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Bayesian Approach to PD Calibration and Stress-testing in Low Default Portfolios

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Abstract

Standard approach to low default portfolio (LDP) probability of default (PD) calibration is to add conservative add-on that should cover the gap with scarce default event data. The most prominent approaches to add-on calibration are based on an assumption about the level of the conservatism (quantile of default event distribution), but there is no transparent way to calibrate it or to relate the level of conservatism to a risk profile of the Bank. Over conservative assumptions can lead to undue shrinkage in LDP and negative shift in the overall risk-profile. Described in the paper PD calibration framework is based on Bayesian inference. The main idea is to calibrate conjugate prior using "closest" available portfolio (CPP) with reliable default statistics. The form of the prior, criteria for CPP selection, application of the approach to real life and artificial portfolios are described in the paper. The advantage of the approach is an elimination of the arbitrary "level of conservatism assumption". The level of conservatism is transparently restricted by CPP portfolio, the general principle is the more data one have for LDP portfolio, the less weight model puts on CPP risk profile. Proposed approach could be also extended for stress-testing purposes.

JEL classification numbers: C01
Keywords: probability of default, credit risk, PD calibration, stress-testing

1 Introduction

Let us assume that there is a low default portfolio (LDP), for which we know for each time period t=1..T the number of borrowers at the beginning of each period n_t and the number of defaulted borrowers (d_t) during each period.
The goal is to estimate expected default rate through the credit cycle (TTC $\overline{PD}$) or so-called Central Tendency (CT) for the portfolio. CT should be non-zero even in case zero default events had been observed in the portfolio.

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Let us also assume that observations are independent between time periods and the number of defaults in a portfolio follows binomial distribution:

$$P(D\ \ defaults\ \ in\ \ porfolio) = \binom{N}{D}\mathrm{pd}^{D}(1-\mathrm{pd})^{\mathrm{N}-D} \tag{1}$$

, where probability of default (pd) is the parameter that we should estimate, $D = \sum_{t=1}^{T} d_t$ and $N = \sum_{t=1}^{T} n_t$ are the total number defaults and borrowers in the portfolio respectively.

Maximum likelihood estimator (MLE) gives us the following answer to (1):

$$pd_{MLE} = \frac{D}{N} \tag{2}$$

In case of LDP portfolios both D and N could be very small numbers, D could be even equal to zero. As will be later proved by Monte-Carlo simulations, MLE estimator could significantly underestimate true default rate. The level of underestimation could be very significant in case of high correlation between default events and short observation periods.

The most widely used approach to tackle pd underestimation problem in LDP was proposed by K.Pluto and D.Tasche [1] (further – P&T model). Generalized rule for PD calibration under original P&T model could be described as search of default rate estimate (PD) under which with the given confidence level (γ) one can reject hypotheses that we are able to observe less than historical number of defaults D:

$$1-\gamma \le P_{PD}[Less\ \ than\ \ 'D'\ \ defaults\ \ observed] \tag{3}$$

In case of assumption of independent default events (1), (3) could be expressed as:

$$1-\gamma \le \sum_{i=0}^{D}\binom{N}{i}\mathrm{pd}^{i}(1-\mathrm{pd})^{\mathrm{N}-i} \tag{4}$$

This approach could be extended to correlated defaults case. Following [2], change in the company's assets V_t in year t could be modelled as:

$$\mathrm{V}_t = \sqrt{\rho}\mathrm{S}_t + \sqrt{1-\rho}\xi_t \tag{5}$$

where ρ stands for the so-called asset correlation, S_t is the realization of the systematic factor in year t, and ξ_t denotes the idiosyncratic (or borrower-specific) component of the change in asset value. The cross-sectional dependence of the default events stems from the presence of the systematic factor S_t . Both systematic and idiosyncratic factors are standard normally distributed, idiosyncratic factors are i.i.d., while joint distribution of S_t is multivariate normal and therefore is completely determined by the correlation matrix.

Borrower defaults in year t if assets change in year t falls below threshold c:

$$V_t < c \tag{6}$$

where default threshold c could be calibrated from unconditional PD:

$$c = \Phi^{-1}(pd) \tag{7}$$

with Φ denoting the standard normal distribution function.

Following [3], probability of a default, given particular realization of systematic factor $\mathrm{S_t}$ is:

$$\mathrm{G}(\mathrm{pd}, \varrho, \mathrm{S_t}) = \Phi\left(\frac{\Phi^{-1}(\mathrm{pd}) - \sqrt{\rho}\mathrm{S_t}}{\sqrt{1-\rho}}\right) \tag{8}$$

Under assumption that default events are conditionally independent given particular realization of systematic factor, inequality (4) becomes:

$$1-\gamma \leq \int_{-\infty}^{+\infty} \sum_{i=0}^{D} \binom{N}{i} \mathrm{G}(\mathrm{pd}, \varrho, S)^{i} \left(1-\mathrm{G}(\mathrm{pd}, \varrho, S)\right)^{\mathrm{N}-i} \phi(S) dS \qquad (9)$$

, where ϕ is a standard normal density function.

Infimum of solutions to the inequality (9) will give us required pd estimate of the Central Tendency for portfolio.

According to [1], the approach could be extended to multi-period case, but as shown in [4], multi-period case is very sensitive to renewal of the portfolio and therefore could give too volatile results. Therefore, further in the article simple multi-period version of the approach (so-cooled Pooled approach) is used. According to Pooled approach, observation within the time periods are treated as independent and therefore aggregated to one time window (omitting $\mathrm{S_t}$ time dependence).

Another model, proposed in [5], is based on Bayesian inference. The main idea of the approach is to apply uninformed or conservative prior in order to add conservatism to PD estimates. The author also demonstrates that in the case of independent default events the upper confidence bounds (P&T model), can be represented as quantiles of a Bayesian posterior distribution based on a prior that is slightly more conservative than the uninformed prior.

Bayesian estimator approach, proposed in [5], has the same drawbacks, - there is no clear guidelines how to choose the prior in order to get the reasonable level of conservatism or the level of conservatism that is connected to the risk profile of a bank. Due similarity and coincidence with the P&T (in case of uniform prior) this approach is not analyzed in the article separately.

Another approach to PD estimation in LDP portfolios could be based on so-called «duration» treatment of migration matrixes (see [6] for details). The core of the approach is $R \times R$ generator or intensity matrix Λ. Based on generator matrix, migration probability matrix $M(t)$ for a given term t could be found as:

$$M(t) = e^{\Lambda t} \qquad (10)$$

where the exponential is a matrix exponential, and the entries of Λ satisfy $\lambda_{ij} \geq 0 \ \forall\, i \neq j$; $\lambda_{ii} = -\lambda_i = -\sum_{i \neq j} \lambda_{ij}$. These entries describe the probabilistic behaviour of the holding time in state i as exponentially distributed with parameter λ_i, where $\lambda_{ii} = -\lambda_i$ and the probability of jumping from state i to j is given by $\frac{\lambda_{ij}}{\lambda_i}$ (11).

Even in case of zero default events in a given rating class, since there are migration to worse rating classes, the approach should produce non-zero PD estimates.

The main disadvantage of the approach is that it lacks any level of conservatism and has serious restrictions:

- It couldn’t be used for standalone portfolios that are covered by a specialized rating model - only low default rating classes of «normal» portfolios could be covered by this methodology.
- Long history of a consistent ranking model application should be in place in order to estimate (10).

2 PD Calibration Framework

Proposed in the article approach (further – CPP approach) is based on principles of Bayesian inference with the following assumptions:

1) Conjugate prior (beta distribution) to binomial default distribution is used.
2) Prior distribution is calibrated from the default rate statistics of the «closest possible portfolio» (further – CPP), which should have reliable default statistics and from economic point of view should be maximally close to LDP portfolio.

The beta prior has the following form:

$$Beta(pd|a,b) = \frac{1}{B(a,b)} pd^{a-1} pd^{b-1} \tag{12}$$

where $B(a,b)$ is a beta function.

Generally, the posterior distribution of the default rate estimate (pd) is:

$$p(pd|\mathcal{D}) = \frac{p(pd|\mathcal{D})p(pd)}{p(D)} \tag{13}$$

In case of assumptions of binomial default distribution (1), beta distributed prior (12) and given defaults statistics for LDP and CPP portfolios ($\mathcal{D}_{LDP}$ *and* $\mathcal{D}_{CPP}$ respectively), the posterior distribution is:

$$p(pd|\mathcal{D}_{LDP},\mathcal{D}_{CPP}) \propto p(\mathcal{D}_{LDP}|pd)p(pd|\mathcal{D}_{CPP}) \propto$$
$$Bin(D|pd,N)Beta(pd|a,b) \propto$$
$$Beta(pd|a+D,N-D+b) \tag{14}$$

Following [7], the mean of the posterior distribution (14) could be estimated as:

$$\overline{pd} = \frac{a+D}{a+b+N} \tag{15}$$

It also could be shown that posterior mean is convex combination of the prior mean and the MLE of LDP portfolio:

$$\mathbb{E}(pd|\mathcal{D}) = \frac{\alpha m + D}{N+\alpha} = \frac{\alpha}{N+\alpha}m + \frac{N}{N+\alpha}\frac{D}{N} = \lambda m + (1-\lambda)\frac{D}{N} \tag{16}$$

where $\alpha = a + b$ is an equivalent sample size of the prior, $m = a/\alpha$ is the prior mean and the “weight” of the prior is:

$$\lambda = \frac{\alpha}{N+\alpha} = \frac{a+b}{N+a+b} \tag{17}$$

More data about LDP we have, the more important MLE becomes since the “weight” of a prior reduces.

The CPP calibration approach, proposed in this article, consists of the following steps:

1) Find the CPP portfolio, that satisfies the following requirements:
   - Default statistics is enough for PD calibration (according to internal validation or regulatory requirements).
   - From the economic point of view, risk drivers for LDP and CPP portfolios should be simmilar (e.g. financial sector companies is a bad CPP for large corporate portfolio since the risk drivers and their level/speed of influence could be quite different).
   - From the economic point of view, LDP portfolio should be at least slightly risky (the central tendency should be higher) than CPP portfolio (e.g. sub-investment grade corporate portfolio could be a good CPP for investment-grade corporate portfolio).
2) Calibrate the parameters to of the prior (12) to historical default rate of the CPP portfolio using MLE o approach.

3) Use estimator (15) to get desired pd value (mode or quantile of the posterior could be also used as an estimators).
4) Apply variable dispersion beta regression model in order to get dependence between prior (12) parameters and macro-variables for stress-testing and point at time pd calibration purposes.

The main challenge of the approach is to find CPP portfolio. The following ideas/examples could be used as guidlines:
1) In case we have to estimate pd for a «high» rating grade category, we can extend the sample up to the rating grades where default events are enough to pass the validation tests for pd estimation. For example, pd for AAA rated counterparties could be estimated using prior calibrated from statistics of counterparties rated from AA up to speculative grades.
2) In case we should estimate Central Tendency for a LDP portfolio, covered by specialized ranking model, segmentation criteria could be relaxed. For example, portfolio of companies with more than 1 bln. USD annual revenue, default statistics of the companies with revenue from 100 mln. USD up to 1 bln. USD could be used as prior.

Beta distribution as a prior has the following properties:
- It is a conjugate prior to (1) and, therefore, allow us effective and simple posterior mean estimation.
- The weight of the prior depends on the level and stability of DR estimates and do not depend on the number of observations in CPP (CPP can dramatically over wait the LDP by number of observations).
- Beta prior can be regressed on macro-variables, so the model can be used seamless for stress-testing purposes. The advantage of variable dispersion beta regression (VDBR) model (see [8] for details) over classical regression model is the ability to predict mean and accuracy of estimates simultaneously depending on different covariates. Therefore, VDBR allows us to model not only the expected increase in PD level, but also the shift of our uncertainty in our estimate given stress situation.

The CPP approach has the following properties:
- The level of conservatism is quite transparent: by using prior we assume that the LDP portfolio is by default not less risky than the closest portfolio for which we have reliable pd estimate.
- The more data we have for LDP portfolio the more wait we will put to LDP data and less to the prior, moreover, as shown below, the wait of the prior could be estimated directly.
- It's very likely that the LDP and CPP portfolios are influenced by the same systematic factors, which contributes the accuracy of estimates.
- It's very likely that the LDP and CPP portfolios are influenced by the same bank's risk appetite policy and strategy.

Further, the results of application of estimators (2), (9) and (15) will be shown on artificial and real data sets.

3 Application of the Framework for Stress-testing Purposes

For stress-tested purposes, shifted prior (12) could be used. The shift could be calibrated using variable dispersion beta regression (VDBR) model (see [8] for details).

In order to apply VDBR model we have to reparametrize the prior (12) in the following way:

$$B(y;\mu,\phi)=\frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma\big((1-\mu)\phi\big)}y^{\mu\phi-1}(1-y)^{(1-\mu)\phi-1} \tag{18}$$

where $\mu=\frac{a}{a+b}, \phi=a+b.$

$\mathbb{E}(y)=\mu$ and $VAR(y)=\mu(1-\mu)/(1+\phi)$, therefore parameter ϕ is known as precision parameter, since for fixed μ, the larger ϕ the smaller the variance of y.

The definition of the VDBR model, given the parametrization (18), is: let the observed default rate of CPP portfolio y_i in year i=1…T is distributed as $B(\mu_i,\phi_i)$ independently and:

$$g_1(\mu_i)=x_i^T\beta \tag{19.1}$$

$$g_2(\phi_i)=z_i^T\gamma \tag{19.2}$$

where β and γ are vectors of regression coefficients in the two equations, x_i and y_i are regressor vectors of macro-variables or other risk drivers, $g_1$and g_2 are link functions (for example, logit).

After we fit model (19.1), (19.2), for example, using MLE approach, in order to apply conditional on macro-variables prior (12), we have to invert re-parametrization of beta distribution (18):

$$a_s=g_1^{-1}(x_s^T\beta)g_2^{-1}(z_s^T\gamma) \tag{20.1}$$

$$b_s=g_2^{-1}(z_s^T\gamma)(1-g_1^{-1}(x_s^T\beta)) \tag{20.2}$$

where x_s and z_s are given by stress macro-variables or other stressed risk drivers

Plugging conditional beta parameters into equation (15) we get stressed pd estimate:

$$\overline{pd_s}=\frac{a_s+D}{a_s+b_s+N} \tag{21}$$

One of the possible obstacles to the this approach is a variable or even negligible equivalent sample size of the conditional prior $\alpha_s=a_s+b_s$. One of the simplest mitigations to the problem is a fixation of the prior weight (17) according to thought the cycle calibration.

Since conservative assumptions are always welcomed in stress-testing models, quantiles (e.g. $\eta=99\%$ or 99.5%) of the prior instead of mean could be used in equation (16) in order to capture uncertainty of our estimates in rare stress situations. Quantiles of the beta distribution will be directly influenced by the values of the second part VDBR model (19.2).

$$pd_s^\eta=\lambda QB(\eta,a_s,b_s)+(1-\lambda)\frac{D}{N} \tag{22}$$

where QB is a quantile function of beta distribution with conditional parameters a_s, b_s.

4 Monte-Carlo Study: Comparison of Approaches

Let us assume that we have two portfolios, the first one is LDP with central tendency pd^{LDP} and the second portfolio with central tendency pd^{CPP}, for which we reliable default statistic. The second portfolio could be treated as CPP to LDP portfolio.
The number of borrowers in all periods $t = 1 \ldots T$ is constant and equal to N_{LDP} and N_{CPP} respectively.
The probability of default for each borrower in each period is given by (8), where the systematic factor $\mathrm{S_t}$ and asset correlation ρ_t is common for both portfolios in each period. The distribution of $\mathrm{S_t}$ is determined by correlation matrix with power ϑ time dependence structure:
$s_{i,j} = \vartheta^{\max(i,j)-\min(i,j)}$
Asset correlation value has random and constant (ρ_{base}) parts. The random part depends on the realization of systematic factor in order to capture effect of higher market correlations during stress events. As the result, in each period ρ_t is determined by the following formula:
$\rho_t = \rho_{base} + \rho_{base}\Phi(\mathrm{S_t})$
where Φ is the standard normal distribution function.

General schema of Monte-Carlo simulations is the following:
1) Simulate $\mathrm{S_t}$ and ρ_t for each period $t = 1 \ldots T$.
2) Using (8) and pd^{CPP}, pd^{LDP} values - determine conditional on $\mathrm{S_t}$ probability of default (pd) in each period (CPP and LDP portfolios share the same values of and $\mathrm{S_t}$ and ρ_t).
3) Simulate using uniformly distributed random variables defaults in each portfolio.
4) Apply estimators (2), (9[2]) and (15) to simulated dataset.
5) For each estimator % of underestimated cases ($pd^{LDP} < pd^{Estimated}$) and mean absolute error $\left|\frac{pd^{LDP} - pd^{Estimated}}{pd^{LDP}}\right|$ (MAE) are computed.

Monte-Carlo simulations were held for 3 different CPP portfolios, for each CPP portfolio 3 different values of ρ_{base} were used:
- Independent assets dynamics assumption: $\rho_{base} = 0$.
- Basel II range of possible correlation values: $\rho_{base} = 12\%$, therefore ρ_t is within the Basel II ([9]) range $12\% \leq \rho_t \leq 24\%$.
- Ultra-high correlation range: $\rho_{base} = 24\%$, $24\% \leq \rho_t \leq 48\%$.

Number of observed periods and time dependence parameter for systematic factor are constant for all portfolios $T = 8$, $\vartheta = 0.3$.
Parameters of LDP portfolio are constant: $pd^{LDP} = 0.001$, $N_{LDP} = 100$, therefore the portfolio is low default due to low expected default rate and low number of observations simultaneously.
The first CPP portfolio (CPP №1) has following parameters $N_{CPP} = 1000, pd^{CPP} = 0.01$, it has proportionally higher number of observations and expected default frequency than

[2]Confidence level of 0.9 and mean value of asset correlation $\rho_{Tasche} = 1.5 * \rho_{base}$ were used

LDP (10 times higher). Simulation results are provided in the Table 1.

Table 1: Results for CPP №1

	$\boldsymbol{\rho_t = 0}$		$\boldsymbol{12\% \le \rho_t \le 24\%}$		$\boldsymbol{24\% \le \rho_t \le 48\%}$	
	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %
Mean (2)	90%	45%	115%	53%	141%	67%
CPP (15)	544%	0%	129%	26%	118%	63%
P&T (9)	341%	0%	1965%	0%	5742%	0%

The second CPP portfolio (CPP №2) has $N_{CPP} = 5000, pd^{CPP} = 0.01$, this artificial portfolio should provide information regarding sensitivity of the approach to a significant shift (5 times) in N_{CPP}. Simulation results are provided in the Table 2.

Table 2: Results for CPP №2

	$\boldsymbol{\rho_t = 0}$		$\boldsymbol{12\% \le \rho_t \le 24\%}$		$\boldsymbol{24\% \le \rho_t \le 48\%}$	
	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %
Mean (2)	90%	47%	111%	50%	140%	65%
CPP (15)	789%	0%	140%	10%	114%	58%
P&T (9)	333%	0%	1979%	0%	5770%	0%

The third CPP portfolio (CPP №3) has $N_{CPP} = 1000, pd^{CPP} = 0.05$, this artificial portfolio should provide information regarding sensitivity of the approach to a significant shift (5 times) in pd^{CPP}. Simulation results are provided in the Table 3.

Table 3: Results for CPP №3

	$\boldsymbol{\rho_t = 0}$		$\boldsymbol{12\% \le \rho_t \le 24\%}$		$\boldsymbol{24\% \le \rho_t \le 48\%}$	
	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %	MAE, %	Under - estimation, %
Mean (2)	88%	43%	114%	51%	161%	66%
CPP (15)	3009%	0%	260%	1%	158%	29%
P&T (9)	343%	0%	1982%	0%	5896%	0%

Pictures of smoothed densities of estimators and true central tendency values are provided in Appendix 1.

Mean approach (2), for risk management purposes, has the worst results, since it has a clear wrong way risk pattern: the higher the level of correlation the stronger is the underestimation bias for central tendency. Mean estimator always has wiggly pattern (see Appendix 1) since expected number of defaults for all periods is less than one. The other disadvantage is frequent zero central tendency estimates.

One can see that P&T model (9) produces very wiggle (see Appendix 1) estimates since each additional observed default provides significant jump estimated pd value. Moreover, the «magnitude» has high dependence on a confidence level and correlation value. Therefore, the risk profile of the portfolio could be dramatically changed by arbitrary events, such as zero or one default occurrence and the choice of confidence interval.

CPP (or Beta prior) approach (15) is the most conservative for zero correlation case, since default rate volatility in CPP portfolio is very low (due to $\rho_{base} = 0$) and therefore the power of the prior is very high. Because beta prior is fitted to observable default rate in CPP portfolio, sensitivity to disproportion in N_{CPP} and N_{LDP} is low, but the dependence on change in pd^{CPP} is almost linear.

Given the more realistic assumption of correlation range $12\% \leq \rho_t \leq 24\%$, the results of P&T model become very conservative, while CPP approach has reasonable level of conservatism for CPP №1 and CPP №2 and slightly over conservative for CPP №3 (due to 15 times disproportion between LDP and CPP CTs). The level of conservatism is almost independent on the number of borrowers in CPP portfolio. On average, CPP approach has 8 times more accurate estimates than P&T model. Moreover, the level of conservatism under Beta estimator is always restricted by the risk of CPP portfolio and therefore is measurable, understandable and could not be unreasonably high.

For extreme correlation range $24\% \leq \rho_t \leq 48\%$, P&T model is unreasonable conservative, while Beta estimator still has reasonable results for CPP №2, CPP №3 and underestimates risks for CPP №1. Given relatively low central tendency, number of borrowers and just 8 time observation points, CPP №1 can hardly pass validation tests for reliable PD estimates in case of extremely high correlations and therefore, probably, could not be used as CPP portfolios.

As the result, CPP approach could be overly conservative, in case of zero correlation case. In case of «real life» level of asset correlation, Beta approach has reasonable level of conservatism even with CPP portfolios that are 10-15 times more risky. The level of conservatism is significantly lower than in P&T model with 90% confidence level. The sensitivity to the population of CPP portfolio is relatively low (by construction), while the dependence on the central tendency of CPP is very significant, but restricted. If CPP portfolio has enough observations for reliable PD estimation or significant margin of conservatism, CPP approach performs well even in case of extremely high level of correlation.

5 Real Life Example

The task is to estimate central tendency for Aaa rating class given default statistics provided by Moody's Investor Service [10]. Number of observations n_t^r and number of defaults n_t^r by rating classes $r = [Aaa, Aa, A, Baa, Ba, B, C]$ is available since 1920. Nevertheless, due to economic development and shifts it's reasonable to restrict the sample to one or two most recent credit cycles. Since the definition of global credit cycle is very obscure, let's assume the time frame for our task should be restricted by $t = 1998 \ldots 2015$.

To be on a conservative side, let's extend definition of CPP portfolio up to 'Highly speculative' grade B (including). Inclusion of 'Extremely speculative' grade C could be treated as overly conservative. Moreover, 'Extremely speculative' grades could be driven by different economic forces than Investment and Speculative grade portfolios. Therefore, CPP consist of the following rating grades: $[Aa, A, Baa, Ba, B]$.

The results of the application P&T model (9) and Beta (15) estimators are provided on

Figure 1 and Table 4.

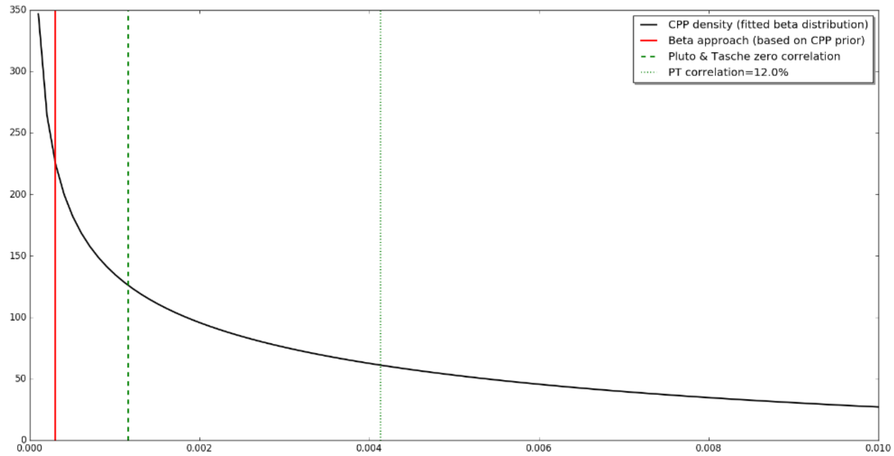

Figure 1: Aaa rating PD calibration results.

Table 4: Aaa rating PD calibration results.

Parameter	**Value**
Number of observed defaults in LDP portfolio	0
Mean default rate in CPP portfolio	0.75%
$\frac{N_{CPP}}{N_{LDP}}$	43.82
MLE fitted prior parameters (a,b)	(0.62, 82)
The weight of the prior (17)	4%
P&T model (zero correlation assumption)	0.11%
P&T model (12% correlation assumption)	0.41%
CPP estimator	0.03%

One can see that the CPP approach is significantly less conservative than P&T model and accidently coincides with Basel II [9] minimum pd value threshold.
Using information from about World (WLD) GDP values[3] (GDP (current US$)) and inflation adjusted oil prices[4], one can try to relate the dynamics of these indicators with the default rate of CPP portfolio for stress-testing purposes (22).
For fitting purposes 'betareg' package was used [11]. The goal of the analysis was not to find the best statistical model for CPP pd prediction, but to demonstrate how the approach (22) could work in practice.
Figure 11 in Appendix 2 provide us information about result of 'betareg' fitting procedure if we try to fit both (19.1) and (19.2) using GDP and Oil dynamics. Let us exclude Oil from (19.2) due to absence of the clear hypnoses about influence of Oil price dynamics on Global default rate (individual correlation of Oil price dynamics and default rates is negative, while

[3]The World Bank database http://databank.worldbank.org/
[4]http://inflationdata.com/inflation/inflation_rate/historical_oil_prices_table.asp

in multivariable fitting procedure the sign of the coefficient is positive) and GDP dynamics from (19.1) due to non-intuitive sign of the coefficient.
Figure 12 in Appendix 2 provide us information about model with restricted set of predictors. All predictors are significant and have economic intuitive sign.
As the result, mean value of the stressed pd is driven by GDP dynamics scenario, while Oil price dynamics has direct influence on the uncertainty of our estimates (and therefore on the quantile).
Using earlier estimated weight of the prior and formulas (21), (22), we could derive conditional pd for rating class Aaa in a given scenario. Results are summarized in Table 5[5].

Table 5. Stress-testing results.

Scenario		**Risk**	
GDP	Oil	Mean	Quantile 0.95
0%	0%	0.06%	0.15%
-3%	-10%	0.09%	0.41%
-3%	+50%	0.09%	0.42%

6 Conclusion

The main assumption of the proposed Beta approach is in similarity of risk characteristics between LDP and more statistically stable CPP portfolio, therefore the level of conservatism of the approach is economically transparent and restricted by the level of risk in CPP portfolio. By construction of the model, the level of conservatism is predominantly driven by the riskiness of the CPP portfolio, the influence of disproportion in number of observations is less pronounced, especially in case of presence of assets correlation between borrowers. The other property of the approach is that the more statistics we get for LDP portfolio, the less influence CPP portfolio has.
Absence of dramatic conservatism increase due to asset correlations between borrowers is a very strong property of the approach, especially for highly volatile economies of developing countries.
The approach has very limited requirements for computing power since all steps, except beta distribution fitting, has closed form solutions. Moreover, in case of simplified approach to betta fitting procedure (methods of moments), all steps could be done in Excel spreadsheet.
The approach could be easily extended for stress-testing purposes and point in time PD assessment (for example, for IFRS 9 purposes).

[5]The influence in Oil dynamics change is dismal, nevertheless it demonstrates additional flexibility of the model.

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[11] Zeileis A, Cribari-Neto F, Gruen B and Kosmidis I, "betareg," 2012. [Online]. Available: http://cran.r-project.org/web/packages/betareg/betareg.pdf. [Accessed 01 12 2016].

Appendix

Appendix 1:

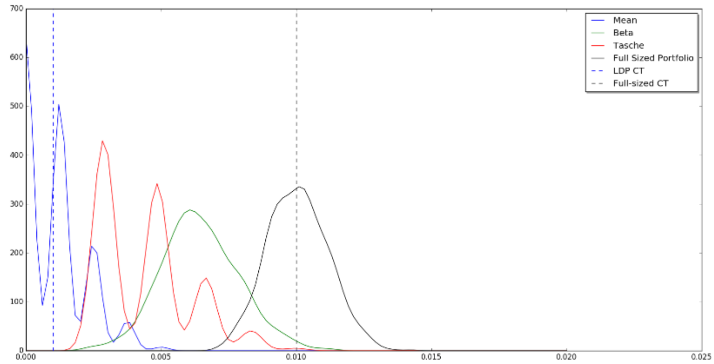

Figure 2: CPP №1 (ρ_{base}= 0%)

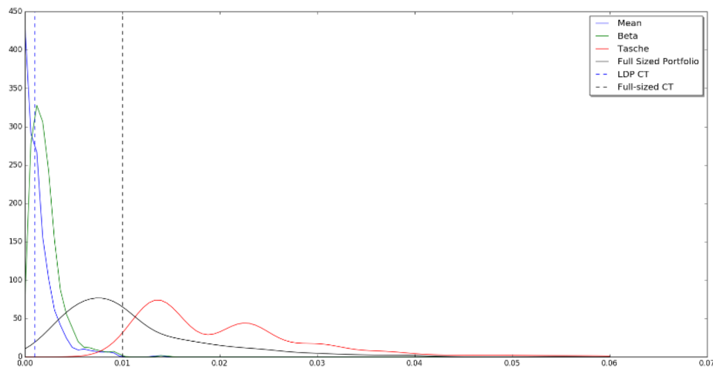

Figure 3: CPP №1 (ρ_{base}= 12%)

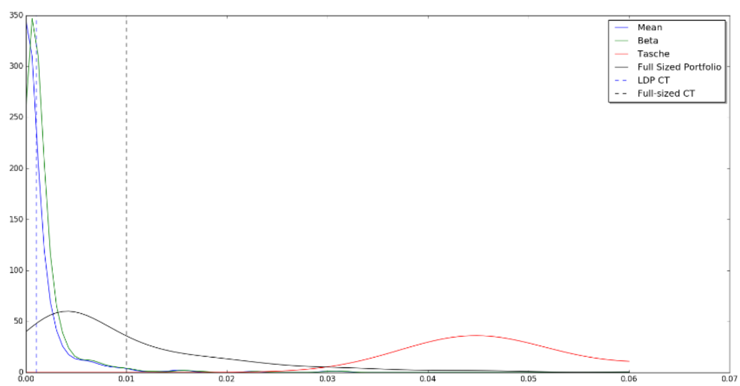

Figure 4: CPP №1 (ρ_{base}= 24%)

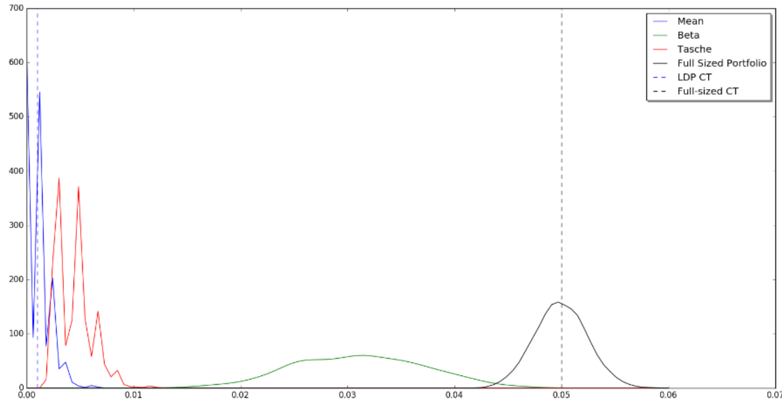

Figure 5: CPP №2 (ρ_{base}= 0%)

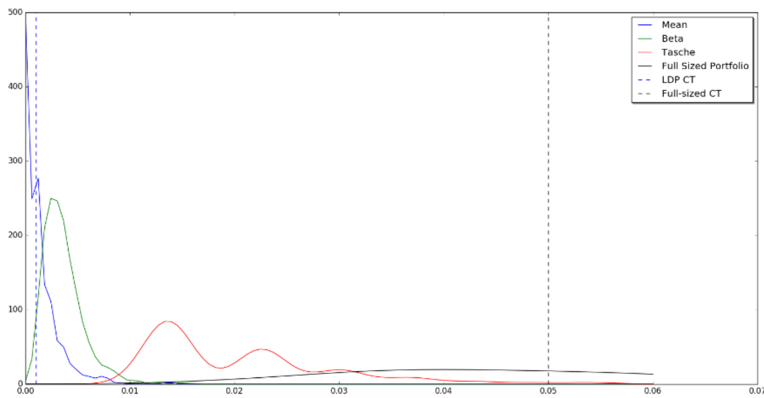

Figure 6: CPP №2 (ρ_{base}= 12%)

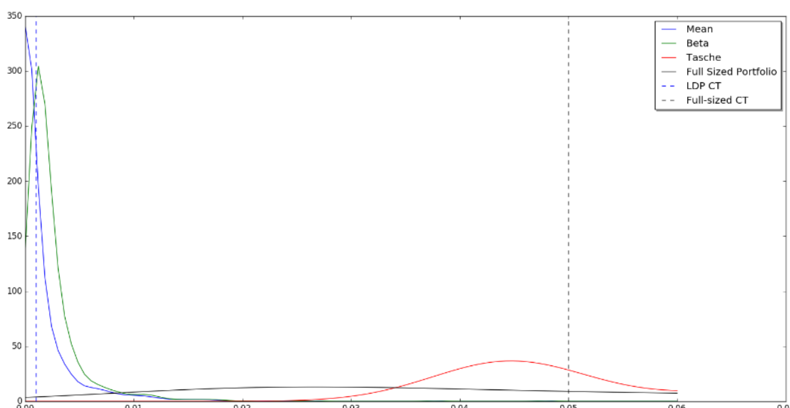

Figure 7: CPP №2 (ρ_{base}= 24%)

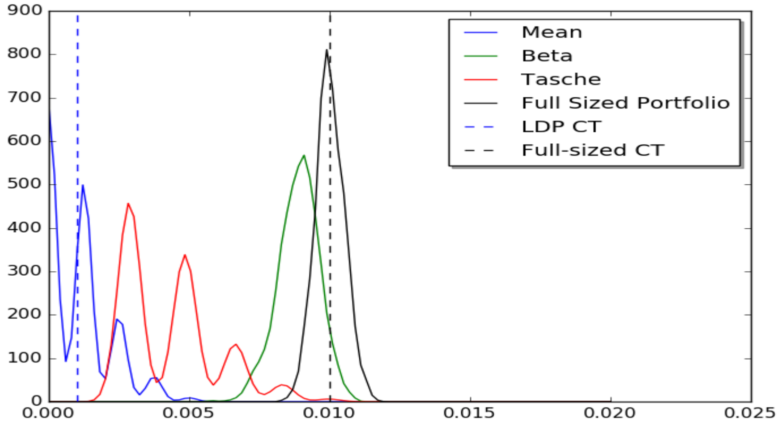

Figure 8: CPP №2 (ρ_{base}= 0%)

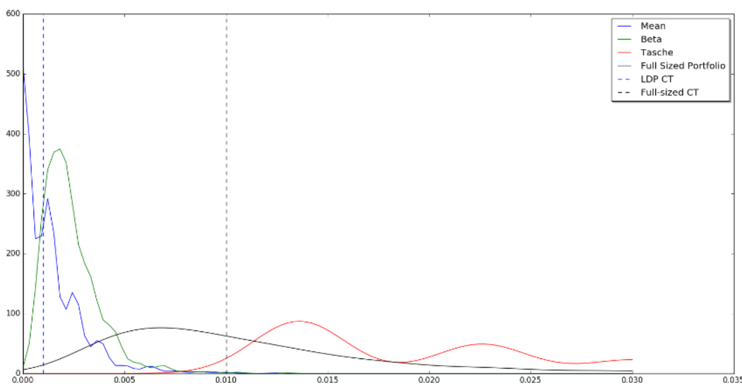

Figure 9: CPP №2 (ρ_{base}= 12%)

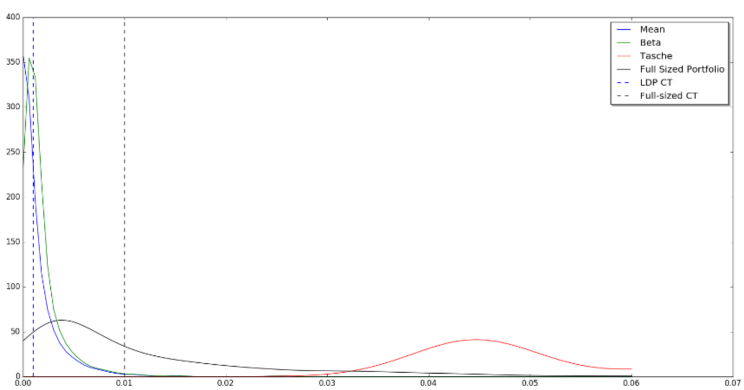

Figure 10: CPP №2 (ρ_{base}= 24%)

Appendix 2

```
betareg(formula = SpecDR ~ GDP + Oil | GDP + Oil, data = dat)

Standardized weighted residuals 2:
    Min      1Q  Median      3Q     Max 
-1.8664 -0.8938  0.1481  0.8228  1.6286 

Coefficients (mean model with logit link):
            Estimate Std. Error z value Pr(>|z|)   
(Intercept)   6.1166     4.3737   1.398  0.16197   
GDP         -11.7530     4.4229  -2.657  0.00788 **
Oil           1.1835     0.4275   2.768  0.00564 **

Phi coefficients (precision model with log link):
            Estimate Std. Error z value Pr(>|z|)   
(Intercept)   19.088      8.019   2.380  0.01730 * 
GDP          -17.667      8.633  -2.046  0.04071 * 
Oil            4.458      1.589   2.805  0.00503 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Figure 11: Output ‘betareg’ package for full set of predictors

```
betareg(formula = SpecDR ~ GDP | Oil, data = dat)

Standardized weighted residuals 2:
    Min      1Q  Median      3Q     Max 
-3.0904 -0.6652  0.1464  0.7313  1.4934 

Coefficients (mean model with logit link):
            Estimate Std. Error z value Pr(>|z|)   
(Intercept)    8.379      4.152   2.018  0.04359 * 
GDP          -12.531      3.982  -3.147  0.00165 **

Phi coefficients (precision model with log link):
            Estimate Std. Error z value Pr(>|z|)  
(Intercept)    2.123      1.270   1.672   0.0946 .
Oil            2.432      1.167   2.084   0.0372 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Figure 12: Output ‘betareg’ package for restricted set of predictors