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On indexed product summability of an infinite series

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Abstract

A theorem on indexed product summability of an infinite series has been established.

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1 Introduction

Let $\sum a_n$ be an infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real constants such that

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$$P_n = p_0 + p_1 + \cdots + p_n \to \infty$$

as $n \to \infty$, $P_{-i} = p_{-1} = 0$.

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_n s_v$$

defines (R, p_n) transform of $\{s_n\}$ generated by $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|R, p_n\right|_k$, $k \geq 1$, if [2]

$$\sum_{n=1}^{\infty} n^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

Similarly, the sequence-to-sequence transforms

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v$$

defines the (N, p_n) transform of $\{s_n\}$ generated by $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|(N, q_n)(N, p_n)\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty ,$$

where $\{\tau_n\}$ defines the sequence of (N, q_n) transform of the (N, p_n) transform of $\{s_n\}$, generated by the sequence $\{q_n\}$ *and* $\{p_n\}$, respectively.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $\left|(N, q_n)(N, p_n), \alpha_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \alpha_n^{k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty ,$$

where $\{\tau_n\}$ defines the sequence of (N, q_n) transform of the (N, p_n) transform of $\{s_n\}$, generated by the sequence $\{q_n\}$ *and* $\{p_n\}$, respectively.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $\left|(N,q_n)\,(N,p_n),\alpha_n;\delta\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \alpha_n^{\delta k+k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty,$$

where $\{\tau_n\}$ defines the sequence of (N,q_n) transform of the (N,p_n) transform of $\{s_n\}$, generated by the sequence $\{q_n\}$ *and* $\{p_n\}$, respectively.

Let f be a function of α_n, if

$$\sum_{n=1}^{\infty} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty,$$

then the series $\sum a_n$ is said to be $\left|(N,q_n)\,(N,p_n),\alpha_n;f\right|_k$, $k \geq 1$, summable .

Clearly for $f(\alpha_n) = \alpha_n^{\delta}$, $\delta \geq 0$,

$$\left|(N,q_n)\,(N,p_n),\alpha_n;f\right|_k = \left|(N,q_n)\,(N,p_n),\alpha_n;\delta\right|_k$$

and for $\delta = 0$

$$\left|(N,q_n)\,(N,p_n),\alpha_n;f\right|_k = \left|(N,q_n)\,(N,p_n),\alpha_n\right|_k.$$

We may assume throughout this paper that $Q_n = q_0 + \cdots + q_n \to \infty$ as $n \to \infty$ and $P_n = p_0 + \cdots + p_n \to \infty$ on $n \to \infty$.

2. Known Results

In 2008, Sulaiman [4] has proved the following theorem.

Theorem 2.1 [4] Let $k \geq 1$ *and* (λ_m) be a sequence of constants.

Define

$$f_v = \sum_{r=v}^{n} \frac{q_r}{P_r}, \quad F_v = \sum_{r=v}^{n} p_r f_r$$

Let $p_n Q_n = O(P_n)$ such that

$$\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k \; Q_{n-1}} = O(\frac{(v q_v^{\;k-1})}{Q_v^k})$$

Then sufficient conditions for the implication $\sum a_n$ is summable $\left|R, r_n\right|_k \Rightarrow \sum a_n \; \lambda_n$ is summable $\left|(R, q_n)\,(R, p_n)\right|_k$ are

$$\left|\lambda_v\right| F_v = O(Q_v)\,,$$

$$\left|\lambda_v\right| = O(Q_n)\,,$$

$$p_v \; R_v \left|\lambda_v\right| = O(Q_v)\,,$$

$$p_v \; q_v \; R_v \left|\lambda_v\right| = O(Q_v \; Q_{v-1}\, r_v)\,,$$

$$p_n \, q_n \, R_n \left|\lambda_n\right| = O(P_n \, Q_n \, r_n)\,,$$

$$R_{v-1} \left|\Delta \lambda_v\right| F_{v+1} = O(Q_v \, r_v)\,,$$

and

$$R_{v-1} \left|\Delta \lambda_v\right| = O(Q_v \, r_v)\,.$$

Subsequently Paikray [1] generalize the above theorem by replacing the (R, p_n) summability by A-summability. He proved:

Theorem 2.2 [1] Let $k \geq 1$, $\{\lambda_n\}$ be a sequence of constants. Let us define

$$f_v = \sum_{r=v}^{n} q_r a_{rv}\,, \quad F_v = \sum_{r=v}^{n} f_r\,.$$

Then the sufficient conditions for the implication $\sum a_n$ is summable $\left|R, r_n\right|_k \Rightarrow \sum a_n \; \lambda_n$ summable $\left|(R, q_n)\,(A)\right|_k$ are

$$\sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k \; Q_{n-1}} = O(\frac{1}{\lambda_v^k})\,, \quad \sum_{r=v}^{n} q_r^{\frac{k}{k-1}} = O(q_v)\,, \quad \sum_{r=v}^{n} a_{r,v}^k = O(v^{k-1})\,,$$

$$R_v = O(r_v)\,, \qquad \frac{q_n}{Q_n} = O(1)\,, \qquad \frac{q_n \; \lambda_n \; a_{n,n}}{Q_{n-1}} = O(1)\,,$$

$\frac{(\Delta\,\lambda_v)^k}{q_v^{k-1}} = O(v^{k-1})$, $\quad \frac{\Delta\lambda_v}{\lambda_v} = O(1)$, $\quad$ and $\quad \frac{\lambda_v^k}{q_v^{k-1}} = O(v^{k-1})$.

In this paper we proved the following theorem on the $\left|(N,q_n)\,(N,p_n),\alpha_n;f\right|_k$, $k \geq 1$, summability of the infinite series $\sum a_n\lambda_n$. We prove:

3 Main Theorem

For the sequences of real constants $\{p_n\}$ *and* $\{q_n\}$and the sequence of positive numbers $\{\alpha_n\}$, we define

$$f_v = \sum_{i=v}^{n} \frac{q_{n-i}\,p_{i-v}}{P_i} \quad \text{and} \quad F_v = \sum_{i=v}^{n} f_i \tag{3.1}$$

Theorem 3.1 Let

$$Q_n = O(q_n\,P_n) \tag{3.2}$$

and

$$\sum_{n=v+1}^{m+1} \frac{\{f(\alpha_n)\}^k (\alpha_n)^{k-1}\, q_n^k}{Q_n^k\, Q_{n-1}} = O(\frac{(v\,q_v)^{k-1}}{Q_v^k}), \text{ as } m \to \infty. \tag{3.3}$$

Then for any sequences $\{r_n\}$ and $\{\lambda_n\}$, the sufficient conditions for the implication $\sum a_n$ is summable $|R,\,r_n|_k \Rightarrow \sum a_n\lambda_n$ is $\left|(N,q_n)\,(N,p_n),\alpha_n;f\right|_k$, $k \geq 1$, summable, are

$$|\lambda_n|\,F_v = O(Q_v), \tag{3.4}$$

$$|\lambda_n| = O(Q_n), \tag{3.5}$$

$$R_v\,F_v\,|\lambda_v| = O(Q_v\,r_v), \tag{3.6}$$

$$q_n R_n F_n |\lambda_n| = O(Q_n Q_{n-1} r_n), \tag{3.7}$$

$$R_{v-1} F_{v+1} |\Delta\lambda_v| = O(Q_v r_v), \tag{3.8}$$

$$R_{v-1} |\Delta\lambda_v| = O(Q_v r_v), \tag{3.9}$$

$$q_n R_n |\lambda_n| = O(Q_n Q_{n-1} r_n), \tag{3.10}$$

$$\sum_{n=1}^{\infty} n^{k-1} |t_n|^k = O(1), \tag{3.11}$$

and

$$\sum_{n=2}^{\infty} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} |t_n|^k = O(1), \tag{3.12}$$

where $R_n = r_1 + r_2 + \cdots + r_n$.

Proof. Let $\{t_n'\}$ be the (R, r_n) transform of the series $\sum a_n$. Then

$$t_n' = \frac{1}{R} \sum_{v=0}^{n} r_v \; s_v \; .$$

Then

$$t_n = t_n' - t_{n-1}' = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} a_v$$

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum a_n \lambda_n$ and $\{\tau_n\}$ the sequence of $(N, q_n)(N, p_n)$-transform of the series $\sum a_n \lambda_n$. Then

$$\tau_n = \frac{1}{Q_n} \sum_{r=0}^{n} q_{n-r} \frac{1}{P_r} \sum_{v=0}^{r} p_{r-v} s_v = \frac{1}{Q_n} \sum_{v=0}^{n} s_v \sum_{r=v}^{n} \frac{q_{n-v} p_{r-v}}{P_r} = \frac{1}{Q_n} \sum_{v=0}^{n} f_v s_v \; .$$

Hence

$$
\begin{aligned}
T_n &= \tau_n - \tau_{n-1} \\
&= \frac{1}{Q_n}\sum_{v=0}^{n} f_v\, s_v - \frac{1}{Q_{n-1}}\sum_{v=0}^{n-1} f_v\, s_v = -\frac{q_n}{Q_n\; Q_{n-1}}\sum_{v=0}^{n} f_v\, s_v + \frac{f_n\, s_n}{Q_{n-1}} \\
&= -\frac{q_n}{Q_n Q_{n-1}}\sum_{r=0}^{n} f_r \sum_{v=0}^{r} a_v\, \lambda_v + \frac{f_n}{Q_{n-1}}\sum_{v=0}^{n} a_v \lambda_v \\
&= -\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}}\sum_{r=v}^{n} f_r + \frac{q_0\, p_0}{P_n Q_{n-1}}\sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}} \\
&= -\frac{q_n}{Q_n\; Q_{n-1}}\{\sum_{v=1}^{n-1}(\sum_{r=1}^{v} R_{r-1} a_r)\, \Delta(\frac{\lambda_v}{R_{v-1}}\sum_{r=v}^{n} f_r) + \frac{\lambda_n}{R_{n-1}} f_n \sum_{v=1}^{n} R_{v-1} a_v\} \\
&\qquad + \frac{p_0 q_0}{P_n Q_{n-1}}\{\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} R_{r-1}\; a_v\right)\Delta(\frac{\lambda_v}{R_{v+1}}) + \frac{\lambda_n}{R_{n-1}}\sum_{v=1}^{n} R_{v-1} a_v\} \\
&= -\frac{q_n}{Q_n\; Q_{n-1}}\{\sum_{v=1}^{n-1}(\lambda_v F_v t_v + \frac{R_{v-1}}{r_v} f_v \lambda_v t_v + \frac{R_{v-1}}{r_v}(\Delta\lambda_v) F_{v+1}\, t_v) + \frac{R_n}{r_n}\lambda_n F_n t_n\} \\
&\qquad + \frac{p_0\; q_0}{P_n\; Q_{n-1}}\{\sum_{v=1}^{n-1}(\lambda_v\, t_v + \frac{R_{v-1}}{r_v}(\Delta\lambda_v) t_v) + \frac{R_n}{r_n}\lambda_n\, t_n\} \\
&= \sum_{i=1}^{7} T_{n,i}
\end{aligned}
$$

In order to prove the theorem, using Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty}\{f(\alpha_n)\}^k (\alpha_n)^{k-1}\left|T_{n,i}\right|^k < \infty,
$$

for $i = 1,2,3,4,5,6,7,\ldots$

Now, on applying Holder's inequality, we have

$$\sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|T_{n,1}\right|^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left| \frac{q_n}{Q_n \, Q_{n-1}} \sum_{v=1}^{n-1} \lambda_v \, F_v \, t_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \frac{q_n^k}{Q_n^k \, Q_{n-1}} \sum_{v=1}^{n-1} \frac{|\lambda_v|^k \, F_v^k \, |t_v|^k}{q_v^{k-1}} \left(\frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{q_v^{k-1}} |\lambda_v|^k \, F_v^k \, |t_v|^k \sum_{n=v+1}^{m+1} \frac{\{f(\alpha_n)\}^k (\alpha_n)^{k-1} \, q_n^k}{Q_n^k \, Q_{n-1}}$$

$$= O(1) \sum_{v-1}^{m} \frac{1}{q_v^{k-1}} |\lambda_v|^k \, F_v^k \, |t_v|^k \, \frac{(v \, q_v)^{k-1}}{Q_v^k} \quad , \quad \text{using (3.3)}$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} |t_v|^k \left(\frac{|\lambda_v| \, F_v}{Q_v} \right)^k$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} |t_v|^k , \quad \text{using (3.4)}$$

$$= O(1) , \text{ as } m \to \infty .$$

Next,

$$\sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|T_{n,2}\right|^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1}}{r_v} \; f_v \lambda_v \, t_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \frac{q_n^k}{Q_n^k \, Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_v^k \, F_v^k \, |\lambda_v|^k}{q_v^{k-1} \, r_v^k} \left(\frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{R_v^k F_v^k \, |\lambda_v|^k \, |t_v|^k}{q_v^{k-1} \, r_v^k} \sum_{n=v+1}^{m+1} \frac{\{f(\alpha_n)\}^k (\alpha_n)^{k-1} \, q_n^k}{Q_n^k Q_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} |t_v|^k \left(\frac{R_v \, F_v \, |\lambda_v|}{r_v \, Q_v} \right)^k$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \, |t_v|^k \quad , \quad \text{using (3.6)}$$

$$= O(1) , \text{ as } m \to \infty .$$

Further,

$$\sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|T_{n,3}\right|^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1}}{r_v} (\Delta\lambda_v) F_{v+1} \, t_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \frac{q_n^k}{Q_n^k \, Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1}^k}{r_v^k \, q_v^{k-1}} \left|\Delta\lambda_v\right|^k F_{v+1}^k \left|t_v\right|^k \left(\frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{R_{v-1}^k \left|\Delta\lambda_r\right|^k}{r_v^k \, q_v^{k-1}} F_{v+1}^k \left|t_v\right|^k \sum_{n=v+1}^{m+1} \frac{\{f(\alpha_n)\}^k (\alpha_n)^{k-1} q_n^k}{Q_n^k \, Q_{n-1}}, \quad \text{by } (3.3)$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \left|t_v\right|^k \left(\frac{R_{v-1} \, F_{v+1} \left|\Delta\lambda_v\right|}{r_v \, Q_v} \right)^k$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \left|t_v\right|^k \quad , \quad \text{using } (3.8)$$

$$= O(1), \text{ as } m \to \infty .$$

Again,

$$\sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|T_{n,4}\right|^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \, \frac{R_n \lambda_n f_n \, t_n}{r_n} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|t_n\right|^k \left(\frac{q_n R_n \, F_n \left|\lambda_n\right|}{Q_n Q_{n-1} \, r_n} \right)^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|t_n\right|^k \left(\frac{q_n R_n \, F_n \left|\lambda_n\right|}{Q_n Q_{n-1} r_n} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|t_n\right|^k \quad , \quad \text{using } (3.7)$$

$$= O(1), \text{ as } m \to \infty .$$

Next,

$$\sum_{n=2}^{m+1}\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\quad\left|T_{n,5}\right|^k$$

$$=\sum_{n=2}^{m+1}\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\left|\frac{p_0q_0}{P_n\;Q_{n-1}}\sum_{v=1}^{n-1}\lambda_v\;t_v\right|^k$$

$$\leq O(1)\sum_{n=2}^{m+1}\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\frac{1}{P_n^k\;Q_{n-1}}\sum_{v=1}^{n-1}\frac{\left|\lambda_v\right|^k}{q_v^{k-1}}\left|t_v\right|^k\left(\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}\frac{\left|\lambda_v\right|^k\left|t_v\right|^k}{q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{\{f(\alpha_n)\}^k(\alpha_n)^{k-1}}{P_n^k\;Q_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}\frac{\left|\lambda_v\right|^k\left|t_v\right|^k}{q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\;q_n^k}{Q_n^k\;Q_{n-1}},\quad\text{using (3.2)}$$

$$=O(1)\sum_{v=1}^{m}v^k\;\left|t_v\right|^k\left(\frac{\left|\lambda_n\right|^k}{Q_v}\right)^k$$

$$=O(1)\sum_{v=1}^{m}v^k\left|t_v\right|^k\;,\quad\text{using (3.7)}$$

$$=O(1)\text{, as }m\to\infty.$$

Again,

$$\sum_{n=2}^{m+1}\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\quad\left|T_{n,6}\right|^k$$

$$=\sum_{n=2}^{m+1}\;\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\left|\frac{p_0q_0}{P_n\,Q_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}}{r_v}(\Delta\lambda_v)\;t_v\right|^k$$

$$\leq O(1)\sum_{n=2}^{m+1}\{f(\alpha_n)\}^k(\alpha_n)^{k-1}\frac{1}{P_n^k\,Q_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}^k}{r_v^k\;q_v^{k-1}}\left|\Delta\lambda_v\right|^k\;\left|t_v\right|^k\left(\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right)^{n-1}$$

$$=O(1)\sum_{v=1}^{m}\frac{R_{v-1}^k\;\left|\Delta\lambda_v\right|^k\;\left|t_v\right|^k}{r_v\;q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{\{f(\alpha_n)\}^k(\alpha_n)^{k-1}}{P_n^k\;Q_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}v^{k-1}\left|t_v\right|^k\left(\frac{R_{v-1}\left|\Delta\lambda_v\right|}{r_v\,Q_v}\right)^k$$

$$=O(1)\sum_{v=1}^{m}v^{k-1}\;\left|t_v\right|^k\;\text{, using (3.9)}\quad R_{v-1}\left|\Delta\lambda_v\right|=O(Q_v\;r_v)$$

$$=O(1)\text{, as }m\to\infty.$$

Finally,

$$\sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|T_{n,7}\right|^k$$

$$= \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} \left|\frac{p_0 q_0}{P_n\, Q_{n-1}} \frac{R_n \lambda_n\, t_n}{r_n}\right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} |t_n|^k \left(\frac{R_n |\lambda_n|}{P_n\, Q_{n-1}\, r_n}\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} |t_n|^k \left(\frac{q_n R_n |\lambda_n|}{Q_n\, Q_{n-1}\, r_n}\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} |t_n|^k \text{ , using (3.10)}$$

$$= O(1), \text{ as } m \to \infty.$$

This completes the proof of the Theorem. □

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