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On Distribution of Goods Using Dynamic Programming Principles With Error Bounds

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Abstract

This work considers the distribution of goods from different factories to a number of ware houses (stores). Dynamic programming principle with error bounds are used under different operational policies to determine the minimum transportation cost. The minimum transportation cost of all the optimal cost of shipping goods from factories to stores was obtained as 6,300,100 Naira, and this was under the first control policy π^1. We also find that the minimum costs of the distribution of the goods with and without error bounds coincide only at infinity. We further find that at various values of β (the parameter that measures the percentage of goods loss in transit), the first policy (i.e., π^1) remains the optimal policy of all the optimal ones.

Mathematics Subject Classification: 49J21, 49K21
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1 Introduction

A distribution company plans to minimize the cost of distributing m kinds of products from m number of factories to n number of stores. The company distributes a particular product from one factory to all the stores (a single product), that is, at factory S_1, product A is distributed to all the n stores. Again, at factory S_2, product B is distributed to all the n stores, and so on. It is also expected that the goods that leave the factories will not come back to the factories(in the case of defective, damage, e.t.c, items). The company considered n number of control policies, $\mu = \{\pi^1, \pi^2, \ldots, \pi^n\}$ to determine which of them will yield the optimum control policy. They also estimated that certain percentage of the products are to reach their final destination successfully at a minimum costs. In a related literature, Mulvey and Vladimirou [7] used the stochastic programming technique of dynamic Programming in financial asset allocation problems for designing low-risk portfolios. Van Roy et al. [15] proposed the idea of using a parsimonious sufficient static in an application of approximate dynamic programming (DP) to inventory management. Powell and Van Roy [13] described an algorithm for computing parameter values to fit linear and separable concave approximations to the value function for large-scale problems in transportation and logistics. Powell and Topaloglu [12] described a more complicated variation of the algorithm that implores execution time and memory requirements. The improvement is critical for practical applications to realistic large-scale problems. Powell [11] used DP for large-scale asset management problems for both single and multiple assets. Topaloglu and Kunnumkal [14] extended an approximate DP method to optimize the distribution operations of a company manufacturing certain products at multiple production plants and shipping to different customer locations for sales. Nwozo and Nkeki [10], used DPP to considered the allocation of buses from single station to different routes in Nigeria for profit maximization. In this paper, we consider the used of DPP for allocation of buses from different stations to different routes in a transportation company in Nigeria in order to maximize expected profit. Nkeki [9] considered the use of dynamic optimization technique for the allocation of buses from different stations to different routes by a transportation company in Nigeria. The result shows that careful planning and effective allocation of the buses will enhance profitability of the operation. In this paper, we consider the distribution of goods from different factories to different stores in a production company. In the next section, we formulate the problem as a dynamic program.

2 Problem Formulation

We consider a problem where there are n states and n control policies. $S^{factory} = \{S_1^f, S_2^f, \ldots, S_m^f\}$, $S^{store} = \{S_1^s, S_2^s, \ldots, S_n^s\}$ for all $S^{factory}, S^{store} \in S$, where S is the state space. The transition matrices corresponding to the control policies $\pi^1, \pi^2, \ldots, \pi^n$ are given as

$$P(\pi^k) = P_{i,j}(\pi^k), i, j = 1, 2, \ldots, n. \tag{1}$$

The transition costs is given by $\varphi(S_t, \pi^k(S_t))$, $k = 1, 2, \ldots, n$; $t \in T$, $S_t \in S$ where T is the set of time period in the planning horizon, S_t is state variable at period t and the discount factor $\beta, (0 \leq \beta < 1)$, where β is the expected percentage loss of the products in transit. We define the function R^{π^k} as the cost corresponding to the control policies $\pi^k, k = 1, 2, \ldots, n$. The function $x_{p,t}^{\pi^k}$ represents the different kinds of products at period t under the policies $\pi^k, k = 1, 2, \ldots, n$.

2.1 One-Period Expected Cost Function

Suppose that the costs of distributing the products from factory S_i, $i = 1, 2, \ldots, m$ to store S_j, $j = 1, 2, \ldots, n$ is $\varphi_t^{i,j}$ at period t, the number of products in factories is $S_t^{f_i}$ at period t and the number of products in stores is $S_t^{s_j}$ at period t, then the obtained costs over T-horizon is

$$\sum_{t=0}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_t^{i,j}(S_t, \mu(S_t)).$$

Let $x_p, p = 1, 2, \ldots, m$ be different kinds of products to be distributed from the factories to the stores. Then,

$$\sum_{p=1}^{n} x_{p,t}^f \geq \sum_{p=1}^{m} x_{p,t}^s, x^f, x^s \in X, t \in T,$$

where $x^f_{p,t}$ is the number of products before distribution at period t and $x^s_{p,t}$ is the number of products that is already in the stores at period t.
The expected minimum cost function obtained under control policies π^k, at period t is given as follows:

$$Q_t^{\pi^k}(S_t) = E_t(\min_{x^f,x^s\in X}\sum_{t=0}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\beta^t\varphi_t^{i,j}(x_t^{\pi^k}(S_{t-1}))), k = 1, 2, \ldots, n$$

subject to:

$$\sum_{p=1}^{n} x^f_{p,t} \geq \sum_{p=1}^{m} x^s_{p,t}, x^f, x^s \in X, t \in T, x^f_{p,t}, x^s_{p,t} \geq 0, p = 1, \ldots, m,$$

where $x^f_{p,t}, x^s_{p,t} \in X$, is the set of feasible solutions of problem (1). We can express (1) above as the expected minimal cost from period t onward as an optimization over $\{x_1, x_2, \ldots, x_T\}$ condition on $S_t = s_t$ as follows:

$$Q_t^{\pi^k}(S_t) = E_t(\min_{x_t,\ldots,x_{T-1}}\{\sum_{t'=t}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\beta^{t'}\varphi_{t'}^{i,j}(x_{t'}^{\pi^k}(S_{t'-1}))|S_t = s_t\}), \qquad (2)$$

where $S_t \in S, x_t \in X, k = 1, 2, \ldots, n,$ subject to:

$$\sum_{p=1}^{n} x^f_{p,t} \geq \sum_{p=1}^{m} x^s_{p,t}, x^f, x^s \in X, t \in T, x^f_{p,t}, x^s_{p,t} \geq 0, p = 1, \ldots, m.$$

For a function $\varphi^{i,j} : S \to \Re^{n\times m}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n,$ if we accumulate the cost of the first T-stage and add to it the terminal cost

$$\varphi_T(S_T) = \sum_{i=1}^{m}\sum_{j=1}^{n}\varphi_T^{i,j}(S_T),$$

then the Equation (2) for $S_t \in S, x_t \in X$ becomes

$$Q_t^{\pi^k}(S_t) = E_t(\min_{x_t}\{\sum_{t'=t}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\beta^{t'}\varphi_{t'}^{i,j}(x_{t'}^{\pi^k}(S_{t'-1})) + \beta^T\varphi_T(S_T)|S_t = s_t\}),$$

subject to:

$$\sum_{p=1}^{n} x^f_{p,t} \geq \sum_{p=1}^{m} x^s_{p,t}, x^f, x^s \in X, t \in T, x^f_{p,t}, x^s_{p,t} \geq 0, p = 1, \ldots, m.$$

2.2 Dynamic Programming Formulation

Let S_t be the state variable at period t and S the state space, we formulate the problem as a dynamic program. The number of product i that leave the factory i to store j at period t is given by $P_{i,j}x^{f_i}_{p,t}$, where $P_{i,j}$ is the transition percentage from factory i to store j. Hence, the total expected cost of goods lost in transit is given by

$$\beta \sum_{p=1}^{m} \sum_{j=1}^{n} P_{i,j} x^{f_i}_{p,t}, t = 1, 2, \ldots, T; i = 1, 2, \ldots, m.$$

Let S^i_{t-1} be the number of products to be distributed from factory i to the stores in period $t-1$, then S^i_t is the expected number of products that will get to the stores from factory i to the stores and let α be the expected percentage of the products that is recovered from the lost ones which are expected to go to the stores at period t, then we have that

$$S^i_t = S^i_{t-1} - (1-\alpha) \sum_{p=1}^{m} \sum_{j=1}^{n} P_{i,j} x^{\pi^k}_{p,t}, \tag{3}$$

for $i = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T$, where S^i_t is the products from factory i that are successfully in the stores from factory i and S^i_{t-1} is the goods that are in factory i before distributing to the stores. Equation (3) is the transformation equation and is a random variable. We can express (3) as follows:

$$S^i_t = S^i_{t-1} - \beta \sum_{p=1}^{m} \sum_{j=1}^{n} P_{i,j} x^{\pi^k}_{p,t},$$

for $i = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T$. The optimal policy can be found by computing the value functions through the optimality equation

$$R^{\pi^k}_t(S^i_t) = \min_{x_t} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi^{i,j}_t(x^{\pi^k}_t(S^i_{t-1})) + \beta E(R^i_{t+1}(S^i_{t+1}))|S^i_t = S_t, \tag{4}$$

for $i = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T$, subject to:

$$\sum_{p=1}^{n} x^f_{p,t} \geq \sum_{p=1}^{m} x^s_{p,t}, x^f, x^s \in X, t \in T, x^f_{p,t}, x^s_{p,t} \geq 0, p = 1, \ldots, m.$$

The Equation (4) can be rewritten as follows:

$$R^{\pi^k}_t(S^i_t) = \min_{x_t} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi^{i,j}_t(x^{\pi^k}_t(S^i_{t-1})) + \beta \sum_{j=1}^{n} P_{i,j}(\pi^k)(R^i_{t-1}(S^i_t)),$$

subject to:

$$\sum_{p=1}^{n} x_{p,t}^{f} \geq \sum_{p=1}^{m} x_{p,t}^{s},\ x^f, x^s \in X,\ t \in T,\ x_{p,t}^{f}, x_{p,t}^{s} \geq 0,\ p = 1, \ldots, m.$$

Equivalently,

$$\textstyle\sum_{p=1}^{n} x_{p,t}^{f_i} = \sum_{p=1}^{m} x_{p,t}^{s_j} + \delta_t^{i,j}, x^f, x^s \in X, t \in T,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$$

where $x_{p,t}^{f}, x_{p,t}^{s} \geq 0, p = 1, \ldots, m$ and $\delta_t^{i,j}$, is the number of products that is lost in transit from factory i to store j at period t. If $\delta_t^{i,j} = 0$, it implies that all the products that left the factories get to the stores successfully without damages or lost. It can be shown that (2) is equal to (4), (see Powell[11]). We may use (2) and (4) interchangeably. We now find the best control policy, μ, that minimize our problem ,i.e, we search for

$$\Phi_t^*(S_t) = \min_{\pi^k \in \mu} \Phi_t^{\pi^k}(S_t), t = 1, 2, \ldots, T; k = 1, 2, \ldots, n; S_t = S_t^i.$$

We do that by solving the optimality equation

$$R_t^{\pi^k}(S_t) = \min_{x_t^f, x_t^s \in X} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_t^{i,j}(x_t^{\pi^k}(S_{t-1})) + \beta \sum_{j=1}^{n} P_{i,j}(\pi^k)(R_{t-1}^i(S_t)), \tag{5}$$

subject to:

$$\sum_{p=1}^{n} x_{p,t}^{f_i} = \sum_{p=1}^{m} x_{p,t}^{s_j} + \delta_t^i, x^f, x^s \in X, t \in T, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,$$

with $x_{p,t}^{f}, x_{p,t}^{s} \geq 0, p = 1, \ldots, m$.
If $\tilde{\varphi}_t^i(S_t^f, \pi^k, S_t^s) = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{\varphi}^{i,j}(S_t^f, \pi^k, S_t^s)$ is the cost of using policy π^k at state $S_t^f = i$ and moving to state $S_t^s = j$ at period t, we use as cost per stage the expected cost $\varphi^i(S^{f_i}, \pi^k, S_t^{s_j}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi^{i,j}(S_t^f, \pi^k, S_t^s)$ given by

$$\begin{aligned} \varphi_t^i(S_t^{f_i}, \pi^k, S_t^{s_j}) &= \sum_{j=1}^{n} P_{i,j}(\pi^k) \tilde{\varphi}^i(S_t^f, \pi^k, S_t^s), t \in T, \\ &= \sum_{j=1}^{n} P_{i,j}(\pi^k) \tilde{\varphi}^i(i, \pi, j), i = 1, 2, \ldots, n, \end{aligned}$$

where for simplicity, we set $m = n$. Using the mapping $\Gamma : R \to \Re^n$ and $\Gamma_\mu : R \to \Re^n$, we express

$$(\Gamma R)(S_t^i) = \min_{\pi \in \mu(S_t)} (\varphi(S_t^f, \pi_t^k, S_t^s) + \beta \sum_{j=1}^{n} P_{i,j}(\pi^k)(R(S_t^j)), i = 1, 2, \ldots, n$$

and

$$(\Gamma_\mu R)(S_t^i) = \min_{\pi\in\mu(S_t)}(\varphi(S_t^f, \pi_t^k, S_t^s) + \beta\sum_{j=1}^{n} P_{i,j}(\pi^k)(R_\mu(S_t^j)), i = 1, 2, \ldots, n$$

Theorem 2.1. (i) Let $B(S)$ be the set of all bounded real-valued functions $R : S \to \Re^n$. The mapping $\Gamma : B(S) \to B(S)$ is a contraction.
(ii) The operator Γ has a unique fixed point (given R^*).
(iii) For any R, $\Gamma^\infty R = R^*$.
(iv) For any R, if $\Gamma R \leq R$, then $\Gamma^t R \geq R^*$, $\forall t = \{1, 2, \ldots\}$.

Proof (see Powell [11]).

□

Theorem 2.2. Let the bounded optimal cost function $R : S \to \Re^n$ be n-dimensional vectors. Then R satisfies

$$R^*(S_0) = \lim_{T\to\infty}(\Gamma^T R)(S_0), \forall S_0 \in S.$$

Proof Let y be a positive integer, $S_0 \in S$ and policy $\mu = \{\pi^1, \pi^2, \ldots, \pi^n\}$, we can decompose the return

$$R^\mu(S_0) = \lim_{T\to\infty} E\{\sum_{t=0}^{T-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)\}$$

into the portion received over the first y stages and over the remaining stages.

$$R^\mu(S_0) = \lim_{T\to\infty} E\{\sum_{t=0}^{y-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)\} + \lim_{T\to\infty} E\{\sum_{t=y}^{T-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)\}.$$

But

$$|\lim_{T\to\infty} E\{\sum_{t=y}^{T-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)\}| \leq \Lambda\sum_{t=y}^{\infty}\beta^t = \frac{\beta^y\Lambda}{1-\beta}.$$

Therefore,

$$R^\mu(S_0) \leq \lim_{T\to\infty} E\{\sum_{t=0}^{y-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)\} + \frac{\beta^y\Lambda}{1-\beta}.$$

Thus, it follows that

$$\begin{aligned} R^\mu(S_0) - \frac{\beta^y\Lambda}{1-\beta} - \beta^y \inf_{S_0\in S}|G(S_0)| &\leq E[\beta^y\varphi(S_{0y}) + \sum_{t=0}^{y-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta^t G_t^{i,j}(x_{t,i,j}^{\pi^k}(S_t)] \\ &\leq R^\mu(S_0) + \frac{\beta^y\Lambda}{1-\beta} + \beta^y \inf_{S_0\in S}|G(S_0)|. \end{aligned}$$

By taking the infimum over μ, we obtain for all S_0 and y,

$$R^{\mu}(S_0)-\frac{\beta^y\Lambda}{1-\beta}-\beta^y\inf_{S_0\in S}|G(S_0)|\leq(\Gamma^yR)(S_0)\leq R^{\mu}(S_0)+\frac{\beta^y\Lambda}{1-\beta}+\beta^y\inf_{S_0\in S}|G(S_0)|.$$

and by taking the limit as $y\to\infty$, we obtain

$$R^*(S_0)\leq(\Gamma^yR)(S_0)\leq R^*(S_0).$$

Hence,

$$R^*(S_0)=\lim_{y\to\infty}(\Gamma^yR)(S_0),\forall S_0\in S.$$

□

This result shows that our optimization problem converges to a fixed point R^* in an infinite horizon. Therefore it follows that

$$R^*(S_0)=\min_{x\in X}E[\sum_{i=1}^{n}\sum_{j=1}^{n}G_t^{i,j}(x(S_0))+\beta R^*(x(S_0))],\forall S_0\in S.$$

3 Dynamic Programming Principle and Error Bounds

In this section, we consider the optimality equation with error bounds. Let

$$R^{\mu}(S_t)=\varphi(S_t,\mu(S_t))+\sum_{t=1}^{\infty}\beta^tE(\varphi(S_t,\mu(S_t))|S_0).$$

But,

$$\Lambda'\leq\|\varphi(S_t,\mu(S_t)\|\leq\Lambda''$$

and

$$\|\lim_{T\to\infty}\beta^tE\varphi(S_t,\mu(S_t))\|\leq\Lambda\sum_{t=T}^{\infty}\beta^t=\frac{\beta^T\Lambda}{1-\beta}.$$

It then follows that

$$\varphi_{\mu}+(\frac{\beta\Lambda'}{1-\beta})e\leq R^{\mu}\leq\varphi_{\mu}+(\frac{\beta\Lambda''}{1-\beta})e. \tag{6}$$

where e is the unit vector and Λ' and Λ'' are such that

$$\Lambda'=\min_{S_t}\varphi(S_t,\mu(S_t)).$$

$$\Lambda'' = \max_{S_t} \varphi(S_t, \mu(S_t)).$$

(6) can be improve upon as follows:

$$(\frac{\Lambda'}{1-\beta})e \leq \varphi_\mu + (\frac{\beta\Lambda'}{1-\beta})e \leq R^\mu \leq \varphi_\mu + (\frac{\beta\Lambda''}{1-\beta})e \leq (\frac{\Lambda''}{1-\beta})e.$$

For a vector R, we compute

$$\Gamma_\mu R = \varphi_\mu + \beta P_\mu R, \tag{7}$$

$$R^\mu = \varphi_\mu + \beta P_\mu R^\mu. \tag{8}$$

By subtracting (7) and (8), we have

$$R^\mu = \Gamma_\mu R + \beta P_\mu (R^\mu - R).$$

R^μ is the cost vector associated with the control policy μ and R is the cost per stage vector. It then follows from (6) that for

$$\eta' = \min_{S_t} (\Gamma_\mu R)(S_t) - R(S_t))$$

and

$$\eta'' = \max_{S_t} (\Gamma_\mu R)(S_t) - R(S_t))$$

we have

$$(\frac{\eta'}{1-\beta})e \leq \Gamma_\mu R - R + (\frac{\beta\eta'}{1-\beta})e \leq R^\mu - R \leq \Gamma_\mu R - R + (\frac{\beta\eta''}{1-\beta})e \leq (\frac{\eta''}{1-\beta})e.$$

Equivalently, for the vector R, we have

$$R + \frac{\psi'}{\beta}e \leq \Gamma_\mu R + \psi' e \leq R^\mu \leq \Gamma_\mu R + \psi'' e \leq R + \frac{\psi''}{\beta}e, \tag{9}$$

where $\psi' = \frac{\beta\eta'}{1-\beta}$ and $\psi'' = \frac{\beta\eta''}{1-\beta}$. (9) can be improve upon as follows. From (7), we have $\eta' = \min_{S_t}(\Gamma R)(S_t) - R(S_t))$, It follows that

$$R + \eta' e \leq \Gamma R \tag{10}$$

From (10), we obtain

$$\begin{aligned} \Gamma(\Gamma R - \eta' e) &= \min_{\pi^k} \|\varphi_t^{\pi^k} + \beta P_{\pi^k}(\Gamma R - \eta' e)\| && (11) \\ &= \min_{\pi^k} \|\varphi_t^{\pi^k} + \beta P_{\pi^k}\Gamma R\| - \beta\eta' e && (12) \\ &= \Gamma^2 R - \beta\eta' e. && (13) \end{aligned}$$

Now, combining (10) and (11), we have

$$R + (1-\beta)\beta\eta' e \leq \Gamma^2 R + \beta^2 \eta' e \leq \Gamma^3 R. \tag{14}$$

Substituting (9) into (14), we have

$$R + (1+\beta+\beta^2)\eta' e \leq \Gamma R + (\beta+\beta^2)\eta' e \leq \Gamma^2 R + \beta^2\eta' e \leq \Gamma^3 R.$$

After k steps, we have

$$R + (\sum_{i=0}^{k} \beta^i)\eta' e \leq \Gamma R + (\sum_{i=1}^{k} \beta^i)\eta' e \leq \Gamma^2 R + (\sum_{i=2}^{k} \beta^i)\eta' e \leq \cdots \leq \Gamma^{k+1} R.$$

As $k \to \infty$, we have

$$R + \frac{\beta\eta' e}{\beta(1-\beta)} \leq \Gamma R + \frac{\beta\eta' e}{1-\beta} \leq \Gamma^2 R + \frac{\beta^2\eta' e}{1-\beta} \leq \cdots \leq R^*.$$

Let $\psi_1' = \frac{\beta\eta'}{1-\beta}$, we have

$$R + \frac{\psi_1' e}{\beta} \leq \Gamma R + \psi_1' e \leq \Gamma^2 R + \beta\psi_1' e \leq R^*. \tag{15}$$

Let $R = \Gamma^{k+1} R$ in (15), we have

$$\Gamma^{k+1} R + \psi_{k+1}' e \leq R^*.$$

From (10), we have that $\min_{S_t}((\Gamma^2 R)(S_t) - (\Gamma R)(S_t)) \geq \beta\eta'$ and $\beta\psi_1' = \frac{\beta^2\eta'}{1-\beta} \leq \psi_2'$. Therefore, we have

$$\Gamma R + \psi_1' e \leq \Gamma^2 R + \psi_2' e \leq R^*.$$

It implies that

$$(\Gamma^k R)(S_t) + \psi_k' \leq (\Gamma^{k+1} R)(S_t) + \psi_{k+1}' \leq R^*(S_t).$$

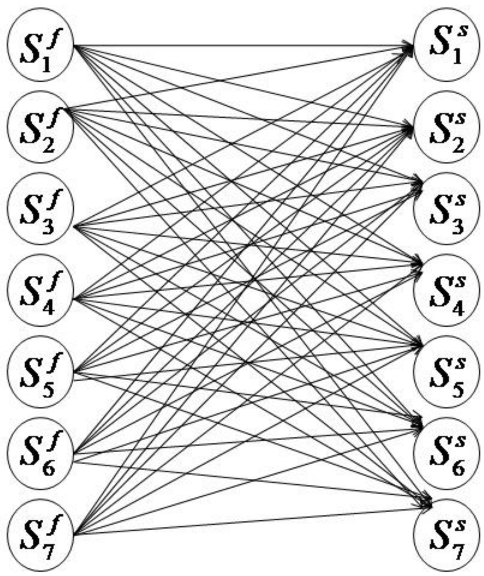
Similarly, $\eta'' = \|\Gamma R - R\|_\infty$. We therefore have that

$$\begin{aligned}(\Gamma^k R)(S_t) + \psi_k' &\leq (\Gamma^{k+1} R)(S_t) + \psi_{k+1}' \\ &\leq R^*(S_t) \leq (\Gamma^{k+1} R)(S_t) + \psi_{k+1}'' \leq (\Gamma^k R)(S_t) + \psi_k''.\end{aligned}$$

This is the dynamic programming (value iteration) that is bounded above and below. The value $(\Gamma^k R)(S_t) + \psi_k'$, is the value iteration with minimum error bound and $(\Gamma^k R)(S_t) + \psi_k''$ is the value iteration with maximum error bound.

4 Numerical Simulation

In this section, we present the numerical simulation of the company problem. The state transition diagrams of the distribution of goods from factories to stores, under the control policies $\pi^1, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \pi^7$ are given below:



The transition matrices corresponding to the control policies

$$\pi^1, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \pi^7$$

are presented below:

$$P(\pi^1) = \begin{pmatrix} \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{4} & \frac{9}{40} & \frac{1}{5} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} \end{pmatrix}, \quad P(\pi^2) = \begin{pmatrix} \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{10} & \frac{1}{5} & \frac{3}{20} & \frac{9}{40} & \frac{1}{4} \end{pmatrix},$$

$$P(\pi^3) = \begin{pmatrix} \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \\ \frac{1}{20} & \frac{1}{10} & \frac{1}{40} & \frac{1}{4} & \frac{9}{40} & \frac{3}{20} & \frac{1}{5} \end{pmatrix}, \quad P(\pi^4) = \begin{pmatrix} \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \\ \frac{1}{10} & \frac{1}{4} & \frac{9}{40} & \frac{1}{20} & \frac{1}{40} & \frac{1}{5} & \frac{3}{20} \end{pmatrix},$$

$$P(\pi^5) = \begin{pmatrix} \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \\ \frac{9}{40} & \frac{3}{20} & \frac{1}{4} & \frac{1}{40} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} \end{pmatrix}, \quad P(\pi^6) = \begin{pmatrix} \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{5} & \frac{1}{20} & \frac{9}{40} & \frac{1}{4} & \frac{1}{40} & \frac{1}{10} \end{pmatrix},$$

$$P(\pi^7) = \begin{pmatrix} \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \\ \frac{1}{5} & \frac{1}{40} & \frac{3}{20} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & \frac{9}{40} \end{pmatrix},$$

The transition cost of the goods are given in the Table 1.1.

Table 1.1: Transition Costs

r	$1^\star$	$2^\star$	$3^\star$	$4^\star$	$5^\star$	$6^\star$	$7^\star$
1	120	110	78	59	108.2	105.6	122
2	56	80	105	140	101	112.6	192
3	100	120	64	92	85.5	90.1	80.25
4	75	60	132	112.8	72.05	80.4	80
5	80.5	65	84.2	62	76.5	79.5	77
6	90	70	93	69	66.4	67.5	75.2
7	92	75	95	75.3	65.3	59	69.9

$1^\star: \varphi(S_1, \pi^r) \times 100000$ (Naira) $2^\star: \varphi(S_2, \pi^r) \times 100000$ (Naira)
$3^\star: \varphi(S_3, \pi^r) \times 100000$ (Naira) $4^\star: \varphi(S_4, \pi^r) \times 100000$ (Naira)
$5^\star: \varphi(S_5, \pi^r) \times 100000$ (Naira) $6^\star: \varphi(S_6, \pi^r) \times 100000$ (Naira)
$7^\star: \varphi(S_7, \pi^r) \times 100000$ (Naira)

The company estimates $\beta = 0.1$ of the goods in transit to be lost or damage. But in this paper, we take different values of β to demonstrate the behavior of the cost function. The cost of the damage goods are added to the cost of the shipment. MATLAB was used to solve the problem. The results are presented in the tables below.

Table 1.2: The Costs of Distributing the Goods without Error Bound

k	$\Gamma^k(R)^\star$	$\Gamma^k(R)^{\star\star}$	$\Gamma^k(R)^{\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star\star}$
0	-	-	-	-	-	-	-
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	5.6000	6.0000	6.4000	5.9000	6.5300	5.9000	6.9900
2	6.2314	6.6187	7.0267	6.5038	7.1494	6.5194	7.6094
3	6.2932	6.6808	7.0883	6.5659	7.2116	6.5816	7.6716
4	6.2994	6.6870	7.0946	6.5721	7.2178	6.5878	7.6778
5	6.3000	6.6876	7.0951	6.5727	7.2184	6.5884	7.6784
∞	6.3001	6.6877	7.0952	6.5728	7.2184	6.5884	7.6784

$^\star\Gamma^k(R)(S_1) \times 100000$ (Naira) $^{\star\star}\Gamma^k(R)(S_2) \times 100000$ (Naira)
$^{\star\star\star}\Gamma^k(R)(S_3) \times 100000$ (Naira) $^{\star\star\star\star}\Gamma^k(R)(S_4) \times 100000$ (Naira)
$^{\star\star\star\star\star}\Gamma^k(R)(S_5) \times 100000$ (Naira) $^{\star\star\star\star\star\star}\Gamma^k(R)(S_6) \times 100000$ (Naira)
$^{\star\star\star\star\star\star\star}\Gamma^k(R)(S_7) \times 100000$ (Naira)

Table 1.3: The Costs of Distributing the Goods with Minimum Error Bound

k	$\Gamma^k(R)^\star$	$\Gamma^k(R)^{\star\star}$	$\Gamma^k(R)^{\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star\star}$
0	-	-	-	-	-	-	-
1	6.2222	6.6222	6.0222	5.5222	7.1522	6.5222	7.6122
2	6.2985	6.6858	7.0938	6.5709	7.2165	6.5865	7.6765
3	6.3000	6.6877	7.0952	6.5728	7.2184	6.5884	7.6784
∞	6.3001	6.6877	7.0952	6.5728	7.2184	6.5884	7.6784

$^\star\Gamma^k(R)(S_1) + \psi' \times 100000$ (Naira) $^{\star\star}\Gamma^k(R)(S_2) + \psi' \times 100000$ (Naira)
$^{\star\star\star}\Gamma^k(R)(S_3) + \psi' \times 100000$ (Naira) $^{\star\star\star\star}\Gamma^k(R)(S_4) + \psi' \times 100000$ (Naira)
$^{\star\star\star\star\star}\Gamma^k(R)(S_5) + \psi' \times 100000$ (Naira) $^{\star\star\star\star\star\star}\Gamma^k(R)(S_6) + \psi' \times 100000$ (Naira)
$^{\star\star\star\star\star\star\star}\Gamma^k(R)(S_7) + \psi' \times 100000$ (Naira)

Table 1.4: The Costs of Distributing the Goods with Maximum Error Bound

k	$\Gamma^k(R)^{\star}$	$\Gamma^k(R)^{\star\star}$	$\Gamma^k(R)^{\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star}$	$\Gamma^k(R)^{\star\star\star\star\star\star\star}$
0	-	-	-	-	-	-	-
1	6.3767	6.7767	7.1767	6.6767	7.3067	6.6767	7.7667
2	6.3016	6.6888	7.0969	6.5739	7.2196	6.5896	7.6796
3	6.3001	6.6877	7.0953	6.5728	7.2185	6.5885	7.6784
∞	6.3001	6.6877	7.0952	6.5728	7.2184	6.5884	7.6784

$^{\star}\Gamma^k(R)(S_1) + \psi'' \times 100000$ (Naira) $^{\star\star}\Gamma^k(R)(S_2) + \psi'' \times 100000$ (Naira)
$^{\star\star\star}\Gamma^k(R)(S_3) + \psi'' \times 100000$ (Naira) $^{\star\star\star\star}\Gamma^k(R)(S_4) + \psi'' \times 100000$ (Naira)
$^{\star\star\star\star\star}\Gamma^k(R)(S_5) + \psi'' \times 100000$ (Naira) $^{\star\star\star\star\star\star}\Gamma^k(R)(S_6) + \psi'' \times 100000$ (Naira)
$^{\star\star\star\star\star\star\star}\Gamma^k(R)(S_7) + \psi'' \times 100000$ (Naira)

Table 1.1 contained the transition costs of the distributions of the goods from the factories to stores. Table 1.2 contained the minimum costs of distributing the goods from the factories to stores without error bounds. Table 1.3 contained the minimum costs of distributing the goods with minimum error bound. Table 1.4 contained the minimum costs of the distributions with maximum error bound. We found from Table 1.2 to Table 1.4, that the minimum costs with and without error bounds coincide only at infinity. Again, the result shows that the optimal cost of the distributions of the goods from factories to stores to be 6,300,100 Naira under the policy π^1, 6,687,700 Naira under policy π^2, 7,095,200 Naira under policy π^3, 6,572,800 Naira under policy π^4, 7,218,400 Naira under policy π^5, 6,588,400 Naira under policy π^6 and 7,678,400 Naira under policy π^7. We therefore recommend that if it is possible to produce the all the products in a single factory, the company should adopt the first policy in distributing their goods to stores.

From Figure 1, at $\beta = 0.05$, we observed that the costs of the distribution reduces, which is expected result, since the lower the numbers damage goods, the lower the costs of the distributions. It shows that the optimal costs to be 5,932,000 Naira under policy π^1, 6,325,700 under policy π^2, 6,729,600 under policy π^3, 6,218,200 under policy π^4, 6,856,100 under policy π^5, 6,226,100 under policy π^6 and 7,316,100 under policy π^7.
In Figure 2, we take $\beta = 0.15$, we found the optimal cost to be 6,710,700 Naira under policy π^1, 7,092,400 under policy π^2, 7,503,400 under policy π^3,

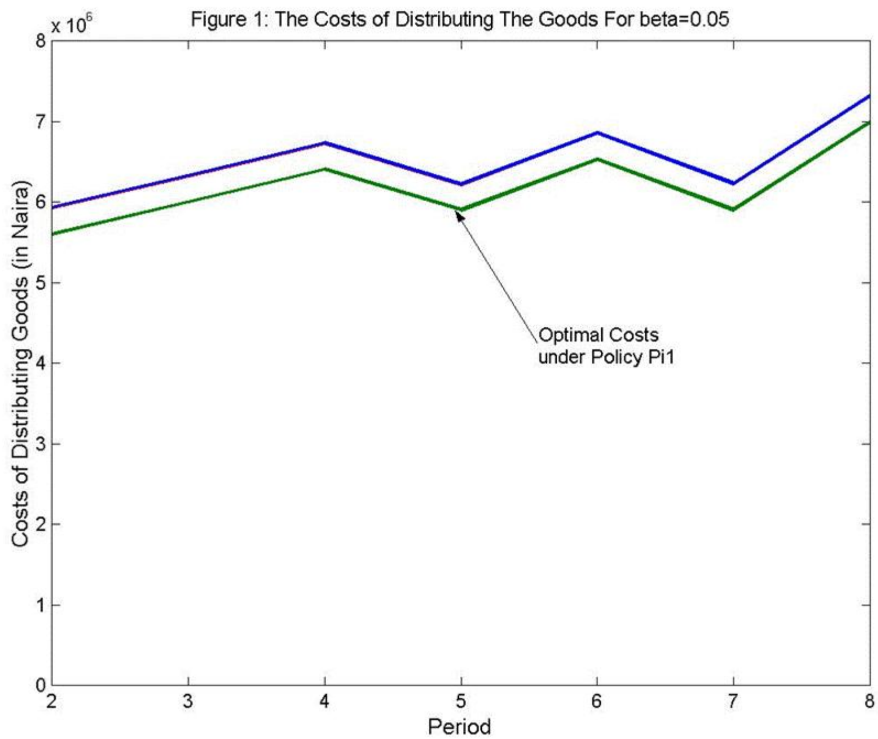


Figure 1: The Costs of Distributing The Goods For beta=0.05
x 10^6
Costs of Distributing Goods (in Naira)
Period
Optimal Costs
under Policy Pi1

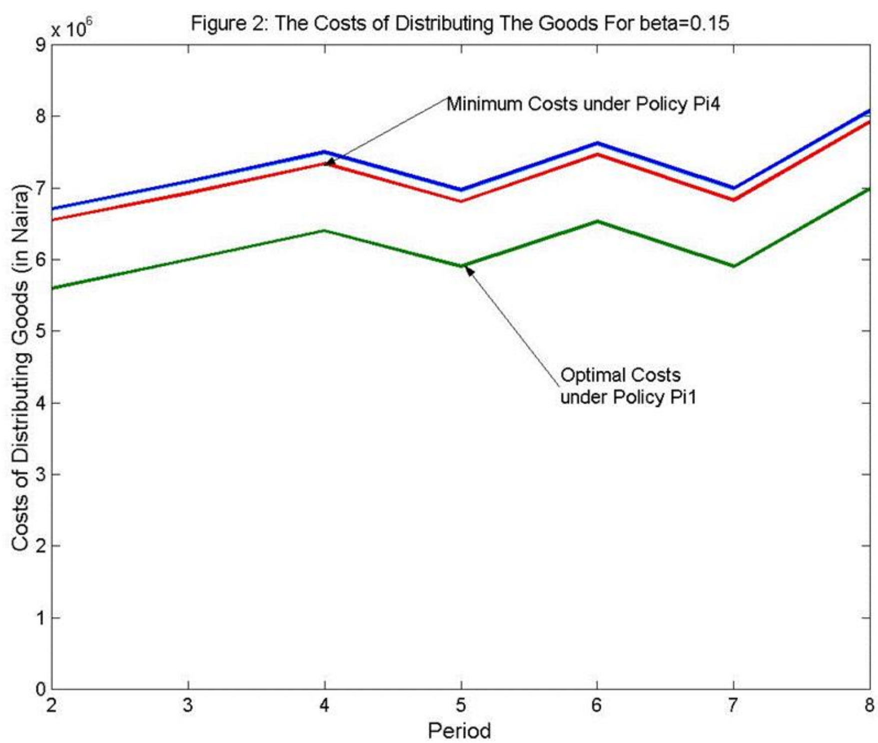


Figure 2: The Costs of Distributing The Goods For beta=0.15
x 10^6
Minimum Costs under Policy Pi4
Costs of Distributing Goods (in Naira)
Period
Optimal Costs
under Policy Pi1

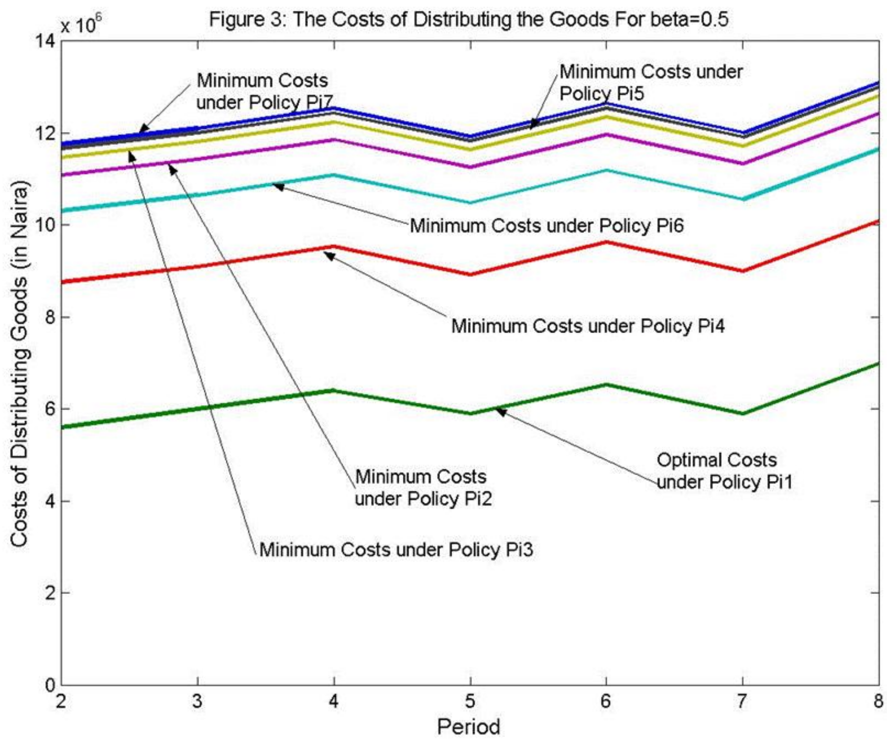


Figure 3: The Costs of Distributing the Goods For beta=0.5
x 10^6
Costs of Distributing Goods (in Naira)
Period
Minimum Costs under Policy Pi7
Minimum Costs under Policy Pi5
Minimum Costs under Policy Pi6
Minimum Costs under Policy Pi4
Optimal Costs under Policy Pi1
Minimum Costs under Policy Pi2
Minimum Costs under Policy Pi3

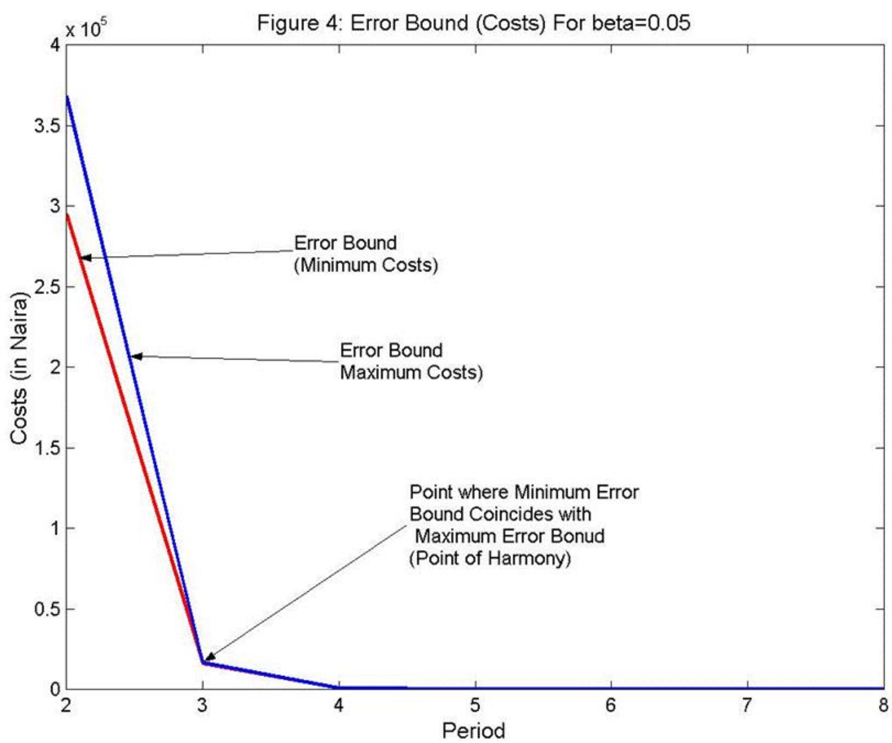


Figure 4: Error Bound (Costs) For beta=0.05
x 10^5
Costs (in Naira)
Period
Error Bound (Minimum Costs)
Error Bound Maximum Costs)
Point where Minimum Error Bound Coincides with Maximum Error Bonud (Point of Harmony)

6,970,100 under policy π^4, 7,623,500 under policy π^5, 6,993,500 under policy π^6 and 8,083,500 under policy π^7.
In Figure 3, we set $\beta = 0.5$ and the result shows that the optimal costs is 11,754,000 Naira under policy π^1, 12,100,000 under policy π^2, 12,528,000 under policy π^3, 11,925,000 under policy π^4, 12,633,000 under policy π^5, 12,003,000 under policy π^6 and 13,093,000 under policy π^7. In all, the optimal minimum costs are obtained under the first policy (i.e., π^1). We have that higher the value of β, higher the costs of distributions of the goods. Hence, for the company to spend less costs on shipment, they must ensure that their goods get to their destination successfully or with little damages.
Figure 4 shows that after four iterations, the minimum and the maximum error bounds coincided. We referred to this point as "Point of Harmony" (i.e., a point where the errors canceled out). At infinity, the the minimum and maximum error bounds vanishes.

5 Conclusion

We have that the optimal cost of shipping the goods from the factories to the stores are 6,301,000 naira under policy π^1, 6,687,700 Naira under policy π^2, 7,095,200 Naira under policy π^3, 6,572,800 Naira under policy π^4, 7,218,400 Naira under policy π^5, 6,588,400 Naira under policy π^6 and 7,678,400 Naira under policy π^7. The company is therefore advised to maintain the first policy, π^1 if it is possible to produce all the products in a single factory, since it yields the minimum transportation cost. We also found that the minimum costs of the distributions of the goods with and without error bounds coincide only at infinity. We further found that at various values of β, the first policy (i.e., π^1) remain the optimum policy of all the optimal policies.

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