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On the study of an SEIV epidemic model concerning vaccination and vertical transmission

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Abstract

In this work, we study an epidemic model with vaccination and vertical transmission. We get the basic reproduction number R_0 of the system and carry out a bifurcation analysis and obtain the conditions ensuring that the system exhibits backward bifurcation.

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1 Introduction

At present, vaccination is a commonly used method for controlling disease[1,2], but in fact, for many infectious disease, the immunity which is acquired either by preventive vaccine or by infection will wane. In [3] and [4], Moghadas and J. Hui have presented a study of models with non-permanent immunity respectively. Mathematical models including vaccination aim at deciding on a

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vaccination strategy and at determining changes in qualitative behavior that could result from such a control measure[5,6].
Many infectious diseases in nature transmit through both horizontal and vertical modes. These contain such human diseases as rubella, herpes simplex, hepatitis B, and AIDS, etc. Busenberg and Cooke [7] studied a variety of diseases that transmit both horizontally and vertically, and gave a comprehensive survey of the formulation and the mathematical analysis of compartmental models that also incorporate vertical transmission. In this paper, we consider a model not only with non-permanent immunity but also with vertical transmission as following

$$\begin{cases} \frac{dS(t)}{dt} = (1-b)A - \beta SI(1+\alpha I) - \mu S + \omega V - (1-p)\mu I, \\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = bA - \mu V - \omega V + \tau I, \\ \frac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta SI(1+\alpha I) - \mu E - \sigma E, \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \sigma E - \tau I - p\mu I, \end{cases} \tag{1}$$

where $S(t), V(t), E(t)$ and $I(t)$ denote the number of the susceptible individuals, vaccinated individuals, exposed individuals but not yet infectious, and infectious individuals , respectively. All of the parameters are positive and have the following meaning: A is the recruitment rate of people (either by birth or by immigration) into the population (assumed susceptible); b is the fraction of recruited individuals who are vaccinated; β is the rate at which susceptible individuals become infected by those who are infectious; the natural birth rate and death rate are assumed to be identical and denoted by μ; σ is the rate at which exposed individuals become infectious; τ is the rate at which infected individuals are treated; ω is the rate at which vaccine wanes; p is the proportion of the offspring of infective parents that are susceptible individuals.

2 The basic reproduction number

It is easy to see that the region$\{(S,V,E,I)|S > 0, V > 0, E \geq 0, I \geq 0\}$ is positively invariant for the model (1). Summing up the four equations in model (1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(S+V+E+I) = \mu[\frac{A}{\mu} - (S+V+E+I)].$$

Then, $\lim\limits_{t\to\infty}\sup(S+V+E+I) \le \frac{A}{\mu}$.So we study the dynamic behavior of model (1) on the region

$$\Sigma = \{(S,V,E,I)|S>0, V>0, E\ge 0, I\ge 0, S+V+E+I\le \frac{A}{\mu}\},$$

which is a positive invariant set for (1).
Corresponding to $E=I=0$, model (1) always has a disease-free equilibrium, $P_0(\frac{A[\mu(1-b)+\omega]}{\mu(\mu+\omega)}, \frac{bA}{\mu+\omega}, 0, 0)$.
Let $x=(E,I,S,V)^\top$. Then the model (1) can be written as $\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{F}(x) - \mathcal{V}(x)$, where

$$\mathcal{F}(x) = \begin{pmatrix} \beta SI(1+\alpha I) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(x) = \begin{pmatrix} \mu E + \sigma E \\ \tau I + p\mu I - \sigma E \\ -(1-b)A + \beta SI(1+\alpha I) + \mu S - \omega V + (1-p)\mu I \\ \mu V + \omega V - \tau I - bA \end{pmatrix}.$$

We have

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{\beta A[\mu(1-b)+\omega]}{\mu(\mu+\omega)} \\ 0 & 0 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} \mu+\sigma & 0 \\ -\sigma & \tau+p\mu \end{pmatrix},$$

so

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{1}{\mu+\sigma} & 0 \\ \frac{\sigma}{(\mu+\sigma)(\tau+p\mu)} & \frac{1}{\tau+p\mu} \end{pmatrix}.$$

In paper[8], the basic reproduction number is defined as the spectral radius of the next generation matrix $\mathbf{FV^{-1}}(\rho(\mathbf{FV^{-1}}))$. So, according to Theorem 2 in [8], the basic reproduction number of model (1),denoted R_0, is

$$R_0 = \rho(\mathbf{FV^{-1}}) = \frac{\beta\sigma A[\mu(1-b)+\omega]}{\mu(\mu+\omega)(\mu+\sigma)(\tau+p\mu)}.$$

Define

$$R_1 = \frac{2\sqrt{\alpha\mu\beta M[(M-\omega\sigma\tau)+\mu\sigma(1-p)(\mu+\omega)]} - \beta[M-\omega\sigma\tau+\mu\sigma(1-p)(\mu+\omega)]}{\alpha\mu M},$$

$$R_2 = \frac{\beta M - \omega\sigma\beta\tau + \mu\sigma\beta(1-p)(\mu+\omega)}{\alpha\mu M},$$

where $M = (\mu+\sigma)(\mu+\omega)(\tau+p\mu)$.
Remark 2.1. It is easy to see that:
(i)$R_1 \leq 1$;
(ii)$R_2 \leq 1$, if and only if, $R_2 \leq R_1$.

3 Local stability of equilibria and bifurcation analysis

Theorem 3.1. The disease-free equilibrium P_0 is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$.
Proof. The linearized problem corresponding to (1) is $\frac{\mathrm{d}X}{\mathrm{d}t} = JX$, where

$$X = (x_1, x_2, x_3, x_4)^T, (x_1, x_2, x_3, x_4) \in R^4_+,$$

and

$$J = \begin{pmatrix} -\beta I(1+\alpha I) - \mu & \omega & 0 & -\beta S(1+2\alpha I) - (1-p)\mu \\ 0 & -\mu-\omega & 0 & \tau \\ \beta I(1+\alpha I) & 0 & -\mu-\sigma & \beta S(1+2\alpha I) \\ 0 & 0 & \sigma & \tau - p\mu \end{pmatrix}.$$

The Jacobian matrix of (1) at P_0 is

$$J(P_0) = \begin{pmatrix} -\mu & \omega & 0 & -\frac{\beta A[\mu(1-b)+\omega]}{\mu(\mu+\omega)} - (1-p)\mu \\ o & -\mu-\omega & 0 & \tau \\ 0 & 0 & -\mu-\sigma & \frac{\beta A[\mu(1-b)+\omega]}{\mu(\mu+\omega)} \\ 0 & 0 & \sigma & -\tau - p\mu \end{pmatrix}$$

with eigenvalues $\lambda_1 = -\mu, \lambda_2 = -\mu-\omega$, and the roots of the quadratic

$$f(\lambda) = \lambda^2 + (\mu+\sigma+\tau+p\mu)\lambda + (\mu+\sigma)(\tau+p\mu) - \frac{\beta\sigma A[\mu(1-b)+\omega]}{\mu(\mu+\omega)}.$$

Because all the model parameter values are assumed positive, so it follows that $\lambda_1 < 0, \lambda_2 < 0$. Obviously, if $R_0 < 1$ then the roots of $f(\lambda)$ have negative real parts, therefore, P_0 is locally asymptotical stable when $R_0 < 1$; if $R_0 > 1$,then the roots of $f(\lambda)$ are real and one is positive, so that P_0 is unstable.
Theorem 3.2. (a) Let $R_2 < 1$. Then system (1) admits no real equilibria when $R_0 < R_1$, two endemic equilibria for $R_1 < R_0 < 1$,and a unique endemic

equilibrium P^* for $R_0 \geq 1$.
(b) Let $R_2 > 1$. Then system (1) admits no real equilibria when $R_0 < R_1$, no endemic equilibria for $R_1 < R_0 < 1$,and a unique endemic equilibrium P^* for $R_0 \geq 1$.
Proof. The endemic equilibria of system (1) ,denoted $P^*(S^*, V^*, E^*, I^*)$, can be deduces by the system,

$$\begin{cases} (1-b)A - \beta S^* I^*(1+\alpha I^*) - \mu S^* + \omega V^* - (1-p)\mu I^* = 0, \\ bA - \mu V^* - \omega V^* + \tau I^* = 0, \\ \beta S^* I^*(1+\alpha I^*) - \mu E^* - \sigma E^* = 0, \\ \sigma E^* - \tau I^* - p\mu I^* = 0, \end{cases} \tag{2}$$

From (2), we can get $S^* = \frac{(\sigma+\mu)(\tau+p\mu)}{\alpha\beta(1+\alpha I^*)}, E^* = \frac{(\tau+p\mu)I^*}{\sigma}, V^* = \frac{bA+\tau I^*}{\mu+\omega}$, and I^* is positive which satisfies the equation $a_1 I^{*2} + a_2 I^* + a_3 = 0$, where

$$\begin{aligned} a_1 &= \alpha\beta(\omega\sigma\tau - M) - \mu\sigma\alpha\beta(1-p)(\mu+\omega), \\ a_2 &= M(\mu\alpha R_0 - \beta) + \omega\sigma\beta\tau - \mu\sigma\beta(1-p)(\mu+\omega), \\ a_3 &= \mu M(R_0 - 1). \end{aligned}$$

It is easy to see that $a_1 < 0; a_2 > 0 \Leftrightarrow R_0 > R_2; a_3 > 0 \Leftrightarrow R_0 > 1$.
By the Descartes' rules of sings, we can see that when $a_3 > 0$ there is a unique endemic equilibrium ; when $a_3 < 0, a_2 > 0, a_2^2 - 4a_1a_3 > 0$ there are two endemic equilibria, and there are no endemic equilibria otherwise.
Furthermore, we find that there is a bifurcation point when $R_0 = R_1$ i.e., $a_3 < 0, a_2 > 0, a_2^2 - 4a_1a_3 = 0$. In fact,

$$\begin{aligned} a_2^2 - 4a_1a_3 = \big[&\mu\alpha M R_0 + \beta\big(M + \mu\sigma(1-p)(\mu+\omega) - \omega\sigma\tau\big)\big]^2 \\ &- 4\mu\alpha\beta M\big[M + \mu\sigma(1-p)(\mu+\omega) - \omega\sigma\tau\big]. \end{aligned}$$

Thus, $a_2^2 - 4a_1a_3 \geq 0$ whenever $R_0 \geq R_1$. From the above mentioned, (a) and (b) can easily follow.
Let $S = x_1, V = x_2, E = x_3, I = x_4$, the system (1) becomes

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = (1-b)A - \beta x_1 x_4(1+\alpha x_4) - \mu x_1 + \omega x_2 - (1-p)\mu x_4 := f_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = bA - \mu x_2 - \omega x_2 + \tau x_4 := f_2, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = \beta x_1 x_4(1+\alpha x_4) - \mu x_3 - \sigma x_3 := f_3, \\ \frac{\mathrm{d}x_4}{\mathrm{d}t} = \sigma x_3 - \tau x_4 - p\mu x_4 := f_4. \end{cases} \tag{3}$$

We will use the results in [9] to show that system (3) may exhibit a backward bifurcation when $R_0 = 1(\beta = \beta' = \frac{\mu(\mu+\omega)(\mu+\sigma)(\tau+p\mu)}{\sigma A[\mu(1-b)+\omega]})$. The eigenvalues of the matrix,

$$J(P_0,\beta^{'})=\begin{pmatrix} -\mu & \omega & 0 & -\frac{\beta^{'}A[\mu(1-b)+\omega]}{\mu(\mu+\omega)}-(1-p)\mu \\ 0 & -\mu-\omega & 0 & \tau \\ 0 & 0 & -\mu-\sigma & \frac{\beta' A[\mu(1-b)+\omega]}{\mu(\mu+\omega)} \\ 0 & 0 & \sigma & -\tau-p\mu \end{pmatrix},$$

are given by $\lambda_1=-\mu$, $\lambda_2=-\mu-\omega$, $\lambda_3=-(\mu+\sigma+\tau+p\mu)$, $\lambda_4=0$. So $\lambda_4=0$ is a simple zero eigenvalue of the matrix $J(P_0,\beta^{'})$ and the other eigenvalues are real and negative.

We denote a right eigenvector corresponding the zero eigenvalue $\lambda_4=0$ by $\mathbf{w}=(w_1,w_2,w_3,w_4)^T$. It can be deduced by $J(P_0,\beta^{'})(w_1,w_2,w_3,w_4)^T=0$, thus, we have

$$\begin{cases} -\mu w_1+\omega w_2-\big[\frac{\beta^{'}A[\mu(1-b)+\omega]}{\mu(\mu+\omega)}+(1-p)\mu\big]w_4=0, \\ (-\mu-\omega)w_2+\tau w_4=0, \\ (-\mu-\sigma)w_3+\frac{\beta^{'}A[\mu(1-b)+\omega]}{\mu(\mu+\omega)}w_4=0, \\ \sigma w_3+(-\tau-p\mu)w_4=0. \end{cases}$$

It implies $w_1=\frac{\omega\tau\sigma-M-\sigma\mu(1-p)(\mu+\omega)}{\mu(\tau+p\mu)(\mu+\omega)}$, $w_2=\frac{\tau\sigma}{(\tau+p\mu)(\mu+\omega)}$, $w_3=1$,$w_4=\frac{\sigma}{\tau+p\mu}$. Then, the right eigenvector is

$$\mathbf{w}=\big(\tfrac{\omega\tau\sigma-M-\sigma\mu(1-p)(\mu+\omega)}{\mu(\tau+p\mu)(\mu+\omega)},\tfrac{\tau\sigma}{(\tau+p\mu)(\mu+\omega)},1,\tfrac{\sigma}{\tau+p\mu}\big)^{\top}. \tag{4}$$

In the same way, we can get the left eigenvector, denoted $\mathbf{v}=(v_1,v_2,v_3,v_4)$, satisfying $\mathbf{v}\cdot\mathbf{w}=1$ is

$$\mathbf{v}=\big(0,0,\tfrac{\tau+p\mu}{\mu+p\mu+\sigma+\tau},\tfrac{(\tau+p\mu)(\mu+\sigma)}{\sigma(\mu+p\mu+\sigma+\tau)}\big). \tag{5}$$

Evaluating the partial derivatives at P_0, we can get

$$\frac{\partial^2 f_1}{\partial x_1\partial x_4}=\frac{\partial^2 f_1}{\partial x_4\partial x_1}=-\beta,\frac{\partial^2 f_1}{\partial x_4^2}=\frac{-2\alpha\beta A[\mu(1-b)+\omega]}{\mu(\mu+\omega)},$$

$$\frac{\partial^2 f_3}{\partial x_1\partial x_4}=\frac{\partial^2 f_3}{\partial x_4\partial x_1}=\beta,\frac{\partial^2 f_3}{\partial x_4^2}=\frac{2\alpha\beta A[\mu(1-b)+\omega]}{\mu(\mu+\omega)},$$

$$\frac{\partial^2 f_1}{\partial x_4\partial\beta}=\frac{\partial^2 f_1}{\partial\beta\partial x_4}=\frac{-A[\mu(1-b)+\omega]}{\mu(\mu+\omega)},\frac{\partial^2 f_3}{\partial x_4\partial\beta}=\frac{\partial^2 f_3}{\partial\beta\partial x_4}=\frac{A[\mu(1-b)+\omega]}{\mu(\mu+\omega)},$$

all other second-order partial derivatives are equal to zero.
Then, we evaluate the coefficient a and b,

$$\begin{aligned} a &= \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(P_0, \beta^{'}) \\ &= 2v_1 w_1 w_4 \frac{\partial^2 f_1}{\partial x_1 \partial x_4}(P_0, \beta^{'}) + v_1 w_4^2 \frac{\partial^2 f_1}{\partial x_4^2}(P_0, \beta^{'}) \\ &\quad + 2v_3 w_1 w4 \frac{\partial^2 f_3}{\partial x_1 \partial x_4}(P_0, \beta^{'}) + v_3 w_4^2 \frac{\partial^2 f_3}{\partial x_4^2}(P_0, \beta^{'}), \end{aligned}$$

$$b = \sum_{k,i,=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(P_0, \beta^{'}) = 2v_1 w_4 \frac{\partial^2 f_3}{\partial x_4 \partial \beta}(P_0, \beta^{'}) + 2v_3 w_4 \frac{\partial^2 f_3}{\partial x_4 \partial \beta}(P_0, \beta^{'}).$$

Taking into account of (4) and (5), we have

$$a = \frac{2\sigma\beta[\omega\tau\sigma - M - \sigma\mu(1-p)(\mu+\omega) + \sigma\alpha A(\mu(1-b)+\omega)]}{\mu(\tau+p\mu)(\mu+\omega)(\mu+p\mu+\sigma+\tau)},$$

and

$$b = \frac{2\sigma A[\mu(1-b)+\omega]}{\mu(\mu+\omega)(\mu+p\mu+\sigma+\tau)}.$$

Obviously, the coefficient b is positive, so according to the results in [9], the sign of the coefficient a decides the local dynamics around the disease-free equilibrium for $\beta = \beta^{'}$.

Remark 3.5. Let $\alpha^{'} = \frac{M - \omega\sigma\tau + \sigma\mu(1-p)(\mu+\omega)}{\sigma A[\mu(1-b)+\omega]}$, we can get $a > 0$ when $\alpha > \alpha^{'}$. In this case, the direction of the bifurcation of system (1) at R_0 is backward. In fact, when $R_0 = 1$, the condition $\alpha > \alpha^{'}$ is equivalent to the condition $R_2 < 1$. So we have the following theorem.

Theorem 3.6. When $R_0 = 1$, system (1) exhibits a backward bifurcation for $R_2 < 1$; and exhibits a forward bifurcation for $R_2 > 1$.

Theorem 3.7. When $R_0 > 1$, the endemic equilibrium P^* of the system (1) is locally asymptotically stable if $b_3 > 0$ and $b_1 b_2 - b_3 > 0$, where b_1, b_2 and b_3 are presented in the following proof.

Proof. The Jacobian matrix of (1) at P^* is

$$J(P^*) = \begin{pmatrix} -\beta I^*(1+\alpha I^*) - \mu & \omega & 0 & -\beta S^*(1+2\alpha I^*) - (1-p)\mu \\ 0 & -\mu-\omega & 0 & \tau \\ \beta I^*(1+\alpha I^*) & 0 & -\mu-\sigma & \beta S^*(1+2\alpha I^*) \\ 0 & 0 & \sigma & \tau - p\mu \end{pmatrix}.$$

The characteristic equation of the matrix $J(P^*)$ is $(\lambda+\mu)(\lambda^3+b_1\lambda^2+b_2\lambda+b_3) = 0$, where

$$\begin{aligned} b_1 &= \tau + p\mu + 2\mu + \omega + \sigma + \beta I^*(1+\alpha I^*),\\ b_2 &= (\mu+\omega)(\mu+\sigma) + (\tau+p\mu)(2\mu+\omega+\sigma) + \beta I^*(1+\alpha I^*)(\mu+p\mu+\omega+\sigma+\tau)\\ &\quad -\sigma\beta S^*(1+2\alpha I^*),\\ b_3 &= (\mu+\omega)(\mu+\sigma)(\tau+p\mu) + \beta I^*(1+\alpha I^*)[(\mu+\omega)(p\mu+\tau+\sigma)+\sigma\tau]\\ &\quad -(\mu+\omega)\sigma\beta S^*(1+2\alpha I^*). \end{aligned}$$

So

$$\begin{aligned} b_1b_2 - b_3 = {} & [2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)]\tau^2 + \{[2p\mu+2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)][2\mu+\\ & \omega+\sigma+\beta I^*(1+\alpha I^*)] - \sigma\beta S^*(1+2\alpha I^*)\}\tau + [p\mu+2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)]\\ & [(\mu+\omega)(\mu+\sigma)+p\mu(2\mu+\omega+\sigma)+\mu+p\mu+\omega+\sigma-\sigma\beta S^*(1+2\alpha I^*)]\\ & -p\mu(\mu+\omega)(\mu+\sigma) - \beta I^*(1+\alpha I^*)(\mu+\omega)(p\mu+\sigma) + (\mu+\omega)\sigma\beta S^*(1+2\alpha I^*) \end{aligned}$$

Obviously, $b_1 > 0$. Based on Hurwitz criterion, when $R_0 > 1$, the endemic equilibrium P^* of the system (1) is locally asymptotically stable if $b_3 > 0$ and $b_1b_2 - b_3 > 0$.

Theorem 3.8. The system (1) undergos Hopf bifurcation around the positive equilibrium when $R_0 > 1$ and the parameter τ crosses a critical value.

Proof. If Hopf bifurcation takes place, then there exists τ^* satisfied $(i) g(\tau^*) \equiv b_1(\tau^*)b_2(\tau^*) - b_3(\tau^*) = 0, (ii) \frac{\mathrm{d}}{\mathrm{d}\tau} Re(\lambda(\tau)) \mid_{\tau=\tau^*} \neq 0$. The condition $b_1b_2 - b_3 = 0$ is given by $c_1\tau^2 + c_2\tau + c_3 = 0$, where

$$\begin{aligned} c_1 &= 2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*),\\ c_2 &= [2p\mu+2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)][2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)] - \sigma\beta S^*(1+2\alpha I^*),\\ c_3 &= [p\mu+2\mu+\omega+\sigma+\beta I^*(1+\alpha I^*)][(\mu+\omega)(\mu+\sigma)+p\mu(2\mu+\omega+\sigma)+\mu+p\mu+\omega\\ &\quad +\sigma-\sigma\beta S^*(1+2\alpha I^*)] - p\mu(\mu+\omega)(\mu+\sigma) - \beta I^*(1+\alpha I^*)(\mu+\omega)(p\mu+\sigma)\\ &\quad +(\mu+\omega)\sigma\beta S^*(1+2\alpha I^*). \end{aligned}$$

If $\tau = \tau^*$, we can get

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_1b_2 = (\lambda^2+b_2)(\lambda+b_1) = 0, \quad (6)$$

which has three roots $\lambda_1(\tau) = i\sqrt{b_2}, \lambda_2 = -i\sqrt{b_2}, \lambda_3 = -b_1$. For all τ, the roots are in general of the form $\lambda_1(\tau) = \xi_1(\tau) + i\xi_2(\tau), \lambda_2(\tau) = \xi_1(\tau) - i\xi_2(\tau), \lambda_3 = -b_1$.

Substituting $\lambda_1(\tau) = \xi_1(\tau) + i\xi_2(\tau)$ into (6) and calculation the derivative, we have

$$B(\tau)\xi_1'(\tau) - C(\tau)\xi_2' + E(\tau) = 0, C(\tau)\xi_1'(\tau) + B(\tau)\xi_2' + F(\tau) = 0,$$

where

$$\begin{aligned} B(\tau) &= 3\xi_1^2(\tau) + 2b_1(\tau)\xi_1(\tau) + b_2(\tau) - 3\xi_2^2(\tau), \\ C(\tau) &= 6\xi_1(\tau)\xi_2(\tau) + 2b_1(\tau)\xi_2(\tau), \\ E(\tau) &= b_1'(\tau)\xi_1^2(\tau) + b_2'(\tau)\xi_1(\tau) + b_3'(\tau) - b_1'(\tau)\xi_2^2(\tau), \\ F(\tau) &= 2\xi_1(\tau)\xi_2(\tau)b_1'(\tau) + b_2'(\tau)\xi_2(\tau). \end{aligned}$$

Since $C(\tau^*)F(\tau^*) + B(\tau^*)E(\tau^*) \neq 0$, so $\frac{\mathrm{d}}{\mathrm{d}\tau} Re(\lambda_1(\tau)) \mid_{\tau=\tau^*} = -\frac{CF+BE}{B^2+C^2} \mid_{\tau=\tau^*} \neq 0$. By the same way, we can get that $\frac{\mathrm{d}}{\mathrm{d}\tau} Re(\lambda_2(\tau)) \mid_{\tau=\tau^*} \neq 0$. And

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Re(\lambda_3(\tau)) \mid_{\tau=\tau^*} = \frac{\mathrm{d}}{\mathrm{d}\tau} Re(-b_1(\tau)) \mid_{\tau=\tau^*} \neq 0.$$

Thus, the transversal condition holds. This implies that a Hopf bifurcation takes place when $\tau = \tau^*$.

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