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Numerical Solution of Two-Dimensional Volterra Integral Equations by Spectral Galerkin Method

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Abstract

In this paper, we present ultraspherical spectral discontinuous Galerkin method for solving the two-dimensional volterra integral equation (VIE) of the second kind. The Gauss-Legendre quadrature rule is used to approximate the integral operator and the inner product based on the ultraspherical weights is implemented in the weak formulation. Illustrative examples are provided to demonstrate the preciseness and effectiveness of the proposed technique. Moreover, a comparison is made with another numerical approach that is proposed recently for solving two-dimensional VIEs.

Mathematics Subject Classification : 65R20
Keywords: two-dimensional integral equations, spectral discontinuous galerkin, ultraspherical polynomials

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1 Introduction

Integral equations are a well-known mathematical tool for representing physical problems. Historically, they have achieved great popularity among the mathematicians and physicists in formulating boundary value problems of gravitation, electrostatics, fluid dynamics and scatering. It is also well known that initial-value and boundary-value problems for differential equations can often be converted into integral equations and there are usually significant advantages to be gained from making use of this conversion.

In this paper, we consider a general class of nonlinear two-dimensional volterra integral equations (VIE) of the second kind as follows

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{x} K(x,t,y,z,u(y,z))dydz, \qquad (x,t) \in D, \tag{1}$$

where $u(x,t)$ is an unknown real valued function, while $f(x,t)$ (source function) and $K(x,t,y,z,u)$ (kernel function) are the given analytical functions defined on $D = [-1,1] \times [-1,1]$ and

$$E = \{(x,t,y,z,u) :\ -1 \leq y \leq x \leq 1,\ -1 \leq z \leq t \leq 1,\ -\infty \leq u \leq \infty\},$$

respectively.

Trivially, any two-dimensional VIE of the second kind can be transformed into (1) by a set of simple linear transformations. For this purpose, suppose that the following equation is given:

$$\begin{aligned} &\hat{u}(\hat{x},\hat{t}) = g(\hat{x},\hat{t}) + \int_{0}^{\hat{t}}\int_{0}^{\hat{x}} R(\hat{x},\hat{t},\hat{y},\hat{z},\hat{u}(\hat{y},\hat{z}))d\hat{y}d\hat{z}, \\ &\text{with} \quad (\hat{x},\hat{t}) \in \hat{D} = [0,X] \times [0,T] \end{aligned} \tag{2}$$

Now, we use the change of variables $\hat{t} = \frac{T}{2}(1+t)$ and $\hat{x} = \frac{X}{2}(1+x)$ (as done in [11]) to rewrite the VIE (2) as follows

$$u(x,t) = f(x,t) + \int_{0}^{\frac{T}{2}(1+t)}\int_{0}^{\frac{X}{2}(1+x)} R(\frac{X}{2}(1+x), \frac{T}{2}(1+t), \hat{y}, \hat{z}, \hat{u}(\hat{y},\hat{z}))d\hat{y}d\hat{z}, \tag{3}$$

where $t \in [-1,1]$, $x \in [-1,1]$, $u(x,t) = \hat{u}(\frac{X}{2}(1+x), \frac{T}{2}(1+t))$ and $f(x,t) = g(\frac{X}{2}(1+x), \frac{T}{2}(1+t))$. Moreover, to transform the integral intervals $[0, \frac{T}{2}(1+t)]$ and $[0, \frac{X}{2}(1+x)]$ into the intervals $[-1,t]$ and $[-1,x]$, respectively, we use again the two linear transformations $\hat{y} = \frac{X}{2}(1+y)$ and $\hat{z} = \frac{T}{2}(1+z)$, where $y \in [-1, x]$ and $z \in [-1, t]$. Then Eq. (3) will be changed into the Eq. (1), while

$$K(x,t,y,z,u(y,z)) = \frac{T}{2}.\frac{X}{2}R(\frac{X}{2}(1+x), \frac{T}{2}(1+t), \frac{X}{2}(1+y), \frac{T}{2}(1+z), u(y,z)).$$

During the last two decades significant progress has been made in numerical treatment of one-dimensional version of equation (1) (see [11, 4, 5] and references therein). However the analysis of computational methods for several-dimensional integral equations has started more recently, modification of the existing methods and development of new techniques should yet be explored to obtain accurate solutions successfully.

Among the finite difference methods (FDMs), one can point out to the method that explained in [3]. In this paper, the authors has been proposed a class of explicit Runge-Kutta type methods which have the convergence of order 3 for solving equation (1). Also, for solving the above-mentioned equation, bivariate cubic spline functions of full continuity was suggested by Singh [10]. In addition, Brunner and Kauthen [6] developed collocation and iterated collocation techniques for two-dimensional linear VIEs. Moreover, Gouqiang and Hayami [8] introduced the extrapolation idea of iterated collocation solution for two-dimensional nonlinear VIEs. The iterated Galerkin approach was also applied in [9] for solving the linear form of (1).

Recently, Babolian et al. [1] have considered the use of a basis of Haar functions for the numerical solution of a special class of nonlinear two-dimensional VIEs and FIEs (fredholm integral equations). Also, in [2] Banifatemi et al. have introduced two-dimensional Legendre wavelets method for the numerical treatment of nonlinear mixed two-dimensional VFIEs.

Yet so far, to the authors knowledge, spectral discontinuous-Galerkin methods for the nonlinear two-dimensional VIEs (1) have had few results. In this paper, we investigate the ultraspherical spectral discontinuous-Galerkin method for solving Eq. (1). The method consists of expanding the solution in terms of two-dimensional ultraspherical polynomials (as considered test functions) with unknown coefficients. The properties of ultraspherical polynomials and spectral Galerkin method are then utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1). Note that, in our proposed method, we finally solve a nonlinear system of equations in terms of unknown coefficients. The most important reason of spectral Galerkin consideration is that the spectral (Galerkin or collocation) methods provide highly accurate approximations to the solution of operator equations in function spaces, provided that these solutions are sufficiently smooth [7]. Throughout this paper, we assume that the following conditions are satisfied:

(**i**) Eq. (1) has an unique solution $u(x,t) \in C^r(D)$ for a given $r \in N$;

(**ii**) Functions $K(x,t,y,z,u(y,z))$ and $f(x,t)$ are smooth enough.

The outline of this paper is as follows. Section 2 introduces some elementary properties of the ultraspherical polynomials. In Section 3, we propose the ultraspherical spectral Galerkin method for solving two-dimensional VIE (1) in theoretical aspect. In Section 4, using the latter section, we apply the above-mentioned method for solving Eq. (1) numerically. Section 5 presents several illustrative examples which confirms the performance and efficiency of the proposed method. Note that in this section, at the first three examples, we make a comparison between our solutions and another method [1] that presented recently. Finally, Section 6 includes some concluding remarks.

2 Preliminaries

2.1 Properties of ultraspherical polynomials

Jacobi polynomials for which $\alpha = \beta$ are called ultraspherical polynomials and are denoted simply by $P_k^{(\alpha)}(x)$ [7]. They are related to the legendre polynomials via

$$L_k(x) = P_k^{(0)}(x) \tag{4}$$

and to the first kind chebyshev polynomials via

$$T_k(x) = \frac{2^{2k}(k!)^2}{(2k)!} P_k^{(-\frac{1}{2})}(x). \tag{5}$$

Let $w(x) = (1-x^2)^\alpha$ be a weight function in the usual sense, for $\alpha > -1$. As defined in [7] the set of ultraspherical polynomials $\{P_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_w(-1,1)$-orthogonal system, where $L^2_w(-1,1)$ is a weighted space defined by

$$L^2_w(-1,1) = \{v : v \text{ is measurable and } \|v\|_w < \infty\}$$

equipped with the following norm

$$\|v\|_w = (\int_{-1}^{1} |v(x)|^2 w(x) dx)^{\frac{1}{2}}$$

and the inner product is

$$(u,v)_w = \int_{-1}^{1} u(x)v(x)w(x)dx, \quad \forall\, u,\, v \in L^2_w(-1,1).$$

2.2 Tensor-product expansion

The most natural way to build a two-dimensional expansion, with exploiting all the one-dimensional features, is to take tensor products of one-dimensional expansions. Note that the resulting functions are defined on the cartesian product of intervals [7].

Now given the ultraspherical polynomials $\{P^{(\alpha)}_n(x)\}_{n=0}^{\infty}$ as a basis functions on interval $[-1,1]$. A two-dimensional ultraspherical expansion is produced by the tensor product choice

$$\phi_{ij}(x,t) = P^{(\alpha)}_i(x)P^{(\alpha)}_j(t), \quad i,j = 0,1,\ldots,N$$

Orthogonality of one-dimensional family with respect to the weight function $w(x) = (1-x^2)^{\alpha}$ implies orthogonality of the family $\{\phi_{ij}(x,t)\}_{i,j=0}^{\infty}$ with respect to the weight function $W(x,t) = (1-x^2)^{\alpha}(1-t^2)^{\alpha}$. We denote by $L^2_W(D)$, the space of the measurable functions on $D = [-1,1]\times[-1,1]$, that are square integrable, i.e.,

$$L^2_W(D) = \{\phi : \phi \text{ is measurable and } \parallel \phi \parallel_W < \infty, \}$$

where

$$\parallel \phi \parallel_W = (\int_D |\phi(x,t)|^2 W(x,t) dD)^{\frac{1}{2}},$$

and the inner product is

$$(\varphi,\psi)_W = \int_D \varphi(x,t)\psi(x,t)W(x,t)dD \quad \forall\, \varphi,\, \psi \in L^2_W(D).$$

Then the tensor product of the ultraspherical polynomials form an orthogonal basis for $L^2_W(D)$, [7].

3 Ultraspherical spectral galerkin method

In this section we formulate ultraspherical spectral galerkin method for Eq. (1). Introducing the Uryson integral operator defined by

$$(Ku)(x,t) = \int_{-1}^{t}\int_{-1}^{x} K(x,t,y,z,u(y,z))dydz.$$

The Eq. (1) will be changed into the following operator form

$$u = f + Ku. \tag{6}$$

Then, our goal is to find $u = u(x,t)$ such that

$$u(x,t) = f(x,t) + (Ku)(x,t), \quad \forall\ (x,t) \in D, \tag{7}$$

and the weak form is to find $u \in L^2_W(D)$ such that

$$(u,v)_W = (f,v)_W + (Ku,v)_W, \quad \forall\ v \in L^2_W(D). \tag{8}$$

Let $\mathbf{P}_N = \mathbf{P}_N(D)$ be the space of all algebraic polynomials of degree up to the N in terms of variables x and t. Our ultraspherical spectral galerkin approximation of (8) is now defined as finding $u^N \in \mathbf{P}_N$ such that

$$(u^N,v)_W = (f,v)_W + (Ku^N,v)_W, \quad \forall\ v \in \mathbf{P}_N. \tag{9}$$

If P_N denotes the orthogonal projection operator from $L^2_W(D)$ upon $\mathbf{P}_N$, then (9) can be equivalently rewritten : to find $u^N \in \mathbf{P}_N$ such that

$$u^N = P_N f + P_N K u^N.$$

4 Implementation of the ultraspherical discontinuous-galerkin method

Consider the two-dimensional ultraspherical polynomials

$$\phi_{ij}(x,t) = P_i^{(\alpha)}(x)P_j^{(\alpha)}(t), \quad i,j = 0,1,\ldots,N$$

as the basis functions of $\mathbf{P}_N$. Then the approximate solution can be denoted by

$$u(x,t) \approx u^N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij} P_i^{(\alpha)}(x) P_j^{(\alpha)}(t). \tag{10}$$

By substituting (10) in (9) and taking

$$v(x,t) = \phi_{rs}(x,t) = P_r^{(\alpha)}(x) P_s^{(\alpha)}(t), \quad r,s = 0,1,\ldots,N,$$

one can get a system of simultaneous equations for the unknown parameters $\{a_{ij}\}_{i,j=0}^{N}$.

For the linear case of the Eq. (1)

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{x} K(x,t,y,z)u(y,z)dydz, \ \ (x,t) \in D, \tag{11}$$

using orthogonal properties of two-dimensional ultraspherical polynomials

$$\phi_{rs}(x,t) = P_r^{(\alpha)}(x) P_s^{(\alpha)}(t)$$

on the interval $[-1,1] \times [-1,1]$, we obtain the system of linear equations as follows

$$\begin{aligned} c_{rs}a_{rs} = &\int_{-1}^{1}\int_{-1}^{1} W(x,t)f(x,t)\phi_{rs}(x,t)dxdt + \\ &+ \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij} \int_{-1}^{1}\int_{-1}^{1} W(x,t)\phi_{rs}(x,t)\{\int_{-1}^{t}\int_{-1}^{x} K(x,t,y,z)\phi_{ij}(y,z)dydz\}dxdt, \end{aligned} \tag{12}$$

for $r,s = 0,\ldots,N$, where $c_{rs} = (\|P_r^{(\alpha)}\|_W \|P_s^{(\alpha)}\|_W)^2$.

Now, we use suitable quadrature rules to approximate the integrals in (12). For this purpose at first we set

$$y = \frac{x+1}{2}\theta + \frac{x-1}{2} = y_x^{\theta}, \quad z = \frac{t+1}{2}\psi + \frac{t-1}{2} = z_t^{\psi}, \tag{13}$$

where $\theta \in [-1,1]$ and $\psi \in [-1,1]$. It is clear that

$$\int_{-1}^{t}\int_{-1}^{x} K(x,t,y,z)\phi_{ij}(y,z)dydz = \int_{-1}^{1}\int_{-1}^{1} \widetilde{K}(x,t,y_x^{\theta},z_t^{\psi})\phi_{ij}(y_x^{\theta},z_t^{\psi})d\theta d\psi, \tag{14}$$

with $\widetilde{K}(x,t,y_x^\theta,z_t^\psi) = \frac{x+1}{2}\frac{t+1}{2}K(x,t,y_x^\theta,z_t^\psi)$. Using $(N+1)$ points Gauss-Legendre quadrature rule to approximate (14), yields

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\widetilde{K}(x,t,y_x^\theta,z_t^\psi)\phi_{ij}(y_x^\theta,z_t^\psi)d\theta d\psi &\approx \sum_{m=0}^{N}\sum_{n=0}^{N}\widetilde{K}(x,t,y_x^{\theta_m},z_t^{\psi_n})\phi_{ij}(y_x^{\theta_m},z_t^{\psi_n})v_n v_m\\ &= \widetilde{K}_{ij}(x,t),\end{aligned}$$

where $\{\theta_m\}_{m=0}^{N}$ and $\{\psi_n\}_{n=0}^{N}$ are the set of $(N+1)$ Gauss-Legendre points and $\{v_m\}_{m=0}^{N}$ are the set of corresponding weights.

For the remainder integrals in (12) we use $(N+1)$ points Gauss-ultraspherical quadrature rule as follows

$$\int_{-1}^{1}\int_{-1}^{1}W(x,t)\phi_{rs}(x,t)\widetilde{K}_{ij}(x,t)dxdt \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\phi_{rs}(x_m,t_n)\widetilde{K}_{ij}(x_m,t_n)W_m W_n = K_{ij}^{rs}$$

and

$$\int_{-1}^{1}\int_{-1}^{1}W(x,t)f(x,t)\phi_{rs}(x,t)dxdt \approx \sum_{m=0}^{N}\sum_{n=0}^{N}f(x_m,t_n)\phi_{rs}(x_m,t_n)W_m W_n = f^{rs},$$

where $\{x_m\}_{m=0}^{N}$ and $\{t_n\}_{n=0}^{N}$ are the set of $(N+1)$ Gauss-ultraspherical points and $\{W_m\}_{m=0}^{N}$ are corresponding weights. Therefore, the system of linear equations (12) can be rewritten as follows

$$c_{rs}a_{rs} = f^{rs} + \sum_{i=0}^{N}\sum_{j=0}^{N}K_{ij}^{rs}a_{ij}, \quad r,s = 0,1,\ldots,N. \tag{15}$$

For the nonlinear case of Eq. (1), similar to that of (15), we obtain

$$c_{rs}a_{rs} = f^{rs} + \int_{-1}^{1}\int_{-1}^{1}W(x,t)\phi_{rs}(x,t)\{\int_{-1}^{t}\int_{-1}^{x}K(x,t,y,z,u^N(y,z))dydz\}dxdt. \tag{16}$$

Again, using relations in (13), it is clear that

$$\int_{-1}^{t}\int_{-1}^{x}K(x,t,y,z,u^N(y,z))dydz = \int_{-1}^{1}\int_{-1}^{1}\widetilde{K}(x,t,y_x^\theta,z_t^\psi,u^N(y_x^\theta,z_t^\psi))d\theta d\psi, \tag{17}$$

with

$$\widetilde{K}(x,t,y_x^\theta,z_t^\psi,u^N(y_x^\theta,z_t^\psi)) = \frac{x+1}{2}\frac{t+1}{2}K(x,t,y_x^\theta,z_t^\psi,u^N(y_x^\theta,z_t^\psi)).$$

We note that a similar $(N+1)$ points Gauss-Legendre rule can be applied to approximating the integral involved in the right hand side of Equation (17) as follows

$$\int_{-1}^{1}\int_{-1}^{1}\widetilde{K}(x,t,y_x^{\theta},z_t^{\psi},u^N(y_x^{\theta},z_t^{\psi}))d\theta d\psi \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\widetilde{K}(x,t,y_x^{\theta_m},z_t^{\psi_n})u^N(y_x^{\theta_m},z_t^{\psi_n})v_n v_m$$
$$= \widetilde{K}(x,t)$$

and we have

$$\int_{-1}^{1}\int_{-1}^{1}W(x,t)\phi_{rs}(x,t)\widetilde{K}(x,t)dxdt \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\phi_{rs}(x_m,t_n)\widetilde{K}(x_m,t_n)W_m W_n = K^{rs}$$

Actually, we solve the system of nonlinear equations as follows

$$c_{rs}a_{rs} = f^{rs} + K^{rs}, \quad r,s = 0,\dots,N.$$

Remark. Note that our proposed method can be applied also to the following general form of the mixed Volterra-Fredholm integral equation by a similar procedure

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{1}K(x,t,y,z,u(y,z))dydz,$$

with $(x,t) \in D = [-1,1]\times[-1,1]$.

5 Illustrative Examples

In this section, several numerical examples are considered to demonstrate the efficiency and accuracy of the proposed method. We note that the two special class of ultrapherical polynomials are the Legendre and Chebyshev polynomials (that explained in section 2 briefly), and hence, we consider both of these for solving the above-mentioned examples. In all examples we set parameters $T = 1$, $X = 1$ and in the case of Legendre polynomials consider $\{\theta_m\}_{m=0}^{N}$ as the Legendre-Gauss points with the corresponding weights $v_m = \dfrac{2}{(1-\theta_m^2)[L'_{N+1}(\theta_m)]^2}$, $m = 0,1,\dots,N$, where $L_{N+1}(x)$ is the

$(N+1)th$ Legendre polynomial. Also, in the case of Chebyshev polynomials consider $\{x_m\}_{m=0}^{N}$ as the Chebyshev-Gauss points with the corresponding weights $w_m = \frac{\pi}{N+1}, \quad m = 0, 1, \ldots, N.$

All calculations are designed in MAPLE 14 and run on a Pentium 4 PC Laptop with 2.5 GHz of CPU and 2 GB of RAM.

Table 1: Numerical Results for Example 5.1 by Chebyshev Polynomials

$(x,t)=$ $(\frac{1}{2^l}, \frac{1}{2^l})$	N=1	N=2	N=3	N=4	(m=32) haar wavelet method [1]
l=1	1.6×10^{-2}	2.9×10^{-5}	7.8×10^{-5}	6.2×10^{-8}	1.4×10^{-2}
l=2	3.2×10^{-3}	1.2×10^{-3}	1.8×10^{-5}	1.8×10^{-6}	7.9×10^{-3}
l=3	3.5×10^{-4}	3.3×10^{-4}	1.6×10^{-5}	1.6×10^{-6}	4.1×10^{-3}
l=4	6.5×10^{-4}	1.6×10^{-5}	3.5×10^{-6}	7.5×10^{-7}	2.2×10^{-3}
l=5	3.8×10^{-4}	7.4×10^{-5}	5.9×10^{-7}	9.4×10^{-8}	1.2×10^{-3}
l=6	1.4×10^{-4}	5.8×10^{-5}	1.0×10^{-6}	7.2×10^{-8}	9.3×10^{-9}

Table 2: Numerical Results for Example 5.1 by Legendre Polynomials

$(x,t) = (\frac{1}{2^l}, \frac{1}{2^l})$	N=1	N=2	N=3	N=4
l=1	1.0×10^{-2}	2.0×10^{-5}	5.3×10^{-5}	3.6×10^{-8}
l=2	1.0×10^{-3}	8.1×10^{-4}	1.9×10^{-5}	6.6×10^{-7}
l=3	1.3×10^{-3}	6.7×10^{-5}	1.1×10^{-5}	1.5×10^{-6}
l=4	1.0×10^{-3}	1.7×10^{-4}	1.0×10^{-6}	3.4×10^{-7}
l=5	5.4×10^{-4}	1.6×10^{-4}	3.3×10^{-6}	2.3×10^{-7}
l=6	1.9×10^{-4}	1.1×10^{-4}	2.5×10^{-6}	2.7×10^{-7}

Example 5.1. [1] Consider the following two-dimensional VIE

$$u(x,t) = f(x,t) + \int_0^t \int_0^x (xy^2 + cos(z))u^2(y,z)dydz, \quad 0 \le x,\, t \le 1,$$

where

$$f(x,t) = xsin(t)(1 - \frac{x^2 sin^2(t)}{9}) + \frac{x^6}{10}(\frac{sin(2t)}{2} - t).$$

The exact solution is $u(x,t) = xsin(t)$. Tables 1 and 2 illustrate the numerical results for this example. For solving this example, we apply both of the Chebyshev and Legendre Ultraspherical polynomials. In Tables 1 and 2

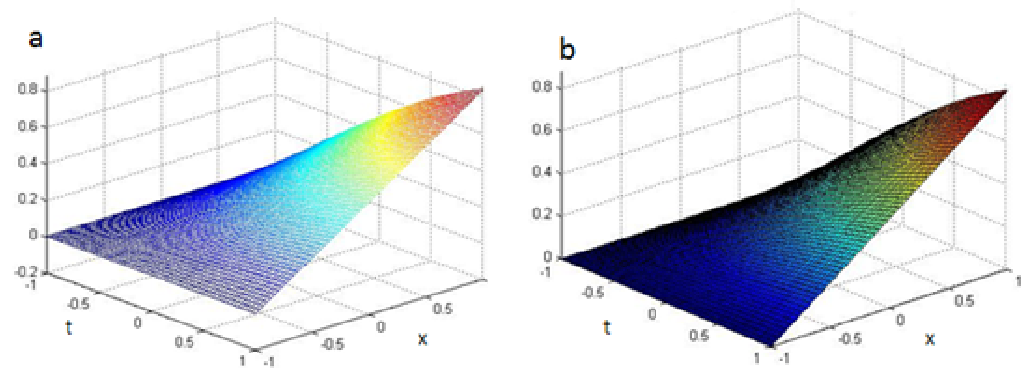

Fig. 1. (a) Numerical solution for N=4,(b) exact solution

at the grids $(\frac{1}{2^l}, \frac{1}{2^l})$, $(l = 1, \ldots, 6)$ we show the absolute value of errors for $N = 1, 2, 3, 4$. In Table 1 (at the last column) we set the best results that obtained in [1]. It can be seen that the errors decay rapidly, which is confirmed by spectral accuracy. Also in Figure 1 the numerical solution

$$\sum_{i=0}^{4}\sum_{j=0}^{4} a_{ij} P_i^{(\alpha)}(x) P_j^{(\alpha)}(t)$$

and the exact solution

$$u(x,t) = xsin(t)$$

are depicted.

Table 3: Numerical Results for Example 5.2 by Chebyshev Polynomials

$(x,t)=$ $(\frac{1}{2^l}, \frac{1}{2^l})$	N=1	N=2	N=3	N=4	(m=32) haar wavelet method [1]
l=1	5.0×10^{-10}	3.9×10^{-10}	1.5×10^{-10}	1.1×10^{-10}	3.1×10^{-2}
l=2	1.4×10^{-10}	3.3×10^{-10}	9.6×10^{-10}	1.7×10^{-10}	3.1×10^{-2}
l=3	4.2×10^{-10}	8.9×10^{-10}	5.6×10^{-10}	8.3×10^{-10}	3.1×10^{-2}
l=4	5.4×10^{-10}	1.1×10^{-9}	2.4×10^{-10}	5.2×10^{-10}	3.1×10^{-2}
l=5	6.1×10^{-10}	1.1×10^{-9}	7.8×10^{-11}	2.5×10^{-10}	3.1×10^{-2}
l=6	6.4×10^{-10}	1.1×10^{-9}	4.2×10^{-12}	1.1×10^{-10}	2.2×10^{-9}

Example 5.2. [1] In this example we are interested to apply our proposed method for numerically solving the following two-dimensional VIE

$$u(x,t) = f(x,t) + \int_0^t \int_0^x (x + t - z - y) u^2(y,z) dydz, \quad 0 \le x,\, t \le 1,$$

Table 4: Numerical Results for Example 5.2 by Legendre Polynomials

$(x,t)=(\frac{1}{2^l},\frac{1}{2^l})$	N=1	N=2	N=3	N=4
l=1	1.0×10^{-9}	1.1×10^{-10}	4.0×10^{-10}	9.2×10^{-10}
l=2	6.9×10^{-10}	3.4×10^{-10}	9.7×10^{-10}	8.0×10^{-10}
l=3	5.2×10^{-10}	7.5×10^{-10}	7.3×10^{-10}	7.0×10^{-10}
l=4	4.3×10^{-10}	9.7×10^{-10}	4.0×10^{-10}	5.3×10^{-10}
l=5	3.9×10^{-10}	1.1×10^{-9}	2.0×10^{-10}	8.0×10^{-10}
l=6	3.7×10^{-10}	1.1×10^{-9}	9.5×10^{-11}	1.2×10^{-9}

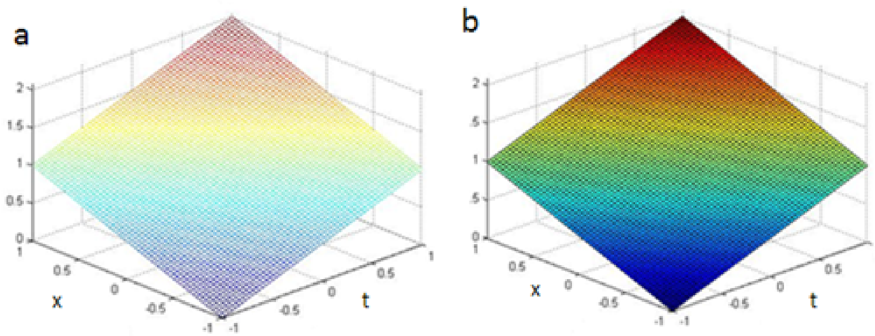

Fig. 2. (a) Numerical solution for N=4, (b) exact solution.

where

$$f(x,t)=x+t-\frac{1}{12}xt(x^3+4x^2t+4xt^2+t^3).$$

The exact solution is

$$u(x,t)=x+t.$$

Similar to Example 5.1 , in Tables 3 and 4 the errors are given and the numerical solution and exact solutions are drawn in Figure 2.

Again we can see the spectral accuracy for not very large values of N. Tables 3 and 4 show that an agreement of ten decimal figures at the abovementioned nodes is obtained. Note that the number of terms in Haar Wavelet method [1] is one less than our approach, and hence $m=N+1$.

Example 5.3. [1] We now turn to another nonlinear example. Consider the following two-dimensional VIE

$$u(x,t)=f(x,t)+\int_0^t\int_0^x u^2(y,z)dydz,\quad (x,t)\in[0,1)\times[0,1),$$

Table 5: Numerical Results for Example 5.3 by Chebyshev Polynomials

$(x,t)=$ $(\frac{1}{2^l},\frac{1}{2^l})$	N=1	N=2	N=3	N=4	(m=32) haar wavelet method [1]
l=1	2.9×10^{-1}	1.0×10^{-10}	6.0×10^{-10}	1.0×10^{-9}	3.2×10^{-2}
l=2	1.3×10^{-1}	6.0×10^{-10}	2.0×10^{-10}	5.0×10^{-10}	1.6×10^{-2}
l=3	3.1×10^{-2}	7.8×10^{-10}	4.7×10^{-10}	8.0×10^{-10}	8.5×10^{-3}
l=4	1.3×10^{-1}	8.6×10^{-10}	6.5×10^{-10}	3.0×10^{-10}	4.6×10^{-3}
l=5	1.9×10^{-1}	8.4×10^{-10}	5.5×10^{-10}	2.0×10^{-10}	2.6×10^{-3}
l=6	2.1×10^{-1}	7.1×10^{-10}	6.6×10^{-10}	2.0×10^{-10}	1.6×10^{-4}

Table 6: Numerical Results for Example 5.3 by Legendre Polynomials

$(x,t)=(\frac{1}{2^l},\frac{1}{2^l})$	N=1	N=2	N=3	N=4
l=1	1.8×10^{-1}	1.0×10^{-10}	4.0×10^{-10}	9.9×10^{-11}
l=2	4.3×10^{-2}	2.0×10^{-10}	5.0×10^{-10}	5.0×10^{-11}
l=3	1.1×10^{-1}	2.2×10^{-10}	5.5×10^{-10}	3.4×10^{-10}
l=4	2.1×10^{-1}	3.2×10^{-10}	5.5×10^{-10}	5.7×10^{-10}
l=5	2.7×10^{-1}	6.6×10^{-10}	2.1×10^{-10}	5.3×10^{-10}
l=6	3.0×10^{-1}	6.9×10^{-10}	2.1×10^{-10}	9.3×10^{-10}

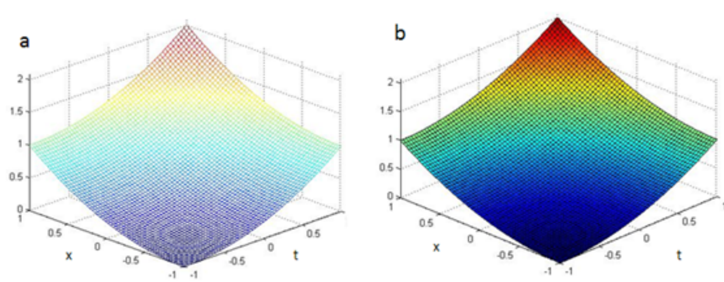

Fig. 3. (a) Numerical solution for N=4, (b) exact solution.

where

$$f(x,t)=x^2+t^2-\frac{1}{45}xt(9x^4+10x^2t^2+9t^4).$$

In this case, the exact solution is

$$u(x,t)=x^2+t^2.$$

The results presented in Tables 5 and 6 show that the high rate of decay of the errors in this example such that only a small number of nodes are needed to obtain very accurate solutions. We plot the numerical and exact solutions in Figure 3.

Table 7: Numerical Results for Example 5.4 by Chebyshev Polynomials

$(x,t)=(\frac{l}{5},\frac{l}{5})$	N=1	N=2	N=3
l=0	5.7×10^{-5}	8.5×10^{-8}	1.3×10^{-9}
l=1	3.4×10^{-6}	3.9×10^{-8}	2.5×10^{-10}
l=2	5.0×10^{-5}	4.3×10^{-8}	4.0×10^{-10}
l=3	1.0×10^{-4}	7.4×10^{-8}	2.0×10^{-10}
l=4	1.6×10^{-4}	3.1×10^{-7}	1.5×10^{-9}
l=5	2.1×10^{-4}	6.7×10^{-7}	7.0×10^{-9}

Table 8: Numerical Results for Example 5.4 by Legendre Polynomials

$(x,t)=(\frac{l}{5},\frac{l}{5})$	N=1	N=2	N=3
l=0	5.5×10^{-5}	6.3×10^{-8}	4.2×10^{-10}
l=1	2.0×10^{-5}	7.5×10^{-8}	2.7×10^{-10}
l=2	9.4×10^{-5}	6.4×10^{-8}	2.0×10^{-10}
l=3	1.7×10^{-4}	9.5×10^{-8}	8.0×10^{-10}
l=4	2.4×10^{-4}	4.0×10^{-7}	1.0×10^{-10}
l=5	3.2×10^{-4}	8.6×10^{-7}	2.9×10^{-9}

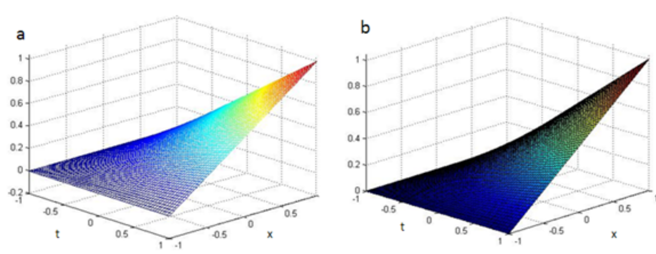

Fig. 4. (a) Numerical solution for N=3,(b) exact solution.

Example 5.4. Finally, in this example we consider the following nonlinear mixed two-dimensional integral equation

$$u(x,t)=f(x,t)+\int_0^t\int_0^1 ze^{u(y,z)}dydz,\quad t\in[0,1],$$

where

$$f(x,t)=xt-e^t+t+1.$$

Here, the exact solution is

$$u(x,t) = xt.$$

From Tables 7 and 8 one can see that our suggested method obtain high accurate solutions (as well as in three previous examples) in this mixed problem and hence our method is applicable to nonlinear mixed two-dimensional integral equations. Also, in Figure 4 the numerical and exact solutions are depicted.

6 Conclusions

The aim of the proposed method is determination of the numerical solution of two-dimensional VIES. The method is based upon the ultraspherical polynomials and Galerkin method. The properties of ultraspherical polynomials together with the Galerkin method are used here to reduce the solution of the two-dimensional VIEs to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Obtaining high accurate solutions with respect to another new method shows the spectral accuracy of the proposed method. Moreover, only a small number of ultraspherical polynomials are needed to obtain satisfactory results. The given numerical examples support this claim.

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