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Comparison of Liao's optimal HAM and Niu's one-step optimal HAM for solving integro-differential equations

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Abstract

In this paper, the Liao's optimal homotopy analysis method is compared with the Niu's one-step optimal homotopy analysis method for solving one system of linear Volterra integro-differential equations and one integro-differential equation. The results reveal that the Liao's optimal HAM has more accuracy to determine the convergence-control parameter than the one-step optimal HAM suggested by Zhao Niu.

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Keywords: Integro-Differential Equations, Optimal HAM

1 Introduction

In 1992, Liao [5], for the first time, used the concept of homotopy to obtain analytic approximations of a nonlinear eqution

$$\mathcal{N}[u(t)] = 0, \tag{1}$$

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by constructing a *one-parameter* family of equations called the zeroth-order deformation equation,

$$(1-q)\mathcal{L}[\phi(t,q)-u_0(t)] = q\mathcal{N}[\phi(t,q)] \tag{2}$$

where $q \in [0,1]$ is an embedding parameter, $\mathcal{N}$ is a nonlinear operator, $u(t)$ is an unknown function, $u_0(t)$ is an initial guess and $\mathcal{L}$ is a linear operator. At $q=0$ and $q=1$, we have $\phi(t,0)=u_0(t)$ and $\phi(t,1)=u(t)$, respectively. So, if the Taylor series

$$\phi(t,q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m, \tag{3}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}\bigg|_{q=0}, \tag{4}$$

converges at $q=1$, we have the so-called homotopy-series solution

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t), \tag{5}$$

which must satisfy the governing equation(1). However, Liao [6, 3] found that the above approach breaks down if the Taylor series (3) diverges at $q=1$. So, to overcome this restriction, in 1997, he introduced [6] a nonzeroth auxiliary parameter c_0, to construct a *two-parameter* family of equations, i.e. the zeroth-order deformation equation:

$$(1-q)\mathcal{L}[\phi(t,q)-u_0(t)] = c_0 q\mathcal{N}(t,q) \tag{6}$$

This auxiliary parameter is also known as a convergence-control parameter [7]. Now, the homotopy-series solution (5) is not only dependent upon the embedding parameter q but also the convergence-control parameter c_0. From[3, 6, 8] we find out that the convergence-control parameter c_0, can provide us a convenient way to ensure the convergence of homotopy series solution and also adjust and control it's convergence region. A simple way of selecting c_0 is to plot the curve of homotopy-series solution's derivatives ("c_0-curves") with respect to c_0 in some points [3, 9, 1]. So, if the homotopy-series solution is unique, all of them converge to the same value and hence there exists a horizontal line segment in it's figure that corresponds to a region of c_0 called the valid region of c_0. It is a pity that the so-called "c_0-curves" approch can not give the optimal value of c_0, to obtain the fastest convergent series. so, during these years, some approches suggested in order to determine the optimal value of c_0. In 2008, Marinca et al. [11, 12] introduced the so-called "homotopy asymptotic method" which is similar to the homotopy analysis method. This approach is based on a homotopy equation

$$[1-q]\mathcal{L}[\phi(t,q)+g(t)] = H(t,q)\mathcal{N}[\phi(t,q)], \tag{7}$$

where

$$H(t,q) = qc_1 + q^2c_2 + ... + q^m c_m(t), \tag{8}$$

is a nonzero auxiliary function for $q \neq 0$,and $H(0,t) = 0$. $c_i, i = 0..m-1$ are auxiliary constants and $c_m(t)$ is a function of t and $u(t)$ is an unknown function. Choosing the control-parameter $c_m(t)$ depends on the given problem. for example, in [10, 12] he used $H(q) = qc_1 + q^2c_2 + q^3c_3$, and in [11] he assumed that $H(q,t) = qc_1 + q^2(c_2 + c_3e^{-2t})$. In Marinca's approach, at the Mth order of approximation, a set of nonlinear algebric equations about c_1, c_2, ...,c_m must be solved in order to find their optimal values and this leads to have a time-concuming approach. However, too many unknowns parameters greatly increase the cpu times and thus make the approach time-consuming. In 2009, to overcome this disadvantage, Zhao Niu [15] and Shijun Liao [4] suggested two new approaches. In this paper, we apply these approaches to solve one system of linear Volterra integro-differential equations and one integro-differential equation, and then, compare the obtained solutions.

2 Preliminary Notes

2.1 A one-step optimal HAM

Consider the equation (1),

$$\mathcal{N}[u(t)] = 0,$$

Niu et al. [15] employed the Liao's [8] zeroth-order deformation equation

$$(1-B(q))\mathcal{L}[\phi(t,q) - u_0(t)] = c_0 A(q)\mathcal{N}[\phi(t,q)] \tag{9}$$

where $A(q)$ and $B(q)$ are the deformation functions satisfying

$$A(0) = B(q) = 0, \quad A(1) = B(1) = 1.$$

The Taylor series of $A(q)$ and $B(q)$ read

$$A(q) = \sum_{m=1}^{+\infty} \mu_m q^m, \quad B(q) = \sum_{m=1}^{+\infty} \sigma_m q^m,$$

which are convergent for $|q| \leq$. He set $H(q) = c_0 A(q)$ and $B(q) = q$ in this (8) equation to construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t,q) - u_0(t)] = H(q)\mathcal{N}[\phi(t,q)] \tag{10}$$

where $\mathcal{L}$ is an auxiliary linear operator, $q \in [0,1]$ is the embedding parameter, $u_0(t)$ an initial approximation of $u(t)$, and $H(q)$ is the convergence-control function satisfying $H(0) = 0$ and $H(0) \neq 0$. Like Liao [6]-[8], defining the vector $\overrightarrow{u_m} = \{u_0(t), u_1(t), ..., u_m(t)\}$, differentiating the equation (10) m times with respect to the embedding parameter q, then divide it by $m!$ and finally set $q = 0$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = \sum_{k=1}^{m} c_k R_{m-k}(t) \tag{11}$$

where

$$R_n(t) = \frac{1}{n!}\frac{\partial^n N[\phi(t,q)]}{\partial q^n} \tag{12}$$

and

$$\chi_m = \begin{cases} 0, & \text{if} \quad m \leq 1, \\ 1, & \text{if} \quad m > 1. \end{cases} \tag{13}$$

Let $\Delta_n(c_n)$ denote the square residual error of the governing equation (1) and express as

$$\Delta_n(c_n) = \int_\Omega \left(\mathcal{N}[\tilde{u}_n(t)]\right)^2 d\Omega, \tag{14}$$

where

$$\overrightarrow{c_m} = c_1, c_2, ..., c_m,$$

and

$$\tilde{u}_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t).$$

At the 1st-order of approximation, Δ_1 is only dependent upon c_1, so, the optimal value of c_1 is obtain by solving the nonlinear algebric equation

$$\frac{d\Delta_1}{dc_1} = 0.$$

At the 2nd-order, since c_1 is known, the square residual error Δ_2 is only dependent upon c_2, thus we can gain the optimal value of c_2 by solving the nonlinear algebric equation

$$\frac{d\Delta_2}{dc_2} = 0.$$

and so on.

2.2 Liao's optimal HAM

This approch is based on the zeroth-order deformation equation (9) and used the *one-parameter* deformation functions

$$A_1(q, c_1) = \sum_{m=1}^{+\infty} \mu_m(c_1) q^m, \quad B_1(q, c_2) = \sum_{m=1}^{+\infty} \sigma_m(c_2) q^m,$$

where $|c_1| < 1$ and $|c_2| < 1$ are constants, and

$$\begin{aligned} \mu_1 &= (1 - c_1), \quad \mu_m = (1 - c_1) c_1^{m-1}, \quad m > 1, \\ \sigma_1 &= (1 - c_2), \quad \sigma_m = (1 - c_2) c_2^{m-1}, \quad m > 1. \end{aligned} \tag{15}$$

So, the new zeroth-order deformation equation is

$$[1 - B_1(q, c_2)]\mathcal{L}[\phi(t, q) - u_0(t)] = c_0 A_1(q, c_1)\mathcal{N}[\phi(t, q)], \quad q \in [0, 1]. \tag{16}$$

Note that c_1 and c_2 are the convergence-control parameters. Like the previous approach, differentiating the equation (16) m times with respect to the embedding parameter q, then dividing it by $m!$ and finally setting $q = 0$, we have the mth-order deformation equation

$$\mathcal{L}\left[u_m(t) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_k(t)\right] = c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) \delta_k(t).$$

Liao [4] found out that, when we use more than one unknown parameters, the cpu time increases exponentially so that the square residual error Δ_n (14) is often inefficient in parctice. So, to decrease the cpu time,he introduced the so-called average residual error

$$E_m = \frac{1}{K} \sum_{j=0}^{K} \left[\mathcal{N} \left(\sum_{k=0}^{m} u_k(j\Delta x) \right) \right]^2, \tag{17}$$

where $\Delta x = \frac{1}{K}$. Now we solve some examples and then we will compare the obtained results.

3 Applications

Example 3.1. Consider the following system of linear Voltra integro-differential equations [16]

$$\begin{cases} u_1' = & 1 + t + t^2 - u_2(t) - \int_0^t (u_1(s) + u_2(s))ds, \quad u_1(0) = 1, \\ u_2' = & -1 - t + u_1(t) - \int_0^t (u_1(s) - u_2(s))ds, \qquad u_2(0) = -1, \end{cases} \tag{18}$$

with exact solutions $u_1(t) = t + exp(t)$ and $u_2(t) = t - exp(t)$. We assume that the solution of above system can be expressed by a set of base function $\{exp(nt)|n \geq 0\}$ in the form

$$u(t) = \sum_{i=0}^{+\infty} a_i t^i \tag{19}$$

where a_i are coefficients to be determined. We choose auxiliary linear operator

$$\mathcal{L}\left[\phi_i(t,q)\right] = \frac{\partial \phi_i(t,q)}{\partial t},$$

with the property

$$\mathcal{L}[b_i] = 0,$$

where b_i are constant. From (18) we define the nonlinear operators

$$\mathcal{N}_1[\phi_1(t,q)] = \frac{\partial \phi_1(t,q)}{\partial t} - (1 + t + t^2) + \phi_2(t,q) + \int_0^t (\phi_1(s,q) + \phi_2(s,q))ds$$

$$\mathcal{N}_2[\phi_2(t,q)] = \frac{\partial \phi_2(t,q)}{\partial t} + (1 + t) - \phi_1(t,q) + \int_0^t (\phi_1(s,q) - \phi_2(s,q))ds$$

according to the boundary conditions (18) and Eq. (19), the initial approximations should be in the form $u_{1,0}(t) = exp(t), u_{2,0}(t) = -exp(t)$. From (12) and (18), we have

$$\mathcal{R}_{1,n}(t) = u'_{1,n-1} - (1-\chi_n)(1+t+t^2) + u_{2,n-1} + \int_0^t (u_{1,n-1} + u_{2,n-1})ds$$
$$\mathcal{R}_{2,n}(t) = u'_{2,n-1} + (1-\chi_n)(1+t) - u_{1,n-1} + \int_0^t (u_{1,n-1} - u_{2,n-1})ds$$

where the prime denotes differentiation with respect to the t. Now, the solution of the mth order deformation equation (11) becomes

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) + \int c_k \mathcal{R}_{i,m-k}(t)dt + b_i, \quad m \geq 1, i = 1, 2,$$

where the constants bi are determined by the initial conditions

$$u_{1,m}(0) = 0, u_{2,m}(0) = 0.$$

So, the first several approximations can be obtained as follows :

$$\tilde{u}_{1,1}(t) = exp(t) - c_1(t + 0.5t^2 + 0.3333t^3),$$
$$\tilde{u}_{2,1}(t) = -exp(t) - c_1(t - 0.5t^2),$$
$$\tilde{u}_{1,2}(t) = exp(t) + 0.912522t + 0.208289t^2 + 0.221517t^3 - 0.008266t^5 - c_2(t + 0.5t^2 + 0.3333t^3),$$
$$\tilde{u}_{1,2}(t) = -exp(t) + 0.912522t - 0.208289t^2 + 0.082667t^3 - 0.008266t^5 - c_2(t - 0.5t^2).$$

It is found that

$$\Delta_{1,1}(t) = \tfrac{37}{10} + \tfrac{3287}{315}c_1 + \tfrac{336059}{45360}c_1^2,$$
$$\Delta_{2,1}(t) = \tfrac{1}{3} + \tfrac{23}{90}c_1 + \tfrac{4687}{9072}c_1^2,$$
$$\Delta_{1,2}(t) = 0.104904 - 1.617464c_2 + 7.408708c_2^2,$$
$$\Delta_{2,2}(t) = 0.078324 - 0.225431c_2 + 0.516645c_2^2.$$

and so on.

At the 1th-order of approximation, in order to determine the optimal value of c_1, each of the equations in (18) is solved separately. So, the obtained values and corresponding square residual errors are,

$$c_1 = -0.704233, \quad \Delta_{1,1}(t) = 0.0256897, \quad \Delta_{2,1}(t) = 0.409589,$$

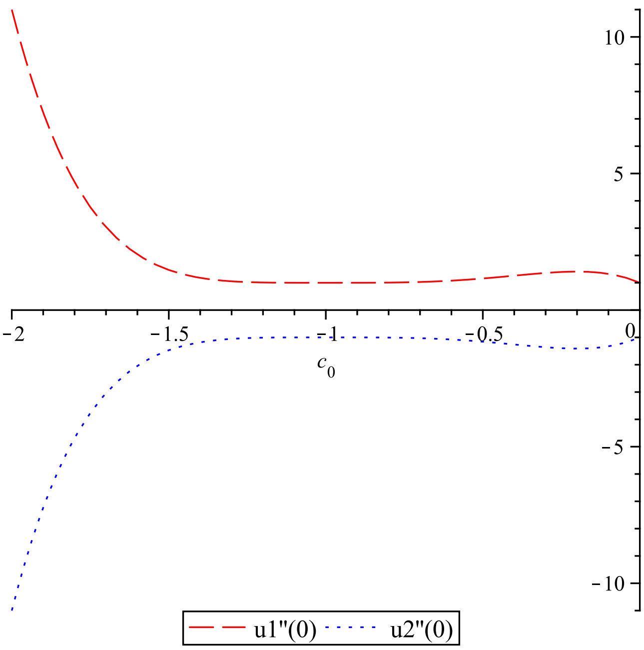

Figure 1: c_0-curve of 4th-order approximation for Example 3.1

for the first equation, and

$$c_1 = -0.247322, \quad \Delta_{1,2}(t) = 1.572389, \quad \Delta_{2,2}(t) = 0.301731,$$

for the second one. So, the minimum of the $\Delta_{1,2}$ and $\Delta_{2,2}$ is correspond to the optimal value of c_1. Thus, $c_1 = -0.704233$ is chosen. This procedure lead to the best approximate solution of the system. The 4th-order of approximate solution is obtained as follows

$$\begin{aligned}\tilde{u}_{1,4}(t) =& exp(t) + 0.907434t + 0.082059t^2 + 0.958009e - 1t^3 - 0.104494e - 1t^4 - \\ & 0.103389e - 1t^5 + 0.6966315e - 3t^6 + 0.394245e - 3t^7 - 0.271121e - 5t^9, \\ \tilde{u}_{2,4}(t) =& - exp(t) + 0.907434t - 0.082059t^2 + 0.4109446e - 1t^3 + 0.104494e - 1t^4 - \\ & 0.615912e - 2t^5 - 0.6966315e - 3t^6 + 0.195207e - 3t^7 - 0.271121e - 5t^9,\end{aligned}$$

Now, we apply the Liao's optimal HAM. First consider Figure 1, the c_0-curve of the 4th-order approximation of Example 3.1. It is obvious that the best region of c_0 is $-1.5 \leq c_0 \leq -0.5$. Note that in this approach we have at most three unknown convergence-control parameter c_0, c_1, c_2. Now we compare the different cases of these parameters.

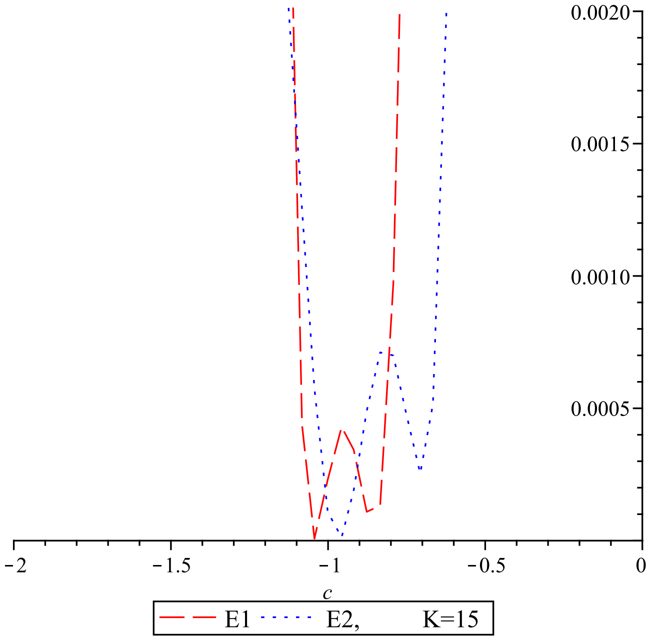

Figure 2: Residual of 4th-order approximation of Example 3.1

Optimal c_0 in case of $c_1 = c_2 = 0$:

This case, is the traditional HAM which was used by Liao (5). In case of $c_1 = c_2 = 0$ there is only one unknown convergence-control parameter c_0, thus, the optimal value of c_0 is found by the minimum of E_4, with $K = 15$, as shown in Table 2. Figure 2 is shown the residual of the 4th-order approximation of this system of equations. The 4th-order of approximate solution through this approach is obtained as follows

$$\tilde{u}_{1,4}(t) = exp(t) + 0.999994t + 0.227256e - 3t^2 + 0.234468e - 2t^3 - 0.705523e - 2t^4 - 0.985020e - 2t^5 + 0.470348e - 3t^6 + 0.782842e - 3t^7 - 0.900634e - 5t^9,$$

$$\tilde{u}_{2,4}(t) = - exp(t) + 0.999994t - 0.227256e - 3t^2 + 0.2193178e - 2t^3 + 0.104494e - 1t^4 - 0.702811e - 2t^5 - 0.470348e - 3t^6 + 0.648456e - 3t^7 - 0.900634e - 5t^9, .$$

Tables 3, 4 shows the fourth approximations of the solutions of system (18) via Liao's optimal approach in two ways and it's comparison with the one-step optimal HAM, HVIM [16] and the exact solutions of (18).

Table 1: Square residual error for Example 3.1

order M	c_n	Δ_1	Δ_2	cpu(s)
1	-0.7042	$0.2569e-1$	0.4096	0.094
2	0.1092	$0.1663e-1$	$0.5987e-1$	0.094
3	$0.3667e-1$	$0.5691e-2$	$0.2925e-1$	0.094
4	$0.2266e-1$	$0.3091e-2$	$0.1343e-1$	1.172

Table 2: Average residual error for Example 3.1

order M	c_0	E_1	E_2	cpu(s)
1	-0.7018	$0.2925e-1$	0.4357	0.125
2	-0.9180	0.1346	$0.2308e-3$	0.141
3	-0.9326	$0.1217e-3$	$0.5986e-2$	0.296
4	-0.9326	$0.5008e-3$	$0.9106e-6$	1.420

The first way of calculation of c_0 is exactly the same as we determine c_1 in one-step optimal approach. But here, we introduce a good way to determine c_0. In this way, at each order of approximation, we solve $E_1 - E_2 = 0$, then, put the solutions in both E_1 and E_2. One that gives less residual, is the optimal one. The approximate solutions are denote by "$*$" in Tables 3, 4.

This way, gives better approximations of the solutions than the first way as shown in Tables 3, 4. Besides, through these Tables we find that the one-step optimal HAM is not suitable approach to determine convergence-control parameters for system of equations.

Optimal $c_1 = c_2$ *in case of* $c_0 = -1$ *:*

Here, we assumed that $c_0 = -1$ and investigate the optimal value of $c_1 = c_2$. In this case, at the 4th-order of approximation, we obtained $E_1 = 0.621164e-3$, $E_2 = 0.574729e-2$ in 3.869s which is not better than the corresponding order's residuals in Table 2. Using more than one convergence-control parameter is not suggested for solving system of equations, because, it becomes more and more complicated to determine the optimal values.

Table 3: Comparison of u_1 given by OHAM, one-step optimal, HVIM, exact for Example 3.1

t_i	$u_{1,4}$OHAM	$u_{1,4}$one step optimal	$u_{1,4}$HVIM	$u_{1,4}$exact	$u^*_{1,4}$OHAM*
0	1	1	1	1	1
0.2	1.421415	1.406918	1.421400	1.421402	1.421403
0.4	1.891730	1.873689	1.891734	1.891824	1.891793
0.6	2.421067	2.414699	2.421423	2.422119	2.421754
0.8	3.021051	3.045654	3.022583	3.025541	3.023801
1	3.705187	3.783876	3.709226	3.718281	3.712749

Table 4: Comparison of u_2 given by OHAM, one-step optimal, HVIM, exact for Example 3.1

t_i	$u_{2,4}$OHAM	$u_{2,4}$one step	$u_{2,4}$HVIM	$u_{2,4}$exact	$u^*_{2,4}$OHAM*
0	-1	-1	-1	-1	-1
0.2	-1.021386	-1.042854	-1.021400	-1.021402	-1.021407
0.4	-1.091615	-1.139148	-1.0917347	-1.091825	-1.091948
0.6	-1.221366	-1.297474	-1.221423	-1.222119	-1.222963
0.8	-1.423969	-1.528951	-1.422583	-1.425541	-1.428827
1	-1.716126	-1.848026	-1.709226	-1.718281	-1.727631

Example 3.2. Consider the nonlinear integro-differential equation [2]

$$u^{'}(t) = -1 + \int_0^t u^2(s)ds, \tag{20}$$

for $t \in [0, 1]$ with the boundary condition $u(0) = 0$.

Consider high-order deformation equation (11) subject to the $u_m(0) = 0$, where

$$\mathcal{R}_n(t) = u_n^{'} + (1 - \chi_{n+1}) - \int_0^t \sum_{j=0}^{n} u_j(s)u_{n-j}(s)ds.$$

The initial approximation should be in the form $u_0(t) = -t$.

So, the 2th-order of approximate solution via one-step optimal HAM is obtained as follows

$$\tilde{u}_2(t) = -\,t + 0.083234t^4 - 0.003699t^7.$$

and

$$\tilde{u}_2(t) = -\,t + 0.083039t^4 - 0.003548t^7,$$

is the 2th-order of approximation which is obtained by OHAM. The comparison between residuals via two approaches is shown in Table 5.

Table 5: Comparison between residuals, via two approches for Example 3.2

order M	c_n	Δ_n	cpu(s)	c_0	E	cpu(s)
1	-0.945677	$0.447006e-5$	0.062	-0.942067	$0.588916e-5$	0.032
2	$0.571303e-3$	$0.718753e-8$	0.062	-0.965568	$0.182630e-8$	0.109
3	$0.260778e-4$	$0.219308e-10$	0.062	-0.971777	$0.247549e-12$	0.281
4	0.147745	$0.872873e-13$	0.062	-0.973225	$0.671736e-16$	1.404

Optimal c_0 in case of $c_1 = c_2 = 0$:

The optimal values of c_0 are shown as Table 5. It is obvious that, in this example, the optimal HAM when $c_1 = c_2 = 0$ has more accuracy than the one-step HAM.
Other cases of these convergence-control parameters is not suggested, because, the obtained residuals are not better than this case (Optimal c_0 in case of $c_1 = c_2 = 0$). For example, in case of $c_0 = -1, c_1 = 0.002572, c_2 = 0$ we have $E = 0.349442e-7$.

4 Conclusion

In this article, first, we have described Liao's optimal analysis method and Niu's one-step optimal analysis method, then we have applied these methods for solving one system of linear Volterra integro-differential equations and one integro-differential equation. In order to illustrate the differences between these methods, we solve two examples. The results compared show that, in both examples, liao's approach gives better approximations than the Niu's approach. In example1, we have introduced a simple way to determine the optimal value of convergence-control parameter in system of equations.

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