On Solving of Common Fixed Point Problems

for Nonexpansive Semigroups in Hilbert Spaces

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Abstract

This paper is concerned with a common fixed point problem of a nonexpansive semigroup in Hilbert spaces. The strong convergence theorem for a nonexpansive semigroup is obtained by a novel general iterative scheme based on the viscosity approximation method and applicability of the results is shown to extend the results of many authors existing in the current literature.

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1 Introduction

Thoughout this paper, we always assume that C be a nonempty closed convex subset of a real Hilbert space H with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Recall that P_C is the *metric projection* of H onto C; that is, for each $x \in H$ there exists the unique point in $P_C x \in C$ such that

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

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A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C,$$

and the mapping $f : C \to C$ is called a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If $C \subset H$ is bounded, closed and convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [1]). A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \{x \in C : T(s)x = x, \ 0 \leq s < \infty\}$. It is known that if $C \subset H$ is bounded, closed and convex, then $F(\mathcal{S})$ is nonempty, closed and convex (see [2]). Construction of fixed points of nonexpansive mappings (and of common fixed points of nonexpansive semigroups) is an important subject in the theory of nonexpansive mappings and finds application in number of applied areas, in particular, in the minimization problem (see, e.g. [3, 4, 5, 6, 7] and the references therein).

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping in a real Hilbert space H:

$$\min_{x \in \Omega} \left\{ \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle \right\},$$

where A is a bounded linear operator on H, Ω is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Recall that A

be a strongly positive bounded linear operator on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

Shimizu and Takahashi [8] introduced an iterative scheme for finding a common fixed points of a nonexpansive semigroup as the following theorem.

Theorem ST. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is the sequence defined by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the conditions $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \longrightarrow \infty} s_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(\mathcal{S})}u$.

Marino and Xu [9] introduced the following an iterative scheme for finding a fixed point of nonexpansive mapping based on the viscosity approximation method introduced by Moudafi [10]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad \forall n \in \mathbb{N}, \tag{1}$$

where $x_1 \in H$, A is a strongly positive bounded linear operator on H, f is a contraction on H and S is a nonexpansive on H. They proved that under some appropriate conditions imposed on the parameters, if $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1) converges strongly to the unique solution $z = P_{F(S)}(I - A + \gamma f)z$ of the variational inequality

$$\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\},$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Futhermore, Plubtieng and Wangkeeree [11] introduced an iterative scheme for finding a common fixed point of a nonexpansive semigroup as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds, \tag{2}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$ and f is a contraction on H. They proved that under some appropriate conditions imposed on the parameters, if $F(\mathcal{S}) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (2) converges strongly to the unique solution $z = P_{F(\mathcal{S})}(I - A + \gamma f)z$ of the variational inequality

$$\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in F(\mathcal{S}).$$

In the same way, Plubtieng and Punpaeng [12] introduced an iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \, ds, \qquad (3)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$, f is a contraction on C and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on C. They proved that under some appropriate conditions imposed on the parameters, if $F(\mathcal{S}) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (3) converges strongly to the unique solution $z = P_{F(\mathcal{S})}f(z)$ of the variational inequality

$$\langle (I - f)z, x - z \rangle \geq 0, \quad \forall x \in F(\mathcal{S}).$$

Very recently, Wangkeeree [13] introduced an iterative scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \, ds, \qquad (4)$$

for all $n \in \mathbb{N}$, where $x_1 \in H$ and f is a contraction on H. We note that their iteration is well defined if we let $C = H$, and the appropriateness of the control condition α_n of their iteration should be $\{\alpha_n\} \subset (0,1)$ (see Thoerem 3.1 in [13]). He proved that under some appropriate conditions imposed on the parameters, if $F(\mathcal{S}) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution $z = P_{F(\mathcal{S})}(I - A + \gamma f)z$.

In this paper, we introduce a novel general iterative scheme by the viscosity approximation method to find a common fixed point of a nonexpansive semigroup in Hilbert space as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \, ds, \quad (5)$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, A and B are two mappings of the strongly positive linear bounded self-adjoint operator mappings, and $f : H \to H$ be a contraction mapping.

As special cases of the iterative scheme (5), we have the following.

(i) If $\mathcal{S} = \{T(s) : 0 \le s < \infty\} = T$, then (5) is reduced to iterative scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)T x_n. \quad (6)$$

(ii) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (5) is reduced to iterative scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + (I - \beta_n B - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (7)$$

(iii) If $B \equiv I$, then (5) is reduced to iterative scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \epsilon_n - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (8)$$

(iv) If $\beta_n = 0$ for all $n \in \mathbb{N}$, then (8) is reduced to iterative scheme:

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \epsilon_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (9)$$

(v) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (8) is reduced to iterative scheme of Wang-keeree [13]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (10)$$

(vi) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (9) is reduced to iterative scheme of Plubtieng and Wangkeeree [11]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (11)$$

(vii) If $\gamma = 1$ and $A \equiv I$, then (10) is reduced to iterative scheme of Plubtieng and Punpaeng [12]:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds. \quad (12)$$

(viii) If $\mathcal{S} = \{T(s) : 0 \le s < \infty\} = T$, then (11) is reduced to iterative scheme of Marino and Xu [9]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T x_n. \quad (13)$$

(ix) If $\beta_n = 0$ for all $n \in \mathbb{N}$ and for $u \in C$ chosen arbitrarily, define the mapping $f : C \to C$ by $f(x) = u$ for all $x \in C$ then (12) is reduced to iterative scheme of Shimizu and Takahashi [8]:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds. \tag{14}$$

We suggest and analyze the iterative scheme (5) above under some appropriate conditions imposed on the parameters, the strong convergence theorem for a common fixed point of a nonexpansive semigroup is obtained and applicability of the results is shown to extend the results of many authors existing in the current literature.

2   Preliminary Notes

We collect the following definition and lemmas which be used in the proof for the main results in the next section.

Definition 2.1. (see [14]) *A space X is said to safisfy Opial's condition if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$, we have*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x.$$

Remark 2.2. *It is well known that Hilbert spaces satisfy Opial's condition.*

Lemma 2.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Then the following inequality holds:*

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad \forall x \in H, \ y \in C.$$

Lemma 2.4. (see [9]) *Let H be a Hilbert space, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and $A : H \to H$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then,*

(1) if $0 < \gamma < \overline{\gamma}/\alpha$, then

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2 , \ \forall x, y \in H;$$

(2) if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.5. (see [8]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \eta_n)a_n + \delta_n, \ n \geq 1,$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\lim_{n \longrightarrow \infty} \eta_n = 0$ *and* $\sum_{n=1}^{\infty} \eta_n = \infty$;

(2) $\limsup_{n \longrightarrow \infty} (\delta_n/\eta_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.6. (see [8]) *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. For $x \in C$ and $t > 0$. Then for any $0 \leq h < \infty$, we have*

$$\lim_{t \longrightarrow \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x \ ds - T(h)\left(\frac{1}{t} \int_0^t T(s)x \ ds\right) \right\| = 0.$$

3 Main Results

Theorem 3.1. *Let H be a real Hilbert space. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on H. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \ ds, \quad (15)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\}, \{\epsilon_n\} \subset [0, 1)$ such that $\epsilon_n \leq \alpha_n$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = \lim_{n \longrightarrow \infty} \beta_n = \lim_{n \longrightarrow \infty} (\epsilon_n/\alpha_n) = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. From $\|B\| = \overline{\beta} \in (0, 1]$, $\{\beta_n\} \subset [0, 1)$, $\epsilon_n \leq \alpha_n$ for all $n \in \mathbb{N}$, (C1), we have $\alpha_n \longrightarrow 0$, $\epsilon_n \longrightarrow 0$ and $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, we may assume without loss of generality that $\alpha_n < (1 - \epsilon_n - \beta_n\|B\|)\|A\|^{-1}$ for all $n \in \mathbb{N}$. Since A and B are two mappings of the linear bounded self-adjoint operators, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$$

and

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\langle((1-\epsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle = (1-\epsilon_n)\langle x, x\rangle - \beta_n \langle Bx, x\rangle - \alpha_n \langle Ax, x\rangle$$
$$\geq 1 - \epsilon_n - \beta_n\|B\| - \alpha_n\|A\|$$
$$> 0.$$

Therefore, we obtain $(1 - \epsilon_n)I - \beta_n B - \alpha_n A$ is positive. Thus, by the strong positively of A and B, we get

$$\|(1-\epsilon_n)I - \beta_n B - \alpha_n A\|$$

$$= \sup\{\langle((1-\epsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\}$$
$$= \sup\{(1-\epsilon_n)\langle x, x\rangle - \beta_n \langle Bx, x\rangle - \alpha_n \langle Ax, x\rangle : x \in H, \|x\| = 1\}$$
$$\leq 1 - \epsilon_n - \beta_n\overline{\beta} - \alpha_n\overline{\delta}$$
$$\leq 1 - \beta_n\overline{\beta} - \alpha_n\overline{\delta}. \tag{16}$$

Define the sequence of mappings $\{P_n : H \rightarrow H\}$ as follows:

$$P_n x = \alpha_n\gamma f(x) + \beta_n Bx + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x\ ds, \ \forall x \in H,$$

for all $n \in \mathbb{N}$. Firstly, we prove that P_n has a unique fixed point in H. Note that for all $x, y \in H$, by (16), the contraction of f, the nonexpansiveness of $T(s)$, and the linearity of A and B, we have

$$\|P_n x - P_n y\| \leq \alpha_n\gamma\|f(x) - f(y)\| + \beta_n\|B\|\|x - y\|$$
$$+ \|(1-\epsilon_n)I - \beta_n B - \alpha_n A\|\frac{1}{s_n}\int_0^{s_n}\|T(s)x - T(s)y\|\ ds$$
$$\leq \alpha_n\gamma\delta\|x - y\| + \beta_n\overline{\beta}\|x - y\| + (1 - \beta_n\overline{\beta} - \alpha_n\overline{\delta})\|x - y\|$$
$$= (1 - (\overline{\delta} - \gamma\delta)\alpha_n)\|x - y\|.$$

Hence, P_n is a contraction with coefficient $1 - (\overline{\delta} - \gamma\delta)\alpha_n \in (0,1)$. Therefore, by Banach contraction principle guarantees that P_n has a unique fixed point in H, and so the iteration (15) is well defined.

Next, we prove that $\{x_n\}$ is bounded. Pick $p \in F(\mathcal{S}) = \bigcap_{s \geq 0} F(T(s))$ and setting $y_n = \frac{1}{s_n} \int_0^{s_n} T(s)x_n\, ds$. By the nonexpansiveness of $T(s)$, we have

$$
\begin{aligned}
\|y_n - p\| &= \left\| \frac{1}{s_n} \int_0^{s_n} T(s)x_n\, ds - \frac{1}{s_n} \int_0^{s_n} T(s)p\, ds \right\| \\
&\leq \frac{1}{s_n} \int_0^{s_n} \|T(s)x_n - T(s)p\|\, ds \\
&\leq \|x_n - p\|.
\end{aligned} \tag{17}
$$

By (16), (17), the contraction of f, and the linearity of A and B, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)y_n - p\| \\
&= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n B(x_n - p) \\
&\qquad + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - p) - \epsilon_n p\| \\
&\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|B\| \|x_n - p\| \\
&\qquad + \|(1 - \epsilon_n)I - \beta_n B - \alpha_n A\| \|y_n - p\| + \epsilon_n \|p\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \beta_n \overline{\beta} \|x_n - p\| \\
&\qquad + (1 - \beta_n \overline{\beta} - \alpha_n \overline{\delta}) \|x_n - p\| + \alpha_n \|p\| \\
&\leq \left(1 - (\overline{\delta} - \gamma\delta)\alpha_n\right) \|x_n - p\| + \alpha_n \left(\|\gamma f(p) - Ap\| + \|p\|\right) \\
&\leq \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\overline{\delta} - \gamma\delta} \right\}.
\end{aligned}
$$

It follows from induction that

$$
\|x_{n+1} - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\overline{\delta} - \gamma\delta} \right\},
$$

for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{f(x_n)\}$.

Put $z_1 = P_{F(\mathcal{S})}x_1$ and set

$$
D = \left\{ z \in H : \|z - z_1\| \leq \|x_1 - z_1\| + \frac{\|\gamma f(z_1) - Az_1\| + \|z_1\|}{\overline{\delta} - \gamma\delta} \right\}.
$$

Then D is a nonempty bounded closed convex subset of H which is $T(s)$-invariant for each $s \in [0, \infty)$ and $\{x_n\}, \{y_n\} \subset D$. Without loss of generality,

we may assume that $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on D. By (C2) and Lemma 2.6, we get

$$\lim_{n \longrightarrow \infty} \|y_n - T(h)y_n\| = 0, \tag{18}$$

for every $h \in [0, \infty)$. For all $x, y \in H$, by Lemma 2.4(2), the nonexpansiveness of $P_{F(\mathcal{S})}$, the contraction of f and the linearity of A, we have

$$\|P_{F(\mathcal{S})}(I - A + \gamma f)x - P_{F(\mathcal{S})}(I - A + \gamma f)y\|$$

$$\begin{aligned}
&\leq & \|(I - A + \gamma f)x - (I - A + \gamma f)y\| \\
&\leq & \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
&\leq & \gamma\delta\|x - y\| + (1 - \overline{\delta})\|x - y\| \\
&= & (1 - (\overline{\delta} - \gamma\delta))\|x - y\|.
\end{aligned}$$

Therefore, $P_{F(\mathcal{S})}(I - A + \gamma f)$ is a contraction with coefficient $1 - (\overline{\delta} - \gamma\delta) \in (0, 1)$, by Banach contraction principle guarantees that $P_{F(\mathcal{S})}(I - A + \gamma f)$ has a unique fixed point, say $w \in H$, that is, $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$. Hence, by Lemma 2.3, we obtain

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \qquad \forall y \in F(\mathcal{S}). \tag{19}$$

Next, we claim that

$$\limsup_{n \longrightarrow \infty} \langle \gamma f(w) - Aw, y_n - w \rangle \leq 0.$$

To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \longrightarrow \infty} \langle \gamma f(w) - Aw, y_n - w \rangle = \lim_{i \longrightarrow \infty} \langle \gamma f(w) - Aw, y_{n_i} - w \rangle . \tag{20}$$

Since $\{y_{n_i}\} \subset D$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\overline{w}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \overline{w}$ as $i \longrightarrow \infty$.

Next, we prove that $\overline{w} \in F(\mathcal{S}) = \bigcap_{s \geq 0} F(T(s))$. Suppose that $\overline{w} \notin F(\mathcal{S})$, that is, $T(h)\overline{w} \neq \overline{w}$ for some $h \in [0, \infty)$. Since $\|y_{n_i} - T(h)y_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$ by (18), therefore, by the nonexpansiveness of $T(h)$ and the Opial's condition, we have

$$\begin{aligned}
\liminf_{i \longrightarrow \infty} \|y_{n_i} - \overline{w}\| &<& \liminf_{i \longrightarrow \infty} \|y_{n_i} - T(h)\overline{w}\| \\
&\leq & \liminf_{i \longrightarrow \infty} \left(\|y_{n_i} - T(h)y_{n_i}\| + \|T(h)y_{n_i} - T(h)\overline{w}\| \right) \\
&\leq & \liminf_{i \longrightarrow \infty} \|y_{n_i} - \overline{w}\|.
\end{aligned}$$

This is a contradiction. So, we obtain $\overline{w} \in F(\mathcal{S})$. Therefore, from (19) and (20), we obtain

$$
\begin{aligned}
\limsup_{n \longrightarrow \infty} \langle \gamma f(w) - Aw, y_n - w \rangle &= \lim_{i \longrightarrow \infty} \langle \gamma f(w) - Aw, y_{n_i} - w \rangle \\
&= \langle (\gamma f - A)w, \overline{w} - w \rangle \le 0. \qquad (21)
\end{aligned}
$$

Next, we prove that $x_n \longrightarrow w$ as $n \longrightarrow \infty$. Since $w \in F(\mathcal{S})$, the same as in (17), we have

$$
\|y_n - w\| \le \|x_n - w\|. \qquad (22)
$$

Therefore, by (16), (22), the contraction of f, and the linearity of A and B, we have

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)y_n - w\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Aw) + \beta_n B(x_n - w) \\
&\qquad + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w) - \epsilon_n w\|^2 \\
&\le \Big(\|\alpha_n(\gamma f(x_n) - Aw) + \beta_n B(x_n - w) \\
&\qquad + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\| + \epsilon_n \|w\| \Big)^2 \\
&= \|\alpha_n(\gamma f(x_n) - Aw) + \beta_n B(x_n - w) \\
&\qquad + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\|^2 + M_n^{(1)} \\
&= \|\beta_n B(x_n - w) + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\|^2 \\
&\qquad + 2\alpha_n \langle ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w), \gamma f(x_n) - Aw \rangle \\
&\qquad + M_n^{(1)} + M_n^{(2)} \\
&\le \Big(\beta_n \|B\| \|x_n - w\| + \|(1 - \epsilon_n)I - \beta_n B - \alpha_n A\| \|y_n - w\| \Big)^2 \\
&\qquad + 2\alpha_n \gamma \langle y_n - w, f(x_n) - f(w) \rangle + M_n^{(1)} + M_n^{(2)} + M_n^{(3)} \\
&\le \Big(\beta_n \overline{\beta} \|x_n - w\| + (1 - \beta_n \overline{\beta} - \alpha_n \overline{\delta}) \|x_n - w\| \Big)^2 \\
&\qquad + 2\alpha_n \gamma \delta \|x_n - w\|^2 + M_n^{(1)} + M_n^{(2)} + M_n^{(3)} \\
&= \Big(1 - 2(\overline{\delta} - \gamma \delta)\alpha_n \Big) \|x_n - w\|^2 + \alpha_n^2 \overline{\delta}^2 \|x_n - w\|^2 \\
&\qquad + M_n^{(1)} + M_n^{(2)} + M_n^{(3)} \\
&\le (1 - \eta_n) \|x_n - w\|^2 + \delta_n,
\end{aligned}
$$

where

$$
\begin{aligned}
M_n^{(1)} &:= 2\epsilon_n \| \alpha_n(\gamma f(x_n) - Aw) + \beta_n B(x_n - w) \\
&\qquad\qquad + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w) \| \|w\| + \epsilon_n^2 \|w\|^2, \\
M_n^{(2)} &:= \alpha_n^2 \|\gamma f(x_n) - Aw\|^2 + 2\alpha_n\beta_n \langle B(x_n - w), \gamma f(x_n) - Aw \rangle, \\
M_n^{(3)} &:= 2\alpha_n \langle y_n - w, \gamma f(w) - Aw \rangle \\
&\qquad\qquad - 2\alpha_n \langle (\epsilon_n I + \beta_n B + \alpha_n A)(y_n - w), \gamma f(x_n) - Aw \rangle, \\
\eta_n &:= (\overline{\delta} - \gamma\delta)\alpha_n \in (0, 1), \\
\delta_n &:= \alpha_n^2 \overline{\delta}^2 \|x_n - w\|^2 + M_n^{(1)} + M_n^{(2)} + M_n^{(3)}.
\end{aligned}
$$

By (C1), (C2), $\lim_{n \to \infty} \epsilon_n = 0$ and (21), we can found that $\lim_{n \to \infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\limsup_{n \to \infty} (\delta_n/\eta_n) \leq 0$. Therefore, by Lemma 2.5, we obtain $\{x_n\}$ converges strongly to w. This completes the proof. $\qquad\square$

Remark 3.2. *The iteration* (15) *is the difference with many others as the following.*

1. *Two mappings A and B of the strongly positive linear bounded self-adjoint operator mappings are used in the iteration of $\{x_n\}$, which be used only one mapping A by many others.*

2. *Three parameters α_n, β_n and ϵ_n are used in the iteration of $\{x_n\}$, which be used only two parameters α_n and β_n by many others.*

4   Applications

Theorem 4.1. *Let H be a real Hilbert space. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $T : H \to H$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)T x_n,
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\}, \{\epsilon_n\} \subset [0, 1)$ such that $\epsilon_n \leq \alpha_n$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = \lim_{n\longrightarrow\infty} \beta_n = \lim_{n\longrightarrow\infty}(\epsilon_n/\alpha_n) = 0;$

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to $w \in F(T)$ where $w = P_{F(T)}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(T).$$

Proof. It is concluded from Theorem 3.1 immediately, by putting $\mathcal{S} = \{T(s) : 0 \leq s < \infty\} = T.$ $\square$

Theorem 4.2. *Let H be a real Hilbert space. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on H. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + (I - \beta_n B - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = \lim_{n\longrightarrow\infty} \beta_n = 0;$

(C2) $\lim_{n\longrightarrow\infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 3.1 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. $\square$

Theorem 4.3. *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with*

coefficient $\delta \in (0,1)$. *Let* $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* H. *Assume that* $F(\mathcal{S}) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \epsilon_n - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \ ds,$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ *such that* $\epsilon_n \leq \alpha_n$ *and* $\{s_n\} \subset (0,\infty)$ *satisfying the following conditions:*

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = \lim_{n \longrightarrow \infty} \beta_n = \lim_{n \longrightarrow \infty}(\epsilon_n/\alpha_n) = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $w \in F(\mathcal{S})$ *where* $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 3.1 immediately, by putting $B \equiv I$. $\square$

Theorem 4.4. *Let* H *be a real Hilbert space. Let* $A : H \to H$ *be a strongly positive linear bounded self-adjoint operator mapping with coefficient* $\overline{\delta} \in (0,1]$ *such that* $\overline{\delta} \leq \|A\| \leq 1$, *and let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$. *Let* $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* H. *Assume that* $F(\mathcal{S}) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \epsilon_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \ ds,$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,1)$ *such that* $\epsilon_n \leq \alpha_n$ *and* $\{s_n\} \subset (0,\infty)$ *satisfying the following conditions:*

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = \lim_{n \longrightarrow \infty}(\epsilon_n/\alpha_n) = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $w \in F(\mathcal{S})$ *where* $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.3 immediately, by putting $\beta_n = 0$ for all $n \in \mathbb{N}$. $\qquad\square$

Theorem 4.5. (Wangkeeree [13]) *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0,1]$ such that $\overline{\delta} \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on H. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \; ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = \lim_{n \longrightarrow \infty} \beta_n = 0;$

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.3 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. $\qquad\square$

Theorem 4.6. (Plubtieng and Wangkeeree [11]) *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0,1]$ such that $\overline{\delta} \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on H. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \; ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.4 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 4.7. (Plubtieng and Punpaeng [12]) *Let C be a nonempty closed convex of a real Hilbert space H. Let $f : C \rightarrow C$ be a contraction mapping with coefficient $\delta \in (0, 1)$ and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Assume that $F(\mathcal{S}) \neq \emptyset$. For $x_1 = u \in C$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n \; ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = \lim_{n \longrightarrow \infty} \beta_n = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}f(w)$ is a unique solution of the variational inequality

$$\langle (I - f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.5 immediately, by putting $\gamma = \overline{\delta} = 1$ and $A \equiv I$. □

Theorem 4.8. (Marino and Xu [9]) *Let H be a real Hilbert space. Let $A : H \rightarrow H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$, and let $f : H \rightarrow H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $T : H \rightarrow H$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(T)$ where $w = P_{F(T)}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \qquad \forall y \in F(T).$$

Proof. It is concluded from Theorem 4.6 immediately, by putting $\mathcal{S} = \{T(s) : 0 \leq s < \infty\} = T$. $\qquad\qquad\square$

Theorem 4.9. (Shimizu and Takahashi [8]) *Let C be a nonempty closed convex of a real Hilbert space H. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Assume that $F(\mathcal{S}) \neq \emptyset$. For $x_1 = u \in C$, suppose that $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \; ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0$;

(C2) $\lim_{n \longrightarrow \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}u$ is a unique solution of the variational inequality

$$\langle w - u, y - w \rangle \geq 0, \qquad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.7 immediately, by putting $\beta_n = 0$ for all $n \in \mathbb{N}$ and define the mapping $f : C \to C$ by $f(x) = u$ for all $x \in C$. $\quad\square$

References

[1] W. Takahashi, *Nonlinear functional analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama Japan, 2000.

[2] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proceedings of the National Academy of Sciences*, **54**, (1965), 1041 - 1044.

[3] F. Deutsch and I. Yamada, Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, *Numerical Functional Analysis and Optimization*, **19**(1 - 2), (1998), 33 - 56.

[4] ———, Iterative algorithms for nonlinear operators, *Journal of the London Mathematical Society*, **66**(1), (2002), 240 - 256.

[5] ———, An iterative approach to quadratic optimization, *Journal of Optimization Theory and Applications*, **116**(3), (2003), 659 - 678.

[6] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, D. Butnariu, Y. Censor, and S. Reich, Eds., **8**, (2001), 473 - 504.

[7] I. Yamada, N. Ogura, Y. Yamashita and K. Sakaniwa, Quadratic optimization of fixed points of nonexpansive mappings in Hilbert spaces, *Numerical Functional Analysis and Optimization*, **19**(1 - 2), (1998), 165 - 190.

[8] T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, **211**(1), (1997), 71 - 83.

[9] G. Marino and H.-K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **318**(1), (2006), 43 - 52.

[10] A. Moudafi, Viscosity approximation methods for fixed-points problems, *Journal of Mathematical Analysis and Applications*, **241**(1), (2000), 46 - 55.

[11] S. Plubtieng and R. Wangkeeree, A general viscosity approximation method of fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, *Bulletin of the Korean Mathematical Society*, **45**(4), (2008), 717 - 728.

[12] S. Plubtieng and R. Punpaeng, Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, *Mathematical and Computer Modelling*, **48**, (2008), 279 - 286.

[13] R. Wangkeeree, A general approximation method for nonexpansive semigroups in Hilbert spaces, *Far East Journal of Applied Mathematics*, **34**(2), (2009), 221 - 232.

[14] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bulletin of the American Mathematical Society*, **73**, (1967), 591 - 597.