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An Arnoldi Based Algorithm for solving coupled Sylvester matrix equations

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Abstract

In the present paper, we present a numerical method for the computation of approximate solutions for coupled Sylvester matrix equations. One advantage of the algorithm proposed is that can be used for large coupled Sylvester matrix equations and it require less storage space in implementation than existing numerical methods. Numerical examples and comparisons with other method are given to illustrate the effectiveness of the proposed method.

Mathematics Subject Classification: 65F05, 65F15
Keywords: Coupled Sylvester matrix equations, Jacobi iteration, Gauss-Seidel iteration, Matrix Krylov subspace, Global generalized minimal residual algorithm

1 Introduction

Many applications such as stability analysis of control systems and robust control [4] require the solution of the coupled Sylvester matrix equations.

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These matrix equations cab be written as

$$\begin{cases} AX + YB = C, \\ DX + YE = F, \end{cases} \tag{1}$$

where $A, D \in \mathbb{R}^{n\times n}$, $B, E \in \mathbb{R}^{p\times p}$, and C, F $\in \mathbb{R}^{n\times p}$ are given constant matrices; X, Y $\in \mathbb{R}^{n\times p}$ are the unknown matrices to be solved.

As we know, the coupled Sylvester matrix equations (1) can be written as a big linear system of equations

$$\mathcal{A}\mathcal{G} = \mathcal{L}, \tag{2}$$

where

$$\mathcal{A} = \begin{bmatrix} I_p \otimes A & B^T \otimes I_n \\ I_p \otimes D & E^T \otimes I_n \end{bmatrix} \in \mathbb{R}^{(2np)\times(2np)},$$

$$\mathcal{G} = \begin{bmatrix} x_1 & . & . & . & x_p & y_1 & . & . & . & y_p \end{bmatrix}^T \in \mathbb{R}^{2np},$$

$$\mathcal{L} = \begin{bmatrix} c_1 & . & . & . & c_p & f_1 & . & . & . & f_p \end{bmatrix}^T \in \mathbb{R}^{2np},$$

and $\otimes$ denotes the Kronecker product. Also, $x_j^T, y_j^T, c_j^T, f_j^T,\ i = 1, ..., p$ denote the jth column of the matrices X, Y, C, F, respectively. Thus the system (1) has a unique solution [5]

$$\mathcal{G} = \mathcal{A}^{-1}\mathcal{L}$$

if and only if the matrix $\mathcal{A}$ is nonsingular and the corresponding homogeneous matrix equation

$$\begin{cases} AX + YB = 0, \\ DX + YE = 0, \end{cases}$$

has a unique solution: $X = Y = 0$. Many numerical methods has been proposed for the solution of these matrix equations. When n and p are small, one can used direct methods to solve (2). Other methods are base on matrix transformations into forms for which solutions may be readily computed; examples of such forms include the Jordan canonical form, the companion form, and the Hessenberg-Schur form [1, 2, 3, 6]. However, the dimensions of the associated matrix $\mathcal{A}$ are very high when n and p are large. In this case, iterative methods [7, 9, 10, 11] are preferred. In [12], Starke and Niethammer reported an iterative method for solutions of coupled Sylvester matrix equations by using the SOR technique. In [5], F. Ding and T. Chen, proposed the gradient-based

iterative algorithms by using the gradient search principle and the hierarchical identification principle. In this paper, we focus on numerical solutions for coupled Sylvester matrix equations (1) and we present an iterative method of projection onto a matrix Krylov subspace to produce low rank approximate solutions to the coupled Sylvester matrix equations. The proposed method is based on the global Arnoldi process[8]. This new algorithm also, can be used for the general coupled matrix equations

$$\sum_{j=1}^{s} A_{ij} X_j B_{ij} = C_i, \quad i = 1, ..., s,$$

where $X_j \in \mathbb{R}^{n \times p}$ are the unknown matrices to be solved.

This paper is organized as follows. In Section 2, we give a brief description the gradient-based iterative algorithms for solving coupled matrix equations. In Section 3, we recall the global Arnoldi process with some properties and show how to extract low rank approximate solutions to the coupled Sylvester matrix equations. In Section 4 we give some numerical experiments. Finally, Section 5 summarizes the main conclusion of this paper.

2 Gradient Based Iterative Method

In this section we give a brief description of the gradient based (GB) iterative algorithm which using the gradient search principle and the hierarchical identification principle. Define two matrices

$$\mathcal{L}_1 = \begin{bmatrix} C - YB \\ F - YE \end{bmatrix}, \tag{3}$$

$$\mathcal{L}_2 = [C - AX, \ F - DX]. \tag{4}$$

Then from (1), we obtain two subsystems

$$\begin{bmatrix} A \\ D \end{bmatrix} X = \mathcal{L}_1,$$

$$Y \begin{bmatrix} B, \ E \end{bmatrix} = \mathcal{L}_2.$$

Let $X(k)$ and $Y(k)$ be the estimates or iterative solutions X and Y, associated with subsystems (3) and (4). Then using the gradient search principle [4] to (3) and (4) leads to following recursive equations:

$$X(k) = X(k-1) + \mu \begin{bmatrix} A \\ D \end{bmatrix}^T \{\mathcal{L}_1 - \begin{bmatrix} A \\ D \end{bmatrix} X(k-1)\}, \tag{5}$$

$$Y(k) = Y(k-1) + \mu\{\mathcal{L}_2 - Y(k-1) \begin{bmatrix} B, E \end{bmatrix}\} \begin{bmatrix} B, E \end{bmatrix}^T. \tag{6}$$

Here, $\mu > 0$ is the iterative step size or convergence factor to be given later. Substituting (3) into (5) and (4) into (6) gives

$$X(k) = X(k-1) + \mu \begin{bmatrix} A \\ D \end{bmatrix}^T \{\begin{bmatrix} C - YB \\ F - YE \end{bmatrix} - \begin{bmatrix} A \\ D \end{bmatrix} X(k-1)\} =$$

$$X(k-1) + \mu \begin{bmatrix} A \\ D \end{bmatrix}^T \begin{bmatrix} C - YB - AX(k-1) \\ F - YE - DX(k-1) \end{bmatrix}, \tag{7}$$

$$Y(k) = Y(k-1) + \mu\{[C - SX, F - DX] - Y(k-1)[B, E]\}[B, E]^T =$$

$$Y(k-1) + \mu[C - AX - Y(k-1)B, F - DX - Y(k-1)E][B, E]^T. \tag{8}$$

Because the expressions on the right-hand sides of (7) and (8) contain the unknown parameter matrices Y and X, it is impossible to realize the algorithm in (7) and (8). According to the hierarchical identification principle, the unknown variables Y in (7) and X in (8) are replaced by their estimates $Y(k-1)$ and $X(k-1)$ at time $k-1$. Hence, the iterative solutions $X(k)$ and $Y(k)$ for coupled Sylvester equation (1) are as follows:

$$X(k) = X(k-1) + \mu \begin{bmatrix} A \\ D \end{bmatrix}^T \begin{bmatrix} C - AX(k-1) - Y(k-1)B \\ F - DX(k-1) - Y(k-1)E \end{bmatrix}, \tag{9}$$

$$Y(k) = Y(k-1) + \mu[C - AX(k-1) - Y(k-1)B, F - DX(k-1) - Y(k-1)E][B, E]^T. \tag{10}$$

The convergence factor may be taken to satisfy

$$0 < \mu < \frac{2}{\lambda_{max}(A^T A) + \lambda_{max}(D^T D) + \lambda_{max}(BB^T) + \lambda_{max}(EE^T)}.$$

Theorem 2.1. *If the coupled Sylvester matrix equation (1) has unique solutions X and Y, then for any initial values $X(0)$ and $Y(0)$, the iterative solutions $X(k)$ and $Y(k)$ given by the algorithm (gradient based) in (9)-(10) converge to the solutions X and Y.*

Proof. See [5]. □

3 Low Rank Approximate Solutions To Eq.(1)

In this section, we present an Arnoldi based (AB) algorithm for solving the coupled Sylvester matrix equation (1). As we know, Eq.(1) has a unique solution if and only if the matrix

$$\mathcal{A} = \begin{bmatrix} I_p \otimes A & B^T \otimes I_n \\ I_p \otimes D & E^T \otimes I_n \end{bmatrix} \in \mathbb{R}^{(2np)\times(2np)},$$

is nonsingular and the corresponding homogeneous matrix equation

$$AX + YB = 0, \quad DX + YE = 0$$

has a unique solution:$X = Y = 0$. Throughout this paper, we suppose that this condition is verified. Let $\mathcal{S}$ be the operator from $\mathbb{R}^{2n\times p}$ onto $\mathbb{R}^{2n\times p}$ defined as follows:

$$\mathcal{S}(\begin{bmatrix} X \\ Y \end{bmatrix}) = \begin{bmatrix} AX + YB \\ DX + YE \end{bmatrix}.$$

Thus, by this definition, the coupled Sylvester matrix equation (1) can be expressed as following equivalent form

$$\mathcal{S}(\begin{bmatrix} X \\ Y \end{bmatrix}) = \begin{bmatrix} C \\ F \end{bmatrix}.$$

Let

$$\begin{bmatrix} V \\ U \end{bmatrix} \in \mathbb{R}^{2n\times p}$$

be a rectangular matrix, as we see in [8], the global Arnoldi process construct an F-orthonormal basis

$$\begin{bmatrix} V_1 \\ U_1 \end{bmatrix}, \begin{bmatrix} V_2 \\ U_2 \end{bmatrix}, ..., \begin{bmatrix} V_m \\ U_m \end{bmatrix}$$

of matrix Krylov subspace

$$\mathcal{K}_m(\mathcal{S}, \begin{bmatrix} V \\ U \end{bmatrix}) = \{\begin{bmatrix} V \\ U \end{bmatrix}, \mathcal{S}(\begin{bmatrix} V \\ U \end{bmatrix}), ..., \mathcal{S}^{m-1}(\begin{bmatrix} V \\ U \end{bmatrix})\},$$

i.e.

$$(\begin{bmatrix} V_i \\ U_i \end{bmatrix}, \begin{bmatrix} V_j \\ U_j \end{bmatrix})_F = \delta_{ij}, \quad i, j = 1, ..., m,$$

where δ_{ij} denotes the classical Kronecker symbol and $\mathcal{S}^i(\begin{bmatrix} V \\ U \end{bmatrix})$ is defined recursively as

$$\mathcal{S}^i(\begin{bmatrix} V \\ U \end{bmatrix}) = \mathcal{S}(\mathcal{S}^{i-1}(\begin{bmatrix} V \\ U \end{bmatrix})).$$

It is clear that, each

$$\begin{bmatrix} Z \\ W \end{bmatrix} \in \mathcal{K}_m(\mathcal{S}, \begin{bmatrix} V \\ U \end{bmatrix})$$

can be written by the following formula

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \sum_{i=0}^{m-1} \alpha_i \mathcal{S}^i(\begin{bmatrix} V \\ U \end{bmatrix}),$$

where $\alpha_i,\ i = 0, 1, ..., m-1$ are scalers. The modified global Arnoldi algorithm for constructing an F-orthogonal basis

$$\begin{bmatrix} V_1 \\ U_1 \end{bmatrix}, \begin{bmatrix} V_2 \\ U_2 \end{bmatrix}, ..., \begin{bmatrix} V_m \\ U_m \end{bmatrix}$$

of the Krylov subspace

$$\mathcal{K}_m(\mathcal{S}, \begin{bmatrix} V \\ U \end{bmatrix})$$

is as follows:

Algorithm 3.1 Modified global Arnoldi algorithm.

1- Choose a nonzero matrix $\begin{bmatrix} V \\ U \end{bmatrix} \in \mathbb{R}^{2n \times p}$. Set $\begin{bmatrix} V_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} V \\ U \end{bmatrix} / \| \begin{bmatrix} V \\ U \end{bmatrix} \|_F$.

2- For $j = 1, ..., m$ do:

$$\begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} = \mathcal{S}(\begin{bmatrix} V_j \\ U_j \end{bmatrix}),$$

for $i = 1, ..., j$, do

$$h_{i,j} = (\begin{bmatrix} V_i \\ U_i \end{bmatrix}, \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix}),$$

$$\begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} = \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} - h_{ij} \begin{bmatrix} V_i \\ U_i \end{bmatrix},$$

end for

$$h_{j+1,j} = \| \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \|_F,$$

$$\begin{bmatrix} V_{j+1} \\ U_{j+1} \end{bmatrix} = \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} / h_{j+1,j}$$

End for.

Let us collect the matrices $\begin{bmatrix} V_i \\ U_i \end{bmatrix}$ constructed by the Algorithm 3.1 in the $2n \times mp$ and $2n \times (m+1)p$ orthonormal matrices

$$\begin{bmatrix} \mathcal{V}_m \\ \mathcal{U}_m \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} V_1 \\ U_1 \end{bmatrix}, \begin{bmatrix} V_2 \\ U_2 \end{bmatrix}, ..., \begin{bmatrix} V_m \\ U_m \end{bmatrix} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathcal{V}_{m+1} \\ \mathcal{U}_{m+1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathcal{V}_m \\ \mathcal{U}_m \end{bmatrix}, \begin{bmatrix} V_{m+1} \\ U_{m+1} \end{bmatrix} \end{bmatrix}$$

and also we denote by H_m the upper $m \times m$ Hessenberg matrix whose entries are the scalers h_{ij} and the $(m+1) \times m$ matrix $\bar{H}_m$ is the same as H_m except for an additional row whose only nonzero element is $h_{m+1,m}$ in the $(m+1, m)$ position. From Algorithm 3.1 we can prove following theorem:

Theorem 3.1. *The following relations are satisfied:*

a. $\left[\mathcal{S}(\left[\begin{array}{c} V_1 \\ U_1 \end{array} \right]), ..., \mathcal{S}(\left[\begin{array}{c} V_m \\ U_m \end{array} \right]) \right] = \left[\begin{array}{c} \mathcal{V}_m \\ \mathcal{U}_m \end{array} \right] (H_m \otimes I_p) + h_{m+1,m}[0_{n\times p}, ..., 0_{n\times p}, V_{m+1}].$

b. $\left[\mathcal{S}(\left[\begin{array}{c} V_1 \\ U_1 \end{array} \right]), ..., \mathcal{S}(\left[\begin{array}{c} V_m \\ U_m \end{array} \right]) \right] = \left[\begin{array}{c} \mathcal{V}_{m+1} \\ \mathcal{U}_{m+1} \end{array} \right] (\bar{H}_m \otimes I_p).$

Proof. See [8]. □

Let $\left[\begin{array}{c} X_0 \\ Y_0 \end{array} \right]$ be an initial $2n \times p$ matrix guess to the solution $\left[\begin{array}{c} X \\ Y \end{array} \right]$ of equation (1) and $\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right] = \left[\begin{array}{c} C \\ F \end{array} \right] - \mathcal{S}(\left[\begin{array}{c} X_0 \\ Y_0 \end{array} \right])$ its associated residual. At the mth iterate of global generalized minimal residual algorithm, a correction $\left[\begin{array}{c} Z_m \\ \tilde{Z}_m \end{array} \right]$ is determined in the matrix Krylov subspace

$$\mathcal{K}_m = \mathcal{K}_m(\mathcal{S}, \left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right]) = span\{\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right], \mathcal{S}(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right]), , ..., \mathcal{S}^m(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right])\}$$

such that the new residual is F-orthogonal to $\mathcal{K}_m$, i.e.,

$$\left[\begin{array}{c} X_m \\ Y_m \end{array} \right] - \left[\begin{array}{c} X_0 \\ Y_0 \end{array} \right] = \left[\begin{array}{c} Z_m \\ \tilde{Z}_m \end{array} \right] \in \mathcal{K}_m(\mathcal{S}, \left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right]),$$

$$\left[\begin{array}{c} R_m \\ \tilde{R}_m \end{array} \right] = \left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right] - \mathcal{S}(\left[\begin{array}{c} Z_m \\ \tilde{Z}_m \end{array} \right]) \perp_F \mathcal{K}_m(\mathcal{S}, \mathcal{S}(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right])),$$

where

$$\mathcal{K}_m(\mathcal{S}, \mathcal{S}(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right])) = span\{\mathcal{S}(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right]), \mathcal{S}^2(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right]), ..., \mathcal{S}^m(\left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right])\}.$$

Therefore, $\left[\begin{array}{c} X_m \\ Y_m \end{array} \right]$ can be obtained as a solution of the following minimization problem:

$$min_{\left[\begin{array}{c} X \\ Y \end{array} \right] - \left[\begin{array}{c} X_0 \\ Y_0 \end{array} \right] \in \mathcal{K}_m(\mathcal{S}, \left[\begin{array}{c} R_0 \\ \tilde{R}_0 \end{array} \right])} \| \left[\begin{array}{c} C \\ F \end{array} \right] - \mathcal{S}(\left[\begin{array}{c} X \\ Y \end{array} \right]) \|_F .$$

Any approximate solution $\begin{bmatrix} X_m \\ Y_m \end{bmatrix}$ of the equation (1) in $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \mathcal{K}_m$ can be written as

$$\begin{bmatrix} X_m \\ Y_m \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \begin{bmatrix} \mathcal{V}_m \\ \mathcal{U}_m \end{bmatrix} * y_m,$$

where $y_m \in \mathbb{R}^m$ is the solution of the following small least squares problem

$$min_{y\in\mathbb{R}^m} \| \| \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} \|_F \, e_1 - \bar{H}_m y \|_2, \tag{11}$$

and e_1 is the first unit vector in $\mathbb{R}^{m+1}$. By consideration the QR decomposition of $\bar{H}_m$, we have: $\bar{R}_m = Q_m \bar{H}_m$, where $\bar{R}_m$ is upper triangular and Q_m is unitary. Thus, at step m, the residual R_m produced by the global generalized minimal residual algorithm for for equation (1) has the following properties [8]:

$$\begin{bmatrix} R_m \\ \tilde{R}_m \end{bmatrix} = \gamma_{m+1} \begin{bmatrix} \mathcal{V}_{m+1} \\ \mathcal{U}_{m+1} \end{bmatrix} (Q_m^T e_{m+1} \otimes I_p) \tag{12}$$

and

$$\| \begin{bmatrix} R_m \\ \tilde{R}_m \end{bmatrix} \|_F = | \, \gamma_{m+1} \, |, \tag{13}$$

where γ_{m+1} is the last component of the vector $g_m = \| \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} \|_F \, Q_m e_{m+1}$ and e_{m+1} is the last unit vector in $\mathbb{R}^{m+1}$.

Thus, the restated global generalized minimal residual algorithm for solving coupled Sylvester matrix equation (1) can be summarized as follows:

Algorithm 3.2. Arnoldi based (AB) algorithm for Eq(1).

1. Choose an initial approximate solution $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$ and a tolerance ϵ.
2. Compute $\begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} = \begin{bmatrix} C \\ F \end{bmatrix} - \mathcal{S}(\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix})$, $\beta = \| \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} \|_F$ and $\begin{bmatrix} V_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} / \beta$.
3. If $\| \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} \|_F < \epsilon$ then exit.

4. For $j = 1, ..., m$ apply Algorithm 1 to compute the F- orthonormal basis of $\mathcal{K}_m(\mathcal{S}, \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix})$.
5. Determine y_m the minimizer of the least square problem (11).
6. Compute $\begin{bmatrix} X_m \\ Y_m \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \begin{bmatrix} \mathcal{V}_m \\ \mathcal{U}_m \end{bmatrix} (y_m \otimes I_p)$.
7. Compute the residual $\begin{bmatrix} R_m \\ \tilde{R}_m \end{bmatrix}$ and $\| \begin{bmatrix} R_m \\ \tilde{R}_m \end{bmatrix} \|_F$ using relations (12) and (13).
8. If $\| \begin{bmatrix} R_m \\ \tilde{R}_m \end{bmatrix} \|_F < \epsilon$ stop; else set $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$, $\beta = \| \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} \|_F$, $\begin{bmatrix} V_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} R_0 \\ \tilde{R}_0 \end{bmatrix} / \beta$ and Go to 4.

4 Numerical Experiments

In this section, we present the results of numerical experiments. Computations were carried out using MATLAB 6.5 codes on a personal computer Pentium 3-800EB MHs. The tests were stopped as soon as $\| \begin{bmatrix} R_k \\ \tilde{R}_k \end{bmatrix} \|_F \leq 10^{-9}$, or the maximum number of iterations reached to 100000. For all the experiments, the initial guess was $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The matrix $\begin{bmatrix} C \\ F \end{bmatrix}$ is defined so that a true solution of equation (1) is the matrix of all one. For the Arnoldi based(AB) and gradient based(GB) algorithms, we used the matrices
$A = [(2^{-1}-1)I_n + diag(1, 2, ..., n) + U_n^T] + [(2^{-1}-1)I_n + diag(1, 2, ..., n) + U_n^T]^T$,
$B = [I_p + 2^{-1}U_p] + [I_p + 2^{-1}U_p]^T$,
$D = [(2^{-2}-1)I_n + diag(1, 2, ..., n) + U_n^T] + [(2^{-2}-1)I_n + diag(1, 2, ..., n) + U_n^T]^T$
and $E = [I_p + 2^{-2}U_p] + [I_p + 2^{-2}U_p]^T$
where, U_n and U_p are the $n \times n$ and $p \times p$ matrices respectively with unit entries below the diagonal and all other entries zero. The results were reported in Table 1 and Table 2 with $m = 2$,

$$\mu = \frac{2}{1 + tr(AA^T) + tr(BB^T) + tr(DD^T) + tr(EE^T)},$$

and different values of n, p.

Table 1: Number of iterations and CPU times for AB algorithm.

n	p=100	p=200	p=300	p=400	p=500
500	3(0.797)	3(1.929)	3(3.373)	3(5.282)	5(13.45)
1000	3(2.703)	5(10.51)	4(13.73)	3(14.21)	9(50.65)
1500	5(9.656)	8(34.18)	5(33.67)	4(37.87)	9(123.5)
2000	5(16.42)	7(49.37)	3(31.29)	9(146.7)	8(172.4)
2500	3(14.61)	9(98.27)	7(117.4)	7(164.7)	7(217.7)

Table 2: Number of iterations and CPU times for AB and GB algorithms.

n	Method	p=10	p=15	p=20	p=25
50	AB	2(0.011)	2(0.012)	2(0.015)	2(0.018)
	GB	23456(12.2)	10119(7.53)	5911(6.04)	3960(5.3)
100	AB	3(0.025)	3(0.031)	2(0.015)	3(0.038)
	GB	95656(149.11)	4622(90.938)	†	†

† = No solution has been obtained after 100000 iterations.

5 Conclusion

In this paper, we have presented an Arnoldi based algorithm for solving coupled Sylvester matrix equations. This algorithm have extracted from the global generalized minimal residual method. The experiments presented in this paper showed that the solution of the coupled Sylvester matrix equations can be obtained with high accuracy by using this new algorithm. Table 1 and Table 2 showed that this algorithm is more robust and more efficient than the gradient-based iterative algorithm which has been presented by F. Ding and T. Chen [5]. Also, it reduced the computer storage, time and arithmetic work required and can be used for large coupled Sylvester matrix equations.

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