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The Least-squares Monte Carlo method for pricing options embedded in mortgages

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Abstract

This paper studies the pricing problems for options embedded in fixed rate mortgages by simulation. The least-squares Monte Carlo method, which was initiated by Longstaff and Schwartz (Rev. Financ. Stud. 14(1): 113-147, 2001), is applied to price the mortgage default and prepayment options in a financial environment with two stochastic factors: house price and short term interest rate. A series of numerical comparisons for presented methods with the PDE analytical approximation method in (IAENG Int. J. Appl. Math. 39(1): 9, 2009) and the binomial tree method (BTM) (Decis. Econ. Financ. 35(2): 171-202, 2012) are given. The simulation experiments show the efficiency of presented methods and some cross-validation of the obtained simulation results are given.

JEL Classification: C61; C63; G21
Keywords: Least-squares Monte Carlo method; Options embedded in mortgages; Optimal stopping; Dynamic programming

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1 Introduction

A mortgage is a loan that is secured by an underlying real estate property, and a mortgage-backed security is a derivative security based on the cash flows generated by a pool of mortgages. The valuation of a mortgage embedded with its prepayment and default risks is a difficult but widely discussed problem. In this work, we consider such problem from the mortgage borrower's perspective. The fundamental assumption is that when a mortgage payment is due, the borrower will maximize his utility by choosing one of the following options: continuing the scheduled payment, prepayment or default. We focus on the option-based model to be the utility function.

There are many studies in the research of the numerical methods for the default and prepayment option valuation. In summary, pricing of the above-mentioned options can be classified into four categories: analytical method, recombining binomial and trinomial trees [3], finite difference methods [2], and Monte-Carlo simulation methods. Kau *et al* in [9] (1995) applied the implicit difference numerical method in the default option valuation under a two-factor model. Some promising recent attempts by Xie in [14] have been made to obtain an analytical approximation for both of optimal prepayment rate and callable fixed rate mortgage price within the Vasicek and the Cox-Ingersoll-Ross model. Hurlimann in [7] (2012) considered the recombining binomial tree approach to value fixed and variable rate mortgage. On the other hand, In an influential paper [11], Longstaff and Schwartz (2001) introduced the *least-squares Monte Carlo* (LSM) method to solve the American option value through a set of conditional expectations estimated by the least square regressions of the future financial benefits.

In this paper, we mainly apply the LSM method to price the mortgage default and prepayment options in a financial environment with two stochastic factors: house price and short term interest rate. Also, by a series of numerical tests, we compare this method with the analytical method and the binomial tree method. Our works differ from the previous literature of prepayment and default option valuation in following respects. We present a two steps LSM algorithm to solve the optimal stopping problem embedded in fixed rate mortgages. On the other hand, we not only consider the geometric Brownian motion model for the underlying dynamics of house price, but also a jump diffusion model.

The paper is organized as follows. In Section 2, the financial background is introduced, and the related mathematical models are setup. In Section 3, the valuation of a callable mortgage with the risk of prepayment is considered, and the pricing problems with both default and prepayment risks in a financial environment with two stochastic factors are discussed. In Section 4, the Least-Squares Monte Carlo methods for pricing default and prepayment options embedded in mortgages are presented, and the corresponding algorithms are also given. In Section 5, some numerical results are given to support the presented algorithms, and some conclusions about our study are drawn.

2 Amortized fixed rate mortgage

Consider an *amortized fixed rate mortgage* (AFRM) contract on an underlying good, which can be a house or a more general physical good. In an AFRM, the *borrower* borrows from the lender (usually a bank) a capital equal to P at a fixed nominal instantaneous rate $\rho > 0$ at time 0, and pays it back with a continuous intensity A in the time window $[0, T]$, where T is the maturity of contract. We distinguish between a continuous amortization rate A per unit time and a recurring discrete amortization payment A_p at the end of each interest payment cycle of length I_P. We assume that the interval $[0, T]$ is divided into N discrete time steps $[(i-1)h, ih]$, $i = 1, \ldots, N$, with the length $h = T/N$. Set $n_P = T/I_P$ as the number of recurring payments, and $n_I = I_P/h$ as the number of discrete steps in each interest payment cycle. Let $\rho_N = e^{\rho h} - 1$ be the interest rate per discrete time step and let $\rho_P = \rho_N n_I$ be the interest rate per interest payment cycle. In the continuous time setting, we have

$$P = A \int_0^T e^{-\rho t} \mathrm{d}t = A \frac{1 - e^{-\rho T}}{\rho}.$$

In practical, P is calculated by a discrete time formula:

$$P = A_P \sum_{k=1}^{n_P} (1 + \rho_P)^{-k} = \frac{A_P}{\rho_P} \Big(1 - (1 + \rho_P)^{-n_P} \Big),$$

with interest payment cycle $I_P = h$. In this case, $A_P = A\rho_N/\rho$. We assume that the borrower has an option to prepay the mortgage at an arbitrary date

$t < T$. In case of prepayment, the *outstanding loan balance* is respectively given by

$$L_t = A \int_t^T e^{-\rho(u-t)} \mathrm{d}u = A \cdot \frac{1 - e^{-\rho(T-t)}}{\rho},$$

for any continuous time $t \in [0, T]$, and

$$L_{kI_P} = A_P \sum_{j=k+1}^{n_P} (1 + \rho_P)^{-(j-k)} = \frac{A_P}{\rho_P} \Big(1 - (1 + \rho_P)^{k - n_P}\Big),$$

for any discrete time $k = 0, \ldots, n_P - 1$ with $L_T = 0$. If the prepayment option is exercised, the borrower must prepay the current outstanding loan balance plus any accrued interest since the last recurring payment. The charged amount is called *face value* and is given by

$$FV_{kn_I+m} = \big(1 + \rho_N m\big) L_{kI_P}, \tag{1}$$

for each $k = 0, \ldots, n_P - 1$ and $m = 0, \ldots, n_I - 1$ with $FV_N = 0$. Since the option can be exercised any time in $[0, T]$, the prepayment option is a contingent claim of American type.

If the borrower exercises the prepayment option at time t (resp. at time ih in discrete time), the lender receives immediately the face value $FV_t = L_t$ (resp. FV_i in discrete time) instead of the future stream of payments at the amortization rate A per unit time (resp. A_P at the end of each interest payment cycle). The market value at time t (resp. at time ih in discrete time) of these future cash-flows, also called non-callable mortgage price, are given by

$$V_t = A \int_t^T P(t, u) \mathrm{d}u, \quad t \in [t, T], \tag{2}$$

and

$$V_i = A_P \sum_{k=\lfloor i/n_I \rfloor + 1}^{n_P} P(ih, kI_P), \quad i = 0, \ldots, N-1, \tag{3}$$

with $V_N = 0$, respectively, where $P(t, s)$ denotes the price at time t of a zero-coupon bond with maturity $s \geq t$, and $\lfloor \cdot \rfloor$ is the floor function.

Being this the framework, it seems that in order to price a callable mortgage one has simply to price, with the aid of the usual no-arbitrage theory, the corresponding stream of payments, which always include an American-style option corresponding to the prepayment option of each borrower.

Throughout this paper, we always assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a filtered probability space, and assume that $\mathbb{P}$ is the risk-neutral probability measure. We also assume that the process r_t is the short rate process from which the entire term structure $P(t,s)$ can be obtained (e.g. see [1]), and the process H_t is the price of underlying physical good. Let V_t^p be the prepayment option price at time t. Then, we have

$$V_t^p = \operatorname*{ess\,sup}_{\tau\in[t,T]} \mathbb{E}\Big[e^{-\int_t^\tau r_u \mathrm{d}u}(V_\tau - FV_\tau) \mid \mathcal{F}_t\Big],$$

where the esssup is taken over all $\{\mathcal{F}_t\}$-stopping times $\tau \in [t,T]$, and $\mathbb{E}[\cdot \mid \mathcal{F}_t]$ is the conditional expectation (e.g. see [3]). Furthermore, denote V_t^b the joint option price at time t, i.e. including both the prepayment and default options. Similarly, we have

$$V_t^b = \operatorname*{ess\,sup}_{\tau,\delta\in[t,T]} \mathbb{E}\Big[1_{\{\tau<\delta\}}e^{-\int_t^\tau r_u \mathrm{d}u}(V_\tau - FV_\tau) + 1_{\{\delta<\tau\}}e^{-\int_t^\delta r_u \mathrm{d}u}(V_\delta - H_\delta) \mid \mathcal{F}_t\Big].$$

Here τ and δ indicate the stopping times corresponding respectively to a prepayment decision and a default decision. Now, the price of a callable (embedded with prepayment option) mortgage having rate A and a fixed nominal interest rate ρ and maturity T, at time t would simply be (e.g. see [3])

$$V_t^e = V_t - V_t^p, \tag{4}$$

and in the case when also a default decision is available and if the borrower is fully rational,

$$V_t^e = V_t - V_t^b. \tag{5}$$

3 Prepayment and default options

To begin with, we assume that the short rate r_t follows a CIR process, i.e. it satisfies the stochastic differential equation:

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}W_t^r, \tag{6}$$

We concentrate ourselves on the LSM method to solve the problem (4). In the discrete time setting, the problem becomes to evaluate

$$\operatorname*{ess\,sup}_{\tau\in[i,N]} \mathbb{E}\Big[e^{-h\sum_{j=i}^{\tau} r_j}(V_\tau - FV_\tau) \mid \mathcal{F}_i\Big], \tag{7}$$

for each $i = 0, \ldots, N-1$, and $V_\tau - FV_\tau$ is determined by (1) and (3) with the discount bond values

$$P(ih, kI_P) = \mathbb{E}\Big[e^{-h\sum_{j=i}^{k\cdot n_I-1} r_j} \mid \mathcal{F}_i\Big], \tag{8}$$

for any $i = 0, \ldots, N$ and $k = \lfloor i/n_I \rfloor, \ldots, n_P$. Since the short rate r_j build now a Markov chain, the quantities (8) are deterministic functions of i and r_i, thus the conditional expectation in (7) is a function which only depend on i and r_i, as well as $V_i(r_i) - FV_i$. In this case, (7) can be evaluated by backward recursion using Snell envelope as follows (e.g. see [3]):

$$\begin{aligned} V_N^p(r) &= V_N(r) - FV_N(r) \equiv 0, \\ V_i^p(r) &= \max\Big\{V_i(r) - FV_i(r),\ \mathbb{E}\Big[e^{-hr_{i+1}}V_{i+1}^p(r_{i+1}) \mid r_i = r\Big]\Big\}, \end{aligned} \tag{9}$$

for $i = N-1, \ldots, 1, 0$, where the optimal stopping time is determined by

$$\hat{\tau} = \inf\big\{i \le N : V_i^p(r_i) = V_i(r_i) - FV_i(r_i)\big\}. \tag{10}$$

We then consider the situation that there are not only the prepayment risk, but also the default risk. We assume that the "price of the house" evolves as a geometric Brownian motion:

$$\mathrm{d}H_t = (r_t - s)H_t\mathrm{d}t + \sigma_H H_t \mathrm{d}W_t^H, \tag{11}$$

where s is a constant dividend or so-called service fee for the house, σ_H is the instantaneous standard deviation of returns on the house value, and W_t^H is a standard Wiener process, which is not necessarily independent of W_t^r. We assume $\mathrm{d}W_t^H \cdot \mathrm{d}W_t^r = \rho_{Hr}\mathrm{d}t$.

In case of default, the borrower forfeits the house to the lender, which is the right can be exercised any time during the time interval $[0, T]$; thus, the default option can be seen as an option of American style. The prepayment and default option are alternative to each other, it is impossible to evaluate them separately. Instead, their optimal combined value is given in (5). In the discrete time setting, this problem becomes to evaluate

$$V_i^b = \operatorname*{ess\,sup}_{\tau,\delta\in[t,T]} \mathbb{E}\Big[I_{\{\tau<\delta\}}e^{-h\sum_{j=i}^{\tau} r_j}(V_\tau - FV_\tau) + I_{\{\delta<\tau\}}e^{-h\sum_{j=i}^{\delta} r_j}(V_\delta - H_\delta) \mid \mathcal{F}_i\Big], \tag{12}$$

for each $i = 0, \ldots, N-1$. Since r_i is a Markov chain and V_i is deterministic of r, we have

$$\begin{aligned} V_N^b(r,H) &= 0, \\ V_i^b(r,H) &= \max\Big\{(V_i - FV_i)(r), V_i(r) - H, \\ &\qquad \mathbb{E}\Big[e^{-hr_{i+1}}V_{i+1}^b(r_{i+1}, H_{i+1}) \mid r_i = r, H_i = H\Big]\Big\}. \end{aligned} \tag{13}$$

for $i = N-1, \ldots, 1, 0$. Then $V_i^b = V_i^b(r_i, H_i)$ for $i = N, \ldots, 1, 0$, and two optimal stopping times are given by

$$\begin{aligned} \hat{\tau} &= \inf\big\{i \le N : V_i^b(r_i, H_i) = (V_i - FV_i)(r_i)\big\}, \\ \hat{\delta} &= \inf\big\{i \le N : V_i^b(r_i, H_i) = V_i(r_i) - H_i\big\}. \end{aligned}$$

In the real market, we can sometimes see very large returns (positive or negative) of the house price in small time increments. Thus, we can consider that the house price H_t evolves as a jump diffusion process (e.g. see [12]), which could better address the issue of fat tails.

$$\mathrm{d}H_t = u_t^* H_t \mathrm{d}t + \sigma_H H_t \mathrm{d}W_t^H + (J_{N_t} - 1)H_t \mathrm{d}N_t, \tag{14}$$

where N_t is a Poisson process with intensity λ, J_n, $n = 1, 2, \ldots$, are i.i.d log-normal variables, with mean μ_j and variance σ_j^2, and $u_t^* = r_t - \lambda\mathbb{E}[J_n - 1]$ under the risk neutral measure.

4 Least-squares Monte Carlo methods

In this section, we present how to evaluate the backward recursion (9) by means of the LSM method. We denote

$$\begin{aligned} P_i(r_i) &= \big(V_i(r_i) - FV_i(r_i)\big)^+, \\ P_i^c(r) &= \mathbb{E}\Big[e^{-hr_{i+1}}V_{i+1}^p(r_{i+1}) \mid r_i = r\Big]. \end{aligned}$$

Here $P_i(r_i)$ is the payoff of prepayment option at time i. At each exercise time i, the holder decides to exercise the option and get the payoff $P_i(r_i)$ or to continue. In this case, the payoff may only depend on the value of r_i at time i. On the other hand, $P_i^c(r_i)$ denotes the expected option value if the option is not exercised at time i. The major contribution of Longstaff and Schwartz in [11] is to estimate the continuation value $P_i^c(r_i)$ by ordinary least-squares regression.

We consider J path-realizations r_i with results $r_{i,j}$, where $i \in \{0, \ldots, N\}$ and $j \in \{1, \ldots, J\}$. LSM method propose to regress the J continuation values:

$$e^{-hr_{i+1,j}} V_{i+1,j}^p (r_{i+1,j})$$

against the J simulated values for $r_{i,j}$. Given K basis functions $b : \mathbb{R}^{1\times K} \to \mathbb{R}$, the regression starts at the time step $N-1$, i.e. one step before maturity N. The continuation value $P_{i,j}^c$ is approximated by

$$\widehat{P}_{i,j}^c (r_{i,j}) = \sum_{k=1}^{K} a_{k,i}^* b_k(r_{i,j}). \tag{15}$$

The optimal regression coefficients $a_{k,i}^*$ are the result of

$$\min_{a_{1,i},\ldots,a_{K,i}} \frac{1}{J} \sum_{j=1}^{J} \Big(\sum_{k=1}^{K} a_{k,i} b_k(r_{i,j}) - e^{-hr_{i+1,j}} V_{i+1,j}^p (r_{i+1,j}) \Big)^2 \tag{16}$$

In some circumstances, the quality of the regression can be improved upon when restricting the paths involved in the regression to those where the option is in the money. In addition, The optimal regression coefficients $a_{k,i}^*$ are varying under the backward induction, where $i \in \{0, \ldots, N\}$.

One possible choice of basis functions is the set of (weighted) Laguerre polynomials:

$$L_k(r) = e^{-\frac{1}{2}r} \frac{e^r}{k!} \frac{\mathrm{d}^k}{\mathrm{d}r^k} (r^k e^{-r}).$$

Other types of basis function include the Hermite, Legendre, Chebyshev, Gegenbauer and Jacobi Polynomials, etc.

Given that the prepayment option value at maturity equals 0, a dynamic program solves for all values $V_{i,j}^p$, starting at time N and iterating backwards to 0. Based on the value $V^p_{0,j}$, we can compute an approximation to the prepayment option value, which is known as

$$\widehat{V^p_0} = \frac{1}{J}\sum_{j=1}^{J} V^p_{0,j}, \tag{17}$$

We extend the LSM method to evaluate the backward recursion (13). Since only one mortgage option can be exercised at a given time i, we use following denotations:

$$\begin{aligned} D_i(r_i, H_i) &= (V_i(r_i) - H_i)^+, \\ B_i(r_i, H_i) &= \max\{D_i(r_i, H_i),\ P_i(r_i, H_i)\}, \end{aligned}$$

where $D_i(r_i, H_i)$ is the payoff of default option at time i, and $B_i(r_i, H_i)$ is the payoff of joint option at time i.

Now, we consider J simultaneous path-realizations r_i and H_i with results $r_{i,j}$ and $H_{i,j}$, where $i \in \{0, \ldots, N\}$ and $j \in \{1, \ldots, J\}$. The LSM method proposes to regress the J continuation values

$$e^{-hr_{i+1,j}} V^b_{i+1,j}(r_{i+1,j}, H_{i+1,j})$$

against the J simulated values for $r_{i,j}$ and $H_{i,j}$. Given K basis functions $b : \mathbb{R}^{2\times K} \to \mathbb{R}$, the regression start at the time step $N-1$, i.e. one step before maturity N. The continuation value $B^c_i(r_i, H_i)$ is approximated by

$$\widehat{B}^c_{i,j}(r_{i,j}, H_{i,j}) = \sum_{k=1}^{K} a^*_{k,i} b_k(r_{i,j}, H_{i,j}). \tag{18}$$

The optimal regression coefficients $a^*_{k,i}$ are the result of

$$\min_{a_{1,i},\ldots,a_{K,i}} \frac{1}{J}\sum_{j=1}^{J}\Big(\sum_{k=1}^{K} a_{k,i} b_k(r_{i,j}, H_{i,j}) - e^{-hr_{i+1,j}} V^b_{i+1,j}(r_{i+1,j}, H_{i+1,j})\Big)^2 \tag{19}$$

Finally, the joint option value is given by

$$\widehat{V^b_0} = \frac{1}{J}\cdot\sum_{j=1}^{J} V^b_{0,j}$$

In order to distinguish between default option and prepayment option, we use the following dynamic programming: Set $\tau^*_{N,j} = N$, and

$$\tau^*_{i,j} = \begin{cases} i, & \text{if } B_{i,j}(r_i, H_i) \geq B^c_{i,j}(r_i, H_i),\ i = N-1, \ldots, 1, 0, \\ \tau^*_{i+1,j}, & \text{otherwise,} \end{cases}$$

where $j \in \{1, \ldots, J\}$. To distinguish prepayment option and default option, we have to denote $\tau_{n,j}$ as the optimal stopping time of prepayment option and $\delta_{n,j}$ default option. The rule to distinguish them can be listed as follows. Set

$$\tau_{N,j} = \delta_{N,j} = \tau^*_{N,j}$$

Then, if $B_{i,j}(r_{i,j}, H_{i,j}) \geq B^c_{i,j}(r_{i,j}, H_{i,j})$, we set

$$\begin{aligned} \tau_{i,j} &= \tau^*_{i,j}, \quad \delta_{i,j} = \delta_{i+1,j}, \quad \text{if} \quad P_{i,j}(r_{i,j}, H_{i,j}) \geq D_{i,j}(r_{i,j}, H_{i,j}) \\ \delta_{i,j} &= \tau^*_{i,j}, \quad \tau_{i,j} = \tau_{i+1,j}, \quad \text{otherwise;} \end{aligned}$$

and if $B_{i,j}(r_{i,j}, H_{i,j}) < B^c_{i,j}(r_{i,j}, H_{i,j})$, we set

$$\tau_{i,j} = \tau_{i+1,j}, \quad \delta_{i,j} = \delta_{i+1,j}$$

Finally, based on J paths monte-carlo simulation, we can calculate respectively the value of prepayment option and default option by the formula as follows:

$$\begin{aligned} \widehat{V^p_0} &= \frac{1}{J} \sum_{j=1}^{J} e^{-\int_0^{\tau_{0,j}} r_u \mathrm{d}u} V^b_{\tau_{0,j},j}(r_{\tau_{0,j},j}, H_{\tau_{0,j},j}), \\ \widehat{V^d_0} &= \frac{1}{J} \sum_{j=1}^{J} e^{-\int_0^{\delta_{0,j}} r_u \mathrm{d}u} V^b_{\delta_{0,j},j}(r_{\delta_{0,j},j}, H_{\delta_{0,j},j}). \end{aligned}$$

5 The LSM algorism and simulation results

5.1 LSM Algorism

1) Generate J discrete paths of house value and short interest rate, respectively, by the schemes given in [4];

2) Use the J short rate paths to calculate the value of mortgage without options $V_{i,j}$ and the face value $FV_{i,j}(r_{i,j})$ for $i \in \{0, \ldots, N\}$ and $j \in \{1, \ldots, J\}$, and then set the cash-flows at maturity for each path.

3) Starting from t_{N-1}, select in-the-money paths, and then discount those cash-flows to prior exercise point.

4) Estimate condition expectation function using ordinary least squares regression.

5) Use conditional expectation function estimated from Item 4 to compute continuation value and then compare it with intrinsic value at current point to determine optimal exercise decision.

6) For each path, use results of Item 5 to adjust discounted cash-flow at current point. When updating the joint option value $V^b_{i,j}(r_{i,j}, H_{i,j})$ under the case of continuation, the real continuation value

$$e^{-hr_{i+1,j}} V^b_{i+1,j}(r_{i+1,j}, H_{i+1,j})$$

is to be taken and not the estimated $\widehat{B}^c_{i,j}(r_i, H_i)$.

7) Repeat Items 3 – 6 for next exercise point until t_0.

8) Average discounted cash - flows at time t_0 for default paths and prepayment paths respectively.

In Item 1), we can employ the antithetic method in the variance reduction technique. This method relies on reducing variance by introducing a negative or "antithetic" relation between pairs of simulated random components [6]. In Item 2), there are two different methods: 1-step LSM and 2-steps LSM. The unique difference between them is how to compute V_i, which is given by

$$V_i = A_P \sum_{k=i+1}^{N} P\big(ih, kh\big) = A_P P\big(ih, (i{+}1)h\big) + \mathbb{E}\Big[e^{-hr_{i+1}} V_{i+1} \mid r_i\Big], \tag{20}$$

for $i = N{-}1, \ldots, 1, 0$, where we suppose that $I_P = h$. In the 2-steps LSM, the backward recursion (9) is evaluated by two steps least squares regression under the Monte Carlo simulation framework, i.e. the first step is to estimate V_i by least squares regression in (20) and the second step is to estimate the option price (9) by the other least squares regression. We mention here that the two steps of Monte Carlo regressions share the same simulated paths. On the other hand, 1-step LSM calculates V_i by means of bond analytical formula [3] or directly from the simulated path. Meanwhile, in Item 4, as basis functions in the regressions we use Laguerre polynomials evaluate the standardized house price H_t/H_0, the standardized interest rate r_t/r_0, the cross products $(H_t/H_0)(r_t/r_0)$.

Remarks (Simulating the Jump Diffusion Process). For easily generating the house price path, the process (14) can be expressed in terms of $X_t = \log H_t$,

$$\mathrm{d}X_t = (u_t^* - \frac{\sigma_H^2}{2})\mathrm{d}t + \sigma_H \mathrm{d}W_t^H + \log J_{N_t}\mathrm{d}N_t, \tag{21}$$

where $u_t^* = r_t - \lambda\mathbb{E}[J_n - 1]$, and we denote $\kappa = \mathbb{E}[J_n - 1] = e^{(\mu_j + \sigma_j^2)} - 1$ since $\log J_n \sim \mathbb{N}(\mu_j, \sigma_j^2)$. After discretized, the process (21) becomes

$$X_{i+1,j} = X_{i,j} + (r_{i,j} - \lambda\kappa - \frac{\sigma_H^2}{2})h + \sigma_H(W_{i+1,j}^H - W_{i,j}^H) + \sum_{n=N_{i,j}+1}^{N_{i+1,j}} \log J_n$$

where $i \in \{0, \ldots, N-1\}$ time states and $j \in \{1, \ldots, J\}$ paths. The specific algorism for the jump diffusion process

a) Generate $Z_{i+1,j} \sim \mathbb{N}(0,1)$.

b) Generate $N_{i+1,j} - N_{i,j} \sim Poisson(\lambda h); if N_{i+1,j} = N_{i,j}$ set $M_{i+1,j} = 0$ and go to Item 5).

c) Generate $\log J_{N_{i,j}+1}, ..., \log J_{N_{i+1,j}}$, where $\log J_n \sim \mathbb{N}(\mu_j, \sigma_j^2)$ for $n = N_{i,j} + 1, ..., N_{i+1,j}$.

d) Set
$$M_{i+1,j} = \sum_{n=N_{i,j}+1}^{N_{i+1,j}} \log J_n.$$

e) Set $X_{i+1,j} = X_{i,j} + (r_{i,j} - \lambda\kappa - \frac{\sigma_H^2}{2})\sigma_H^2 h + \sigma_H\sqrt{h}Z_{i+1,j} + M_{i+1,j}$.

5.2 Simulation experiments

Now, we illustrate the above-presented algorism with the following basic parameters setting for the CIR model of interest rate (6), the house value process (11) and the mortgage in Table 1, which are the same as that in paper [7] for the comparison of different methods.

Table 1: Model parameters

A	1	κ	0.15	θ	0.05
Loan to value	1	ρ	0.055	s	0.045
σ_H	0.05	σ_r	0.065	r_0	0.055
ρ_{Hr}	0				

The first experiment is to compare the different choice of basis functions used to generate the LSM algorithm. In Table 2 and Table 3, we show respectively how the different types of basis functions and the different numbers of basis functions affects the simulated results. From Table 2, the obtained results are virtually identical with three different basis. Table 3 shows us that few basis functions $K = 3$ are needed to closely approximate the conditional expectation function (15). Therefore, for following experiments, we choose the Laguerre polynomials and $K = 3$ basis functions.

Table 2: Different basic functions to get LSM estimate of prepayment options

Maturity	$T = 5$	$T = 10$	$T = 20$	$T = 30$
Non-callable mortgage V_0				
Analytical Price	4.3853	7.7802	12.549	15.542
Basic polynomials	4.3853	7.7800	12.5516	15.5440
Laguerre polynomials	4.3854	7.7802	12.549	15.5424
Legendre polynomials	4.3853	7.7790	12.5481	15.5368
Prepay Option Price V_0^p				
Analytical Price	0.0593	0.2438	0.818	1.432
Basic polynomials	0.0564	0.2417	0.8172	1.4148
Laguerre polynomials	0.0566	0.2416	0.8157	1.4175
Legendre polynomials	0.0564	0.2415	0.8150	1.4153

Number of steps $N = 1,000$, the simulation paths $J = 100,000$.

The second experiment is to compare the simulation results of relating mortgage values done by 2-steps LSM with BTM, PDE approximation method and 1-step LSM, which is shown in Table 4. The LSM is done with the time intervals $N = 1,000$, the number of paths $J = 100,000$ (50,000 plus 50,000 authentic) and maturities $T = 5, 10, 20, 30$. In general, the convergence of

Table 3: Different numbers of the basic function to get the LSM estimate

	T=5		T=10		T=20		T=30	
	V_0^p	(s.e.)[a]	V_0^p	(s.e.)	V_0^p	(s.e.)	V_0^p	(s.e.)
K=2	0.055	(0.0022)	0.237	(0.0098)	0.7985	(0.0283)	1.3965	(0.0463)
K=3	0.0567	(0.0018)	0.2426	(0.0076)	0.8142	(0.028)	1.4294	(0.044)
K=4	0.0571	(0.0025)	0.242	(0.0094)	0.8231	(0.0335)	1.4285	(0.0445)
K=5	0.0569	(0.0023)	0.2443	(0.0082)	0.8173	(0.0316)	1.4214	(0.0509)
K=6	0.0534	(0.0023)	0.2378	(0.0082)	0.8082	(0.0509)	1.4129	(0.0468)

[a]The standard errors of the simulation estimates (s.e.) are given in parentheses.

2-steps LSM to the Analytical approximation method and BTM for the price of the callable mortgage is comparably persuasive compared with 1-step LSM. The prices of the non-callable mortgage and the prepayment option is simulated more accurately by 1-step LSM compared with 2-steps LSM. In addition, we also make a diagnostic test for the convergence of a simulation algorithm, shown in Table 5. In our context, this can be implemented by estimating the conditional expectation regressions from one set of paths and then applying the regression functions to an out of sample set of paths [11]. Table 5 substantiates that out of sample values that closely approximate the in sample values for the prepayment option.

The third experiment to extend our model to simulate the value of prepayment and default options embedded in mortgage, which is done with the time intervals $N = 1,000$, the number of paths $J = 1,000$ and maturities $T = 5, 10, 20, 30$, repeated 100 times with different initial seeds for the random number generator. Its relating simulation results are found in Table 6. We then consider the case when the house value follows the jump-diffusion process as (14). The LSM is done with the time intervals $N = 1,000$, the number of paths $J = 1,000$ repeated 100 times with different initial seeds for the random number generator, maturities $T = 5$, the jump mean $\mu_j = -0.90$, the jump volatility $\sigma_j = 0.40$, and the jump intensity $\lambda = 0, 0.1, 0.5, 1, 1.5$. The numerical results are shown in Table 7. Considering the financial implication, Table 4 and Table 6, show that the prepayment and default risks cannot be neglected

Table 4: Comparison of different methods to estimate the value of prepayment options

	$T = 5$	$T = 10$	$T = 20$	$T = 30$
Non-callable mortgage V_0				
Analytical approximation[a]	4.3853	7.7802	12.549	15.542
BTM[b]	4.3853	7.7803	12.55	15.544
1-step LSM	4.3851	7.7802	12.549	15.5424
2-steps LSM	4.3843	7.7765	12.5388	15.5277
Callable mortgage V_0^e				
Analytical approximation	4.326	7.5364	11.731	14.11
BTM	4.326	7.5365	11.731	14.112
1-step LSM	4.3288	7.5386	11.7333	14.1249
2-steps LSM	4.3279	7.5361	11.7284	14.1202
Prepayment option V_0^p				
Analytical approximation	0.0593	0.2438	0.818	1.432
BTM	0.0593	0.2438	0.819	1.432
1-step LSM	0.0563	0.2416	0.8157	1.4175
2-steps LSM	0.0564	0.2404	0.8104	1.4075

[a] Analytical approximation method is done in [14].
[b] Binomial tree price is done in [7].

Table 5: Comparison of the in-sample and out-of-sample LSM estimates of the value of the prepayment option

T	$V_0^{p,\text{is}}$	(s.e.)	$V_0^{p,\text{os}}$	(s.e.)	$V_0^{p,\text{is}}$-$V_0^{p,\text{os}}$	$V_0^{p,\text{is}}$-V_{BTM}
5	0.0562	(0.0022)	0.0568	(0.0026)	-0.0006	-0.0031
10	0.2442	(0.0095)	0.2436	(0.009)	0.0006	0.0004
20	0.8196	(0.0278)	0.8138	(0.0318)	0.0058	0.0016
30	1.4202	(0.0442)	1.416	(0.0558)	0.0042	-0.0118

$V_0^{p,\text{is}}$ and $V_0^{p,\text{os}}$ are in-sample and out-of-sample LSM estimates of the prepayment option respectively.

in valuing the mortgage, since they take up a comparably large proportion of the mortgage value, especially when the maturity T becomes longer. Table 7 shows the jump risks for the underlying house value. As expected, jump intensity λ is positively correlated with the default option value.

Table 6: The estimate of prepayment and default options value using 1-step LSM

	T=5	T=10	T=20	T=30
V_0	4.3858	7.7822	12.5475	15.5401
	(0.0042)	(0.0143)	(0.0449)	(0.073)
V_0^e	4.323	7.5135	11.5866	13.7239
	(0.0025)	(0.0075)	(0.0257)	(0.0393)
V_0^d	0.0193	0.088	0.3253	0.6326
	(0.0024)	(0.0094)	(0.028)	(0.0538)
V_0^p	0.0435	0.1808	0.6355	1.1835
	(0.0018)	(0.008)	(0.0225)	(0.039)
V_0^b	0.0628	0.2688	0.9609	1.8162
	(0.0024)	(0.0094)	(0.028)	(0.0538)

V_0 and V_0^e denote the simulated pure mortgage value and the mortgage value with both risks; V_0^d, V_0^p and V_0^b denote simulated default option value, prepayment option value, and joint options value respectively.

Table 7: The estimate of prepayment and default options under a house jump process with different jump intensities λ at T=5

λ	0	0.1	0.5	1	1.5
V_0^d	0.018	0.0275	0.074	0.1116	0.1316
	(0.0013)	(0.0043)	(0.012)	(0.0142)	(0.0161)
V_0^p	0.0429	0.1103	0.3119	0.4005	0.4389
	(0.0019)	(0.012)	(0.0233)	(0.0254)	(0.0253)
V_0	4.3852	4.3854	4.385	4.3846	4.3857
	(0.0038)	(0.0042)	(0.004)	(0.0038)	(0.0046)
V_0^e	4.3242	4.2477	3.9991	3.8725	3.8152
	(0.0022)	(0.0116)	(0.0252)	(0.0276)	(0.0312)

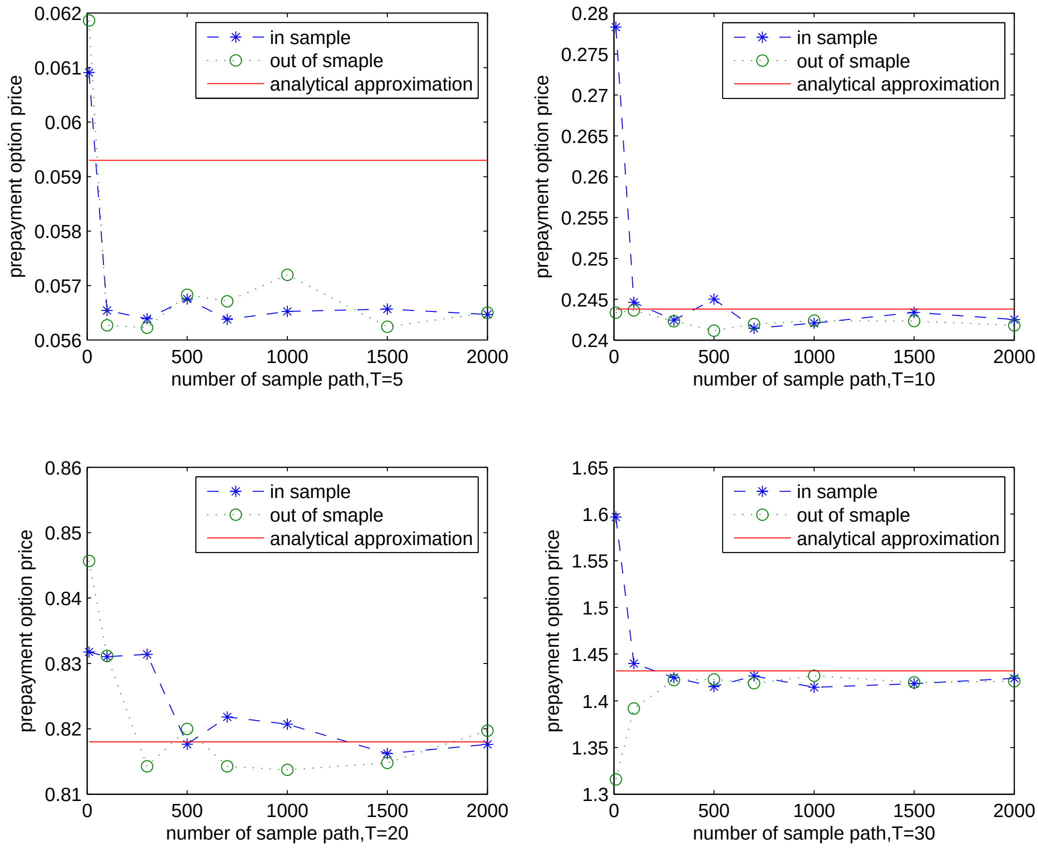

Figure 1: The convergence of in sample and out of sample prepayment option price under different numbers of sample path

The last experiment is is to show the convergence of our LSM algorism to estimate the prepayment options, under the increasing numbers of sample path and numbers of time intervals, shown in Figure 1 and Figure 2 respectively. From these figures, we clearly see that LSM method is a Lower bounds estimator for the prepayment options values. Figure 2 also implicates the convergence of Bermudan style prepayment options to the American style options.

6 Conclusions

To sum up, in this paper, we have developed a model to quantify the pre-

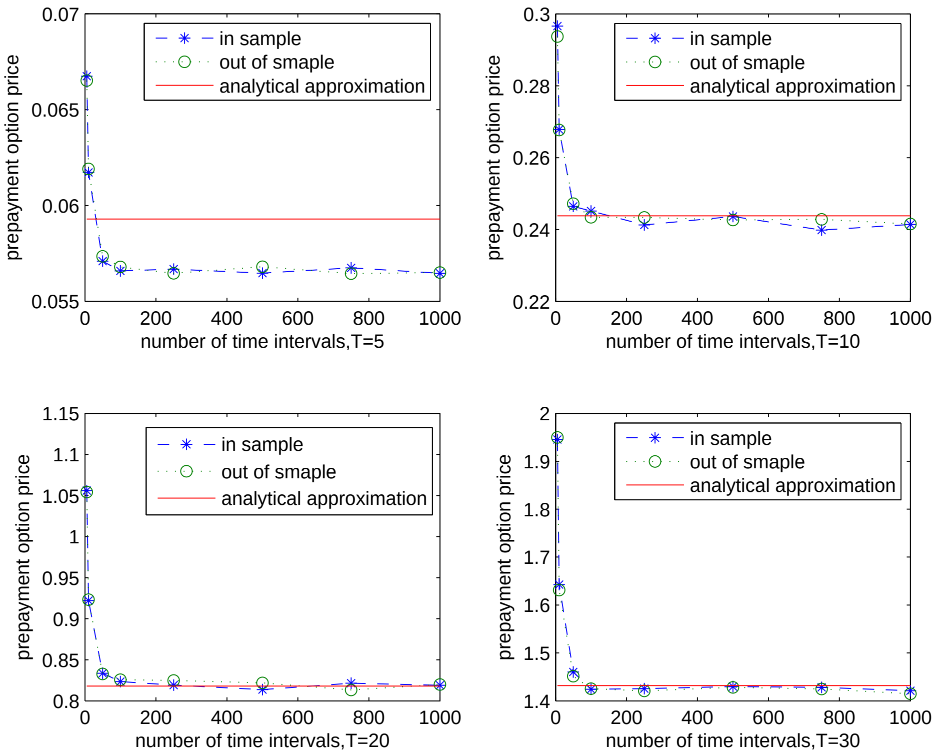

Figure 2: The convergence of in sample and out of sample prepayment option price under different numbers of time interval

payment and default risks embedded in mortgages within the context of the optimal stopping theory. By means of least squares Monte Carlo method, optimal stopping time problem of American-style option can be well solved. We apply this idea to our problem and obtain comparably persuasive numerical results, which are compared with that done by binomial tree method and analytical approximation. To better solve the problem concerning the prepayment option, we novelly present a method called two steps least squares Monte Carlo method. After obtaining the numerical results, we make the financial analysis and conclude: the prepayment and default risks embedded in mortgages should not be neglected and the jump risk for the house price will cause the intensity of default.

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