.3. *Suppose that $(H_1) - (H_4)$ hold. If $h \in S^pAA(\mathbb{X}) \cap C(\mathbb{R}, \mathbb{X})$ for $p > 1$, then the operator Λ defined by*

$$(\Lambda u)(t) = \int_{-\infty}^t U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_t^{+\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma, \quad t \in \mathbb{R}$$

maps $PAA(\mathbb{X})$ into its self.

Proof of Lemma: The proof is similar to that one of [12, Theorem 3.3], so, we omit it. □

Proof of Theorem: To prove that u satisfies equation (7) for all $t \geq s$, all $s \in \mathbb{R}$, we let

$$u(s) = -f(s,u(s)) + \int_{-\infty}^{s} U(s,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{s}^{+\infty} U_Q(s,\sigma)Q(\sigma)h(\sigma)d\sigma \quad (9)$$

Multiply both sides of (9) by $U(t,s)$ for all $t \geq s$, then

$$\begin{aligned} U(t,s)u(s) &= -U(t,s)f(s,u(s)) + \int_{-\infty}^{s} U(t,\sigma)P(\sigma)h(\sigma)d\sigma \\ &- \int_{s}^{+\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &= -U(t,s)f(s,u(s)) + \int_{-\infty}^{t} U(t,\sigma)P(\sigma)h(\sigma)d\sigma \\ &- \int_{s}^{t} U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &- \int_{s}^{t} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &= -U(t,s)f(s,u(s)) + u(t) + f(t,u(t)) - \int_{s}^{t} U(t,\sigma)h(\sigma)d\sigma \end{aligned}$$

Hence u is a mild solution to equation (7).

In $PAA(\mathbb{X})$ define the operator $\Gamma : PAA(\mathbb{X}) \to C(\mathbb{R},\mathbb{X})$ by setting

$$(\Gamma u)(t) = -f(t,u(t)) + \int_{-\infty}^{t} U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma.$$

From previous assumptions one can easily see that Γu is well defined and continuous. Moreover, from Theorem (2.9) and Lemma (3.3) we deduce that $\Gamma u \in PAA(\mathbb{X})$, that is $\Gamma : PAA(\mathbb{X}) \mapsto PAA(\mathbb{X})$.

It remains to prove that Γ is a strict contraction on $PAA(\mathbb{X})$.
For $u, v \in PAA(\mathbb{X})$ we get

$$\|(\Gamma u)(t) - (\Gamma v)(t)\| = \|f(t,u(t)) - f(t,v(t))\| \leq K_f\|u(t) - v(t)\| \leq K_f\|u - v\|_\infty$$

Since $K_f < 1$, it follows that Γ is a strict contraction, and by the Banach fixed-point principle, there exists a unique mild solution to (1) which obviously is pseudo almost automorphic. □

Definition 3.4. *A mild solution to Equation (2) is a continuous function* $u : \mathbb{R} \to \mathbb{X}$ *satisfying*

$$u(t) = -f(t, u(t)) + U(t, s)[u(s) + f(s, u(s))] + \int_s^t U(t, \sigma)g(\sigma, u(\sigma))d\sigma \quad (10)$$

for all $t \geq s$ *and all* $s \in \mathbb{R}$.

Now, we state our second result.

Theorem 3.5. *Assume that* $(H_1) - (H_5)$ *hold. If* $(K_f + \frac{2MK_g}{\delta}) < 1$, *then there exists a unique mild solution* $u \in PAA(\mathbb{X})$ *of equation (2) such that*

$$\begin{aligned} u(t) = -f(t, u(t)) &+ \int_{-\infty}^t U(t, \sigma)P(\sigma)g(\sigma, u(\sigma))d\sigma \\ &- \int_t^{+\infty} U_Q(t, \sigma)Q(\sigma)g(\sigma, u(\sigma))d\sigma, \end{aligned}$$

Proof In $PAA(\mathbb{X})$ define the nonlinear operator $\Gamma : PAA(\mathbb{X}) \to C(\mathbb{R}, \mathbb{X})$ by setting

$$\begin{aligned} (\Gamma u)(t) = -f(t, u(t)) &+ \int_{-\infty}^t U(t, \sigma)P(\sigma)g(\sigma, u(\sigma))d\sigma \\ &- \int_t^{+\infty} U_Q(t, \sigma)Q(\sigma)g(\sigma, u(\sigma))d\sigma, \quad t \in \mathbb{R} \end{aligned}$$

(**i**) From previous assumptions one can easily see that Γ is well defined and continuous. Moreover, let $u(.) \in PAA(\mathbb{X}) \subset S^pPAA(\mathbb{X})$, Using (H_5) and composition theorem on Stepanov-like pseudo almost automorphic functions (Theorem (2.19)), we deduce that $g(., u(.)) \in S^pPAA(\mathbb{X})$. It is easy to check that $g(., u(.)) \in C(\mathbb{R}, \mathbb{Y})$. Applying Lemma (3.3) for $h(.) = g(., u(.))$, it follows that the operator Γ maps $PAA(\mathbb{X})$ into its self.

(**ii**) we will show that $\Gamma : PAA(\mathbb{X}) \mapsto PAA(\mathbb{X})$ has a unique fixed point

For $u, v \in PAA(\mathbb{X})$ we get

$$\begin{aligned}
\|\Gamma(u)(t) - \Gamma(v)(t)\| &\leq K_f \|u(t) - v(t)\| \\
&+ M \int_{-\infty}^{t} e^{-\delta(t-\sigma)} \|g(\sigma, u(\sigma)) - g(\sigma, v(\sigma))\| d\sigma \\
&+ M \int_{t}^{+\infty} e^{-\delta(t-\sigma)} \|g(\sigma, u(\sigma)) - g(\sigma, v(\sigma))\| d\sigma \\
&\leq K_f \|u - v\|_\infty + M K_g \int_{-\infty}^{t} e^{-\delta(t-\sigma)} \|u(\sigma) - v(\sigma)\| d\sigma \\
&+ M K_g \int_{t}^{+\infty} e^{-\delta(t-\sigma)} \|u(\sigma) - v(\sigma)\| d\sigma \\
&\leq (K_f + \frac{2MK_g}{\delta}) \|u - v\|_\infty
\end{aligned}$$

Since $K_f + \frac{2MK_g}{\delta} < 1$, it follows that Γ is a strict contraction, and by the Banach fixed-point principle, Γ has a unique fixed point in $PAA(\mathbb{X})$.
(iii) It remains to prove that u satisfies equation (10) for all $t \geq s$, all $s \in \mathbb{R}$, we let

$$u(s) = -f(s, u(s)) + \int_{-\infty}^{s} U(s, \sigma) P(\sigma) h(\sigma) d\sigma - \int_{s}^{+\infty} U_Q(s, \sigma) Q(\sigma) h(\sigma) d\sigma, \tag{11}$$

Multiply both sides of (11) by $U(t, s)$ for all $t \geq s$, then

$$\begin{aligned}
U(t, s) u(s) &= -U(t, s) f(s, u(s)) + \int_{-\infty}^{s} U(t, \sigma) P(\sigma) h(\sigma) d\sigma \\
&- \int_{s}^{+\infty} U_Q(t, \sigma) Q(\sigma) h(\sigma) d\sigma \\
&= -U(t, s) f(s, u(s)) + \int_{-\infty}^{t} U(t, \sigma) P(\sigma) h(\sigma) d\sigma \\
&- \int_{s}^{t} U(t, \sigma) P(\sigma) h(\sigma) d\sigma - \int_{t}^{+\infty} U_Q(t, \sigma) Q(\sigma) h(\sigma) d\sigma \\
&- \int_{s}^{t} U_Q(t, \sigma) Q(\sigma) h(\sigma) d\sigma \\
&= -U(t, s) f(s, u(s)) + u(t) + f(t, u(t)) - \int_{s}^{t} U(t, \sigma) h(\sigma) d\sigma
\end{aligned}$$

Hence u is a mild solution to equation (10). The proof is complete. $\square$

4 Perturbed nonautonomous neutral partial evolution equations

In this section, we discuss briefly the existence and uniqueness of a pseudo almost automorphic mild solution to perturbed nonautonomous neutral partial evolution equation (3). For this, we need the invariance theorem of Stepanov-like pseudo almost automorphic funnctions under bounded linear operators.

Theorem 4.1. [12] *Let* $u \in S^p PAA(\mathbb{Y})$, $B \in B(\mathbb{Y}, \mathbb{X})$. *If for each* $t \in \mathbb{X}, Bu(t) = v(t)$, *then* $v \in S^p PAA(\mathbb{X})$.

To discuss the existence of pseudo almost automorphic mild solution to (3) we need to set the following assumptions
(H_6) $\mathbb{Y} \hookrightarrow \mathbb{X}$ is continuously embedded, where $\mathbb{Y} := D(A(t))$ for $t \in \mathbb{R}$ (equipped with graph norm).

(H_7) $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ is pseudo almost automorphic and Lipschitz with respect to the second argument uniformly in the first argument in the sence that there exist K_f such that

$$\|f(t,u) - f(t,v)\| \leq K_f \|u - v\|_{\mathbb{Y}}$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$.

(H_8) $g : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ is Stepanov-like pseudo almost automorphic for $p > 1$, jointly continuous and Lipschitz with respect to the second argument uniformly in the first argument in the sence that there exist K_g such that

$$\|g(t,u) - g(t,v)\| \leq K_g \|u - v\|_{\mathbb{Y}}$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$.

(H_9) $B, C \in B(\mathbb{Y}, \mathbb{X})$ with $\max(\|B\|_{B(\mathbb{Y},\mathbb{X})}, \|C\|_{B(\mathbb{Y},\mathbb{X})}) = K'$.

Definition 4.2. *A mild solution to Equation (3) is a continuous function* $u : \mathbb{R} \to \mathbb{Y}$ *satisfying*

$$u(t) = -f(t, Bu(t)) + U(t,s)[u(s) + f(s, Bu(s))] + \int_s^t U(t,\sigma) g(\sigma, Cu(\sigma)) d\sigma$$

for all $t \geq s$ *and all* $s \in \mathbb{R}$.

Lemma 4.3. *Suppose that* $(H_1)-(H_3)$ *and* $(H_6)-(H_9)$ *hold. Then the operator* Ξ *defined by*

$$(\Xi u)(t) = \int_{-\infty}^{t} U(t,\sigma)P(\sigma)g(\sigma, Cu(\sigma))d\sigma - \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)g(\sigma, Cu(\sigma))d\sigma$$

maps $PAA(\mathbb{Y})$ *into its self.*

Proof Let $u(.) \in PAA(\mathbb{Y}) \subset S^pPAA(\mathbb{Y})$, from (H_9) and Theorem (4.1) it is clear that $Cu(.) \in S^pPAA(\mathbb{Y})$. Using (H_8) and composition theorem on Stepanov-like pseudo almost automorphic functions (Theorem (2.19)), we deduce that $g(., Cu(.)) \in S^pPAA(\mathbb{Y})$. It is easy to check that $g(., Cu(.)) \in C(\mathbb{R}, \mathbb{Y})$. Applying Lemma (3.3) for $h(.) = g(., Cu(.))$, it follows that the operator Ξ maps $PAA(\mathbb{Y})$ into its self. $\square$

Now, we state our third result.

Theorem 4.4. *Under Assumptions* $(H_1)-(H_3), (H_6)-(H_9)$, *if* $K'(K_f + \frac{2MK_g}{\delta}) < 1$, *then there exists a unique solution* $u \in PAA(\mathbb{Y})$ *of equation (3) such that*

$$u(t) = -f(t, Bu(t)) + \int_{-\infty}^{t} U(t,\sigma)P(\sigma)g(\sigma, Cu(\sigma))d\sigma$$
$$- \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)g(\sigma, Cu(\sigma))d\sigma$$

Proof The proof is similar to that of Theorem (3.5). So we omit it. $\square$

5 Applications

In this section we provide with two examples to illustrate our abstract results. For that, we first introduce the required background needed in the sequel.

Throughout the rest of this section, we take $\mathbb{X} := L^2([0,\pi])$ equipped with its natural topology and let A be the operator given by

$$A\psi(\xi) := \psi''(\xi) - 2\psi(\xi), \qquad \forall \xi \in [0,\pi], \psi \in D(A),$$

where

$$D(A) = \left\{ \psi \in L^2([0,\pi]) : \psi'' \in L^2([0,\pi]), \psi(0) = \psi(\pi) = 0 \right\}.$$

It is well known [12, 8, 18] that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2-2, n\in\mathbb{N}$, and corresponding normalized eigenfunctions given by

$$\psi_n(\xi) := \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

In addition to the above, the following properties hold:

(**a**) $\{\psi_n : n\in\mathbb{N}\}$ is an orthonormal basis for $\mathbb{X}$.

(**b**) for $\psi\in\mathbb{X}$, $T(t)\psi = \sum_{n=1}^{\infty} e^{-(n^2+2)}\langle\psi,\psi_n\rangle\psi_n$ and
$A\psi = -\sum_{n=1}^{\infty}(n^2+2)\langle\psi,\psi_n\rangle\psi_n$, for every $\psi\in D(A)$.

(**c**) $\|T(t)\|\leq e^{-\pi^2 t}$, for $t\geq 0$.

Now, define a family of linear operators $A(t)$ by

$$\begin{cases} D(A(t)) = D(A), & t\in\mathbb{R} \\ A(t)\psi(\xi) := (A+\cos(\frac{1}{2+\sin(t)+\sin(\pi t)}))\psi(\xi), & \forall\xi\in[0,\pi], \psi\in D(A), \end{cases}$$

Then, $A(t)$ generates an evolution family $(U(t,s))_{t\geq s}$ such that

$$U(t,s)\psi(\xi) = T(t-s)e^{\int_s^t \cos(\frac{1}{2+\sin(\sigma)+\sin(\pi\sigma)})d\sigma}\psi(\xi)$$

and

$$\|U(t,s)\| \leq e^{-(\pi^2+1)(t-s)}, t\geq s$$

It is easy to verify that $A(t)$ satisfy $(H_1)-(H_3)$ with $M=1, \delta=\pi^2+1$.

Example 5.1. Let us investigate pseudo almost automorphic mild solutions to nonautonomous evolution equation

$$\begin{cases} \frac{\partial}{\partial t}\left[u(t,x)+f(t,u(t,x))\right] = \frac{\partial^2}{\partial x^2}\left[u(t,x)+f(t,u(t,x))\right] \\ +(-2+\cos(\frac{1}{2+\sin(t)+\sin(\pi t)}))\left[u(t,x)+f(t,u(t,x))\right]+g(t,u(t,x)), \\ \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi], \\ u(t,x)=0, \text{ for } x=0,\ \pi \text{ and } t\in\mathbb{R}, \\ f(t,u(t,x))=0, \text{ for } x=0,\ \pi \text{ and } t\in\mathbb{R}. \end{cases} \tag{12}$$

where f is pseudo almost automorphic and g is Stepanov-like pseudo almost automorphic for $p>1$, jointly continuous.

The following proposition is an immediate consequence of Theorem 3.5.

Proposition 5.2. *Under assumptions* $(H_4), (H_5)$, *Eq.(12)admits a unique pseudo almost automorphic mild solution if* $L_f + \frac{2L_g}{\pi^2+1} < 1$.

Example 5.3. Let us consider the following perturbed nonautonomous evolution equation

$$\begin{cases} \frac{\partial}{\partial t}\left[u(t,x) + f(t, \frac{\partial^2}{\partial x^2}u(t,x))\right] = \frac{\partial^2}{\partial x^2}\left[u(t,x) + f(t, \frac{\partial^2}{\partial x^2}u(t,x))\right] \\ +(-2+\cos(\frac{1}{2+\sin(t)+\sin(\pi t)}))\left[u(t,x) + f(t, \frac{\partial^2}{\partial x^2}u(t,x))\right] \\ +g(t, \frac{\partial^2}{\partial x^2}u(t,x)), \text{ for } t \in \mathbb{R} \text{ and } x \in [0,\pi], \\ u(t,x) = 0, \text{ for } x = 0, \pi \text{ and } t \in \mathbb{R}, \\ f(t, \frac{\partial^2}{\partial x^2}u(t,x)) = 0, \text{ for } x = 0, \pi \text{ and } t \in \mathbb{R}. \end{cases} \tag{13}$$

Let $\mathbb{Y} := D(A(t))$ denote the space $D(A(t))$ endowed with the graph norm, then $\mathbb{Y} \hookrightarrow \mathbb{X}$ is continuously embedded.
Define the operators B, C by $B\psi = C\psi = \psi''$ with $D(B) = D(C) = D(A)$, then $B, C : \mathbb{Y} \to \mathbb{X}$ are bounded and $\|B\|_{B(\mathbb{Y},\mathbb{X})} = \|C\|_{B(\mathbb{Y},\mathbb{X})} = 1$.

The following proposition is an immediate consequence of Theorem 4.4.

Proposition 5.4. *Under assumptions* $(H_7), (H_8)$, *Eq.(13)admits a unique pseudo almost automorphic mild solution if* $L_f + \frac{2L_g}{\pi^2+1} < 1$.

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