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On Calculating the Determinants of Toeplitz Matrices

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Abstract

We consider the Toeplitz matrices and obtain their unique LU factorizations. As by-products, we get an explicit formula for the determinant of a Toeplitz matrix and the application of inversion of Toeplitz matrices.

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1 Introduction

A Toeplitz matrix is an $n \times n$ matrix:

$$T_n = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ a_2 & a_1 & a_0 & \cdots & a_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

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where $a_{-(n-1)}, \cdots, a_{n-1}$ are complex numbers([3]). The Toeplitz matrix can be applied in signal processing([4]), information theory([2]) and another more applications([1]). In [3], Theorem 1 stated that let T_n be a Toeplitz matrix. If each of the systems of equation $T_n x = f, T_n y = e_1$ is solvable, $x = (x_1, x_2, \cdots, x_n)^T, y = (y_1, y_2, \cdots, y_n)^T$, then
(a) T_n is invertible;
(b) $T_n^{-1} = T_1 U_1 + T_2 U_2$, where

$$T_1 = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & -x_n \\ & & & 1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_n \\ x_n & \cdots & x_2 & x_1 \end{bmatrix} \quad and \quad U_2 = \begin{bmatrix} 0 & y_n & \cdots & y_2 \\ & 0 & \ddots & \vdots \\ & & \ddots & y_n \\ & & & 0 \end{bmatrix}.$$

Furthermore, the Remark of [3] stated that let T_n be a circulant Toeplitz matrix. That is to say, the elements of the matrix $T_n = (a_{p-q})_{p,q=1}^n$ satisfy $a_i = a_{i-n}$ for all $i = 1, \cdots, n-1$. It is easy to see that $f = 0$. Thus, $x = T_n^{-1} f = 0$. From (b) of Theorem 1, the authors got

$$T_n^{-1} = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}.$$

In this paper, we will establish the explicit formulae for the LU factorization of T_n and the determinant of T_n. Moreover, we also obtain some by-products by applying Theorem 1 and Remark in [3].

2 The LU factorization of T_n and applications

Our main aim in this note is to give the explicit unique LU factorization of T_n. As by-products of the main algorithm, we get some corollaries including determinants of Toeplitz matrices and inversion of Toeplitz matrices.

Before stating the main algorithm, we introduce the following definitions.

$$e_2 :={}^{0,0}_{0,0}\ ;\ f_2 :={}^{1,0}_{1,0}\ ;\ g_2 :={}^{0,-1}_{0,-1}\ ;\ h_2 :={}^{1,-1}_{0,0}\ ;$$

and for $k \geq 2$,

$$e_{k+1} := e_k; e_k; e_k; e_k\ ;$$

$$f_{k+1} := e_k; f_k; f_k; e_k\ ;$$

$$g_{k+1} := e_k; g_k; e_k; g_k\ ;$$

$$h_{k+1} := e_k; h_k; f_k; g_k\ ;$$

and for $n \geq 0$,

$$n \times f_2 :={}^{n,0}_{n,0}\ ;$$

$$n \times f_3 :={}^{0,0}_{0,0};{}^{n,0}_{n,0};{}^{n,0}_{n,0};{}^{0,0}_{0,0}\ ;$$

$$n \times g_2 :={}^{0,-n}_{0,-n}\ ;$$

$$n \times g_3 :={}^{0,0}_{0,0};{}^{0,-n}_{0,-n};{}^{0,0}_{0,0};{}^{0,-n}_{0,-n}\ ;$$

$n \times f_k$ and $n \times g_k$ are defined in the same way. Besides, we need the following operations.

$$S({}^{k,l}_{i,j})(a_s a_t - a_u a_v) := a_{s+i}a_{t+j} - a_{u+k}a_{v+l}\ ;$$

$$S({}^{k,l}_{i,j};{}^{o,p}_{m,n})[(a_s a_t - a_u a_v)(a_w a_x - a_y a_z)] := (a_{s+i}a_{t+j} - a_{u+k}a_{v+l})(a_{w+m}a_{x+n} - a_{y+o}a_{z+p})\ ;$$

$S(e_k)$, $S(f_k)$, $S(g_k)$and $S(h_k)$ are defined in the same way. Furthermore, we define the following recurrence relation:

Let

$$d_2 := a_0^2 - a_1 a_{-1}\ ;$$

for $k \geq 2$,

$$d_{k+1} := d_k \times [S(h_k)d_k] - [S(f_k)d_k] \times [S(g_k)d_k].$$

Algorithm 2.1. For $n \geq 3$, T_n can be factored as $T_n = L_n U_n$, where $L_n = [L_n(i,j)]_{i,j=1,2,\cdots,n}$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]_{i,j=1,2,\cdots,n}$ is an upper triangular matrix, whose entries are defined as follows:

$$L_n(i,j) = \begin{cases} 1, & if\ i = j, \\ 0, & if\ i < j, \\ \frac{a_{i-1}}{a_0}, & if\ j = 1, i \geq 2, \\ \frac{a_{i-2}a_0 - a_{i-1}a_{-1}}{a_0^2 - a_1 a_{-1}}, & if\ j = 2, i \geq 3, \\ \frac{S((i-j)\times f_j)d_j}{d_j}, & if\ j+1 \leq i \leq n, j \geq 3. \end{cases}$$

and

$$U_n(i,j) = \begin{cases} a_{1-j}, & if\ i = 1, \\ 0, & if\ i > j, \\ \frac{a_0 a_{2-j} - a_1 a_{1-j}}{a_0}, & if\ i = 2, j \geq 2, \\ \frac{S((j-3)\times g_3)d_3}{a_0(a_0^2 - a_1 a_{-1})}, & if\ i = 3, j \geq 3, \\ \frac{S((j-i)\times g_i)d_i}{a_0(a_0^2 - a_1 a_{-1})\prod_{t=3}^{i-1} d_t}, & if\ i \geq 4, j \geq i. \end{cases}$$

To illustrate our result, we give an example of the explicit factorization of T_5.

Example 2.2. Let $n = 5$, then $T_5 = L_5 U_5$, where

$$T_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix},$$

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{a_1}{a_0} & 1 & 0 & 0 & 0 \\ \frac{a_2}{a_0} & \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}} & 1 & 0 & 0 \\ \frac{a_3}{a_0} & \frac{a_2 a_0 - a_3 a_{-1}}{a_0^2 - a_1 a_{-1}} & \frac{(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_3 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})} & 1 & 0 \\ \frac{a_4}{a_0} & \frac{a_3 a_0 - a_4 a_{-1}}{a_0^2 - a_1 a_{-1}} & \frac{(a_0^2 - a_1 a_{-1})(a_2 a_0 - a_4 a_{-2}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})} & L_5(5,4) & 1 \end{bmatrix}$$

and

$$U_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ 0 & \frac{a_0^2 - a_1a_{-1}}{a_0} & \frac{a_0a_{-1} - a_1a_{-2}}{a_0} & \frac{a_0a_{-2} - a_1a_{-3}}{a_0} & \frac{a_0a_{-3} - a_1a_{-4}}{a_0} \\ 0 & 0 & U_5(3,3) & U_5(3,4) & U_5(3,5) \\ 0 & 0 & 0 & U_5(4,4) & U_5(4,5) \\ 0 & 0 & 0 & 0 & U_5(5,5) \end{bmatrix},$$

where

$$\begin{aligned} L_5(5,4) = \{&[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_1a_0 - a_4a_{-3}) - (a_3a_0 - a_4a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ &-[(a_0^2 - a_1a_{-1})(a_2a_0 - a_4a_{-2}) - (a_3a_0 - a_4a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\ &\div\{[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0^2 - a_3a_{-3}) - (a_2a_0 - a_3a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ &-[(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\}, \end{aligned}$$

$$U_5(3,3) = \frac{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})}{a_0(a_0^2 - a_1a_{-1})},$$

$$U_5(3,4) = \frac{(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})}{a_0(a_0^2 - a_1a_{-1})},$$

$$U_5(3,5) = \frac{(a_0^2 - a_1a_{-1})(a_0a_{-2} - a_2a_{-4}) - (a_1a_0 - a_2a_{-1})(a_0a_{-3} - a_1a_{-4})}{a_0(a_0^2 - a_1a_{-1})},$$

$$\begin{aligned} U_5(4,4) = \{&[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0^2 - a_3a_{-3}) - (a_2a_0 - a_3a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ &-[(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\ \div\{a_0(a_0^2 - a_1a_{-1})&[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})]\}, \end{aligned}$$

$$\begin{aligned} U_5(4,5) = \{&[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ &\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_3a_{-4}) - (a_2a_0 - a_3a_{-1})(a_0a_{-3} - a_1a_{-4})] \\ &-[(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \end{aligned}$$

$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$\div\{a_0(a_0^2-a_1a_{-1})[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\},$$
$$U_5(5,5)=\{\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_4a_{-4})-(a_3a_0-a_4a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$-\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_1a_0-a_4a_{-3})-(a_3a_0-a_4a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_3a_{-4})-(a_2a_0-a_3a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}\}$$
$$\div\{\{a_0(a_0^2-a_1a_{-1})[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}\}.$$

Since the determinant of a triangular matrix is the product of the entries of the main diagonal, as a first by-product, we get the following immediate corollary which provides an explicit formula for the determinant of T_n.

Corollary 2.3. For $n \geq 3$, the determinant of T_n is as follows:

$$\det T_n = \prod_{i=1}^{n} U_n(i,i) = \frac{d_n}{a_0^{n-2}(a_0^2 - a_1a_{-1})^{n-3}\prod_{i=3}^{n-2} d_i^{n-i-1}}.$$

Example 2.4. Let $n = 5$, then

$$\begin{aligned}
\det T_5 = \prod_{1\leq i\leq 5} U_5(i,i) \\
= \{\{[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0^2 - a_3a_{-3}) - (a_2a_0 - a_3a_{-1})(a_0a_{-2} - a_1a_{-3})] \\
-[(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\
\times\{[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0^2 - a_4a_{-4}) - (a_3a_0 - a_4a_{-1})(a_0a_{-3} - a_1a_{-4})] \\
-[(a_0^2 - a_1a_{-1})(a_2a_0 - a_4a_{-2}) - (a_3a_0 - a_4a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0a_{-2} - a_2a_{-4}) - (a_1a_0 - a_2a_{-1})(a_0a_{-3} - a_1a_{-4})]\} \\
-\{[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_1a_0 - a_4a_{-3}) - (a_3a_0 - a_4a_{-1})(a_0a_{-2} - a_1a_{-3})] \\
-[(a_0^2 - a_1a_{-1})(a_2a_0 - a_4a_{-2}) - (a_3a_0 - a_4a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\
\times\{[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_3a_{-4}) - (a_2a_0 - a_3a_{-1})(a_0a_{-3} - a_1a_{-4})] \\
-[(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\
\times[(a_0^2 - a_1a_{-1})(a_0a_{-2} - a_2a_{-4}) - (a_1a_0 - a_2a_{-1})(a_0a_{-3} - a_1a_{-4})]\}\} \\
\div\{\{a_0^3(a_0^2 - a_1a_{-1})^2[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})]\}\}.
\end{aligned}$$

For any $n \times n$ matrix A, let $\tilde{A}_{ij}$ be the matrix obtained from A by deleting the i-th row and the j-th column. Then

$$A^{-1} = Transpose\ \ of\ \ \frac{(-1)^{i+j}\det(\tilde{A}_{ij})}{\det(A)}.$$

From above, we get the following corollary:

Corollary 2.5. The inverse of T_n is

$$T_n^{-1} = T_1U_1 + T_2U_2 = Transpose\ \ of\ \ \frac{(-1)^{i+j}\det((\tilde{T_n})ij)}{\frac{d_n}{a_0^{n-2}(a_0^2-a_1a_{-1})^{n-3}\prod_{i=3}^{n-2}d_i^{n-i-1}}}.$$

In particular,

$$(-1)^{j+i}\det((\tilde{T_n})ji) = \frac{d_n}{a_0^{n-2}(a_0^2-a_1a_{-1})^{n-3}\prod_{i=3}^{n-2}d_i^{n-i-1}} \times T_n^{-1}(i,j).$$

Corollary 2.6. Let T_n be a circulant Toeplitz matrix. Then

$$\begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix} = Transpose\ \ of\ \ \frac{(-1)^{i+j}\det((\tilde{T_n})ij)}{\frac{d_n}{a_0^{n-2}(a_0^2-a_1a_{-1})^{n-3}\prod_{i=3}^{n-2}d_i^{n-i-1}}}.$$

Example 2.7. Let T_5 be a circulant Toeplitz matrix, i.e.

$$T_5 = \begin{bmatrix} a_0 & a_4 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_4 \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Then

$$\det\begin{bmatrix} a_0 & a_4 & a_3 & a_2 \\ a_1 & a_0 & a_4 & a_3 \\ a_2 & a_1 & a_0 & a_4 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

$$
\begin{aligned}
&= y_1 \times \{\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0^2 - a_3a_2) - (a_2a_0 - a_3a_4)(a_0a_3 - a_1a_2)] \\
&-[(a_0^2 - a_1a_4)(a_1a_0 - a_3^2) - (a_2a_0 - a_3a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_4 - a_2^2) - (a_1a_0 - a_2a_4)(a_0a_3 - a_1a_2)]\} \\
&\times\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0^2 - a_4a_1) - (a_3a_0 - a_4^2)(a_0a_2 - a_1^2)] \\
&-[(a_0^2 - a_1a_4)(a_2a_0 - a_4a_3) - (a_3a_0 - a_4^2)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_3 - a_2a_1) - (a_1a_0 - a_2a_4)(a_0a_2 - a_1^2)]\} \\
&-\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_1a_0 - a_4a_2) - (a_3a_0 - a_4^2)(a_0a_3 - a_1a_2)] \\
&-[(a_0^2 - a_1a_4)(a_2a_0 - a_4a_3) - (a_3a_0 - a_4^2)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_4 - a_2^2) - (a_1a_0 - a_2a_4)(a_0a_3 - a_1a_2)]\} \\
&\times\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_4 - a_3a_1) - (a_2a_0 - a_3a_4)(a_0a_2 - a_1^2)] \\
&-[(a_0^2 - a_1a_4)(a_1a_0 - a_3^2) - (a_2a_0 - a_3a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_3 - a_2a_1) - (a_1a_0 - a_2a_4)(a_0a_2 - a_1^2)]\}\} \\
&\div\{\{a_0^3(a_0^2 - a_1a_4)^2[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)]\}\},
\end{aligned}
$$

$$
\det\begin{bmatrix} a_1 & a_4 & a_3 & a_2 \\ a_2 & a_0 & a_4 & a_3 \\ a_3 & a_1 & a_0 & a_4 \\ a_4 & a_2 & a_1 & a_0 \end{bmatrix}
$$

$$
\begin{aligned}
&= -y_2 \times \{\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0^2 - a_3a_2) - (a_2a_0 - a_3a_4)(a_0a_3 - a_1a_2)] \\
&-[(a_0^2 - a_1a_4)(a_1a_0 - a_3^2) - (a_2a_0 - a_3a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0a_4 - a_2^2) - (a_1a_0 - a_2a_4)(a_0a_3 - a_1a_2)]\} \\
&\times\{[(a_0^2 - a_1a_4)(a_0^2 - a_2a_3) - (a_1a_0 - a_2a_4)(a_0a_4 - a_1a_3)] \\
&\times[(a_0^2 - a_1a_4)(a_0^2 - a_4a_1) - (a_3a_0 - a_4^2)(a_0a_2 - a_1^2)]
\end{aligned}
$$

$$-[(a_0^2-a_1a_4)(a_2a_0-a_4a_3)-(a_3a_0-a_4^2)(a_0a_4-a_1a_3)]$$
$$\times[(a_0^2-a_1a_4)(a_0a_3-a_2a_1)-(a_1a_0-a_2a_4)(a_0a_2-a_1^2)]\}$$
$$-\{[(a_0^2-a_1a_4)(a_0^2-a_2a_3)-(a_1a_0-a_2a_4)(a_0a_4-a_1a_3)]$$
$$\times[(a_0^2-a_1a_4)(a_1a_0-a_4a_2)-(a_3a_0-a_4^2)(a_0a_3-a_1a_2)]$$
$$-[(a_0^2-a_1a_4)(a_2a_0-a_4a_3)-(a_3a_0-a_4^2)(a_0a_4-a_1a_3)]$$
$$\times[(a_0^2-a_1a_4)(a_0a_4-a_2^2)-(a_1a_0-a_2a_4)(a_0a_3-a_1a_2)]\}$$
$$\times\{[(a_0^2-a_1a_4)(a_0^2-a_2a_3)-(a_1a_0-a_2a_4)(a_0a_4-a_1a_3)]$$
$$\times[(a_0^2-a_1a_4)(a_0a_4-a_3a_1)-(a_2a_0-a_3a_4)(a_0a_2-a_1^2)]$$
$$-[(a_0^2-a_1a_4)(a_1a_0-a_3^2)-(a_2a_0-a_3a_4)(a_0a_4-a_1a_3)]$$
$$\times[(a_0^2-a_1a_4)(a_0a_3-a_2a_1)-(a_1a_0-a_2a_4)(a_0a_2-a_1^2)]\}\}$$
$$\div\{\{a_0^3(a_0^2-a_1a_4)^2[(a_0^2-a_1a_4)(a_0^2-a_2a_3)-(a_1a_0-a_2a_4)(a_0a_4-a_1a_3)]\}\},$$

and so on.

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