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Portfolio Optimization is One Multiplication, the Rest is Arithmetic

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Abstract

In this paper we present a rigorous, yet easy to apply method that substitutes those tedious techniques and error prone procedures that are currently used in finding optimal portfolios. Our work is not to support or dispute the applicability of the Mean-Variance optimization method in finance; we simply offer a robust approach to find all the characteristics of any efficient portfolios, with or without bonds. We show that one matrix multiplication provides all the characteristics of all efficient portfolios including risk and return of these optimal portfolios and their corresponding Lagrange multipliers as well as the proportions invested in each asset. The rest is just a few simple elementary arithmetic operations.

JEL classification numbers: G11, G12
Keywords: Mean-Variance optimization, Optimal Portfolio, Minimum Variance Portfolio, Asset Allocation, Portfolio Selection Model, Modern Portfolio Theory

1 Introduction

Mean-Variance optimization method in finance, which is commonly known as Markowitz Portfolio Theory, was introduced by [1]. Since then, the Markowitz method changed name to Modern Portfolio Theory and it has been remarkably enhanced in order to help researchers investigate the effect of complex constraints and market conditions on the original optimization technique. Today with the advent of powerful computers and sophisticated software programs, this work can assist researchers to investigate the applicability of such models in forming investment portfolios more efficiently. Furthermore, if modern portfolio theory is going to be taught in schools or be tried in the marketplaces, this paper offers the easiest way to achieve these goals and would assist academicians in

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the classrooms and the practitioners in the equity and derivative markets.
For the past 50 years Markowitz approach had its fair share of criticisms and praises. The theory has survived, and still today in almost all the investments and portfolio management courses the mean-variance optimization techniques is examined and debated. Statman [2] asks "Is Markowitz Wrong?" and shows that the model works; and it worked even during the 2008-2009 financial crisis. [2] declares that "Mean-Variance portfolio theory is consistent with strategic asset allocation and with tactical asset allocation." While [3] is addressing different issues, it confirms that "The stream of final payoffs obeys a classic mean-variance characterization and Capital Asset Pricing Model equilibrium pricing". Draw on the chaotic hurried efforts amidst the financial crisis [4] offers a risk-adjusted model which supports the conditional optimization approach and states that "optimization with sound and rational investment assumptions produces efficiency". In order to advance Markowitz mean-variance optimization model [5] considers margin trading and adds utility function to the process. Thus, the model presented in [5] "allows investors to consider both volatility tolerance and leverage tolerance in selecting optimal portfolios."
Conversely, however, [6] critically questioned the efficiency of the mean-variance approach and show that 'equal weight' or $\frac{1}{N}$ strategy outperforms optimized Sharpe ratio. More importantly their work gives support to the Black–Litterman Model which was developed by [7]. To further support Black–Litterman Model, [8] and [9] add uncertainty to the Black–Litterman portfolio selection process and give the investors the ability to express their tolerable level of uncertainty and thus limit the deviation of the portfolio's return from the benchmark.
In the following sections we present an algorithm that leads to a concise expression that easily computes optimal portfolios' parameters. In addition, we integrate our findings with [10] and the derivation of "Arrow-Pratt risk aversion measure" Arrow [11] and Pratt [12] to introduce an "investor's risk tolerance factor", δ. This factor, which fittingly ranges from 0 to 1, easily reveals the investor's risk-returns preference and it becomes an integral part of computing portfolio's Lagrange multipliers. Among other things, this factor helps an investor to clearly and accurately express his/her risk-returns preferences to the portfolio managers.
We start from the common derivation of two portfolios and then introduce our model by combining these two portfolios. For simplicity, we pick the minimum-variance portfolio and a tangent portfolio to build our model. The tangent portfolio is the portfolio at the tangent point of a line from the origin tangent to the efficient frontier. The progression of the topics in this paper is organized as follows. In Section 1, we review and examine the familiar conventional optimization methods and reintroduce specifics in Sections 1.1, and 1.2. In Section 2, we present our model in reference to the results obtained from previous sections. In Section 3, we present a numerical example illustrating the application of our model.

1.1 The Minimum Variance Portfolio

Since the efficient frontier in the risk return space is a convex quadratic function, it is therefore possible to find a minimum variance for a given 'n' risky assets. Thus, the Lagrange optimization approach can be structured to find the desired solution as follows.

Minimize: $\sigma_p{}^2 = \frac{1}{2}[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}]$

Subject to: $[\mathbf{1}]^T\,[\mathbf{x}] = 1$

Where, $\sigma_p{}^2$ is the variance of the portfolio of 'n' risky assets; $[\mathbf{x}]^T$ is the row vector of x_1 to x_n representing the proportions invested in each asset. $[\boldsymbol{\Sigma}]$ is an n by n variance-covariance matrix, and $[\mathbf{1}]^T$ is a row vector of 1's that satisfies the condition that the summation of the allocation has to be equal to 1. We take the first derivative of the Lagrange function with respect to x_i and λ, and make them equal to zero.

$$\begin{aligned} &L = \frac{1}{2}[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}] - \lambda\big([\mathbf{1}]^T[\mathbf{x}] - 1\big) \\ &[\boldsymbol{\Sigma}][\mathbf{x}] = \lambda[\mathbf{1}] \\ &[\mathbf{1}]^T[\mathbf{x}] = 1 \end{aligned} \tag{1}$$

Multiplying both side of Equation (1) by $[\boldsymbol{\Sigma}]^{-1}$, we will have:

$$[\mathbf{x}] = \lambda[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}] \tag{2}$$

Multiplying both side of Equation (2) by $[\mathbf{1}]^T$ we get:

$$[\mathbf{1}]^T[\mathbf{x}] = \lambda[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]$$

The summation of the investment allocations has to be equal to 1 or $[\mathbf{1}]^T[\mathbf{x}] = 1$. Therefore, $\lambda = \{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]\}^{-\mathbf{1}}$. Substituting for λ in Equation (2), we get:

$$[\mathbf{x}]_{MVP} = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]} \tag{3}$$

Where, $[\mathbf{x}]_{MVP}$ is the proportions invested within the Minimum Variance Portfolio. Let column vector $[\mathbf{z}]$ stands for $[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]$.

$$[\mathbf{x}]_{MVP} = \frac{[\mathbf{z}]}{[\mathbf{1}]^T[\mathbf{z}]} \tag{4}$$

The numerator of Equation (4) is an n-by-1 column vector of z_i values, and the denominator of Equation (4) is the summation of these n values. Therefore, to get the proportions invested in each asset within the Minimum Variance Portfolio, we simply multiply the inverse of the variance-covariance matrix times a column vector of 1, and divide these values by their summation.

1.2 The Tangent Portfolio

Consider a portfolio on the efficient frontier which is also on the tangent line from the origin. To find the proportions invested in each asset within this tangent portfolio we set to minimize the variance of the portfolio subject to the returns constraint, that is:

Minimize: $\sigma_p{}^2 = \frac{1}{2}[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}]$

Subject to: $[\mathbf{x}]^T\,[\mathbf{k}] = r_p$

$$L = \frac{1}{2}[\mathbf{x}]^T[\Sigma][\mathbf{x}] - \psi\big([\mathbf{x}]^T[\mathbf{k}] - r_p\big)$$

Where, $[\mathbf{k}]$ is a column vector of k_i representing the average return for the i^{th} asset, and r_p is the return of the portfolio. Make the first derivative of the Lagrangian function with respect to x_i and ψ, equal to zero as:

$$\begin{aligned}&[\boldsymbol{\Sigma}][\mathbf{x}] = \psi[\mathbf{k}] \\ &[\mathbf{x}]^T\,[\mathbf{k}] = r_p\end{aligned} \tag{5}$$

Once again we multiply both side of Equation (5) by $[\boldsymbol{\Sigma}]^{-1}$.

$$[\mathbf{x}] = \psi[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}] \tag{6}$$

We multiply both side of Equation (6) by $[\mathbf{1}]^T$ and Since $[\mathbf{1}]^T[\mathbf{x}] = 1$, we have:

$$\psi = \{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]\}^{-\mathbf{1}}$$

Substituting for ψ in Equation (6), we get:

$$[\mathbf{x}]_{TP} = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]}{[\mathbf{1}]^T[\Sigma]^{-1}[\mathbf{k}]} \tag{7}$$

Where, $[\mathbf{x}]_{TP}$ is the proportions invested within the Tangent Portfolio. Let column vector $[\mathbf{w}]$ stands for $[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]$.

$$[\mathbf{x}]_{TP} = \frac{[\mathbf{w}]}{[\mathbf{1}]^T[\mathbf{w}]} = \frac{[\mathbf{w}]}{\sum_{i=1}^{n} w_i} \tag{8}$$

The numerator of Equation (8) is an n-by-1 column vector of w_i values, and its denominator is the summation of these n values. Thus, to find the proportions invested in each asset within the Tangent Portfolio, we multiply the inverse of the variance-covariance matrix times the column vector of asset returns, and divide the results by the summation of these values. Likewise, the Capital Market Line (CML) which is the tangent line from the risk-free rate (r_f) to the efficient frontier, has a very similar solution as the Equation (7). That is, the proportions within the tangent portfolio of the Capital Market Line can be computed by Equation (9).

$$[\mathbf{x}]_{MAX} = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]} \tag{9}$$

Where, $[\mathbf{x}]_{\mathrm{MAX}}$ is the proportions invested in the tangent portfolio from the risk-free rate, and $[\mathbf{c}]$ is an n-by-1 column vector as: $[\mathbf{c}] = \left[[\mathbf{k}] - [\mathbf{1}] \times \mathrm{r_f}\right]$. Once again, the numerator of Equation (9) is an n-by-1 column vector, and its denominator is the summation of these n values. [Please see Appendix A for the derivation of Equation (9)].

2 The Model

In this section, we first develop a model in equity market (no bond) and provide formulas to easily compute all the variables of the portfolios on the Efficient Frontier. Typically, the unknowns of an optimal portfolio are: proportions invest in each asset, the Lagrangian multipliers associated with such portfolio and the risk and return of that portfolio. The optimal portfolios on the efficient frontier are subject to the following two constraints. The optimization system and the Lagrange function can be presented as:

Minimize: $\sigma_{\mathrm{p}}{}^{2} = \frac{1}{2}[\mathbf{x}]^{\mathrm{T}}[\boldsymbol{\Sigma}][\mathbf{x}]$

Subject to: $[\mathbf{1}]^{\mathrm{T}}\,[\mathbf{x}] = 1$

$[\mathbf{x}]^{\mathrm{T}}\,[\mathbf{k}] = \mathrm{r_p}$

$$\mathrm{L} = \frac{1}{2}[\mathbf{x}]^{\mathrm{T}}[\boldsymbol{\Sigma}][\mathbf{x}] - \lambda_1\left([\mathbf{1}]^{\mathrm{T}}[\mathbf{x}] - 1\right) - \lambda_2\left([\mathbf{x}]^{\mathrm{T}}[\mathbf{k}] - \mathrm{r_p}\right)$$

The results of the first partial derivative of this function with respect to $\mathrm{x_i}$, λ_1 and λ_2, can be presented as:

$$\begin{aligned}
&[\boldsymbol{\Sigma}][\mathbf{x}] = \lambda_1[\mathbf{1}] + \lambda_2[\mathbf{k}] \\
&[\mathbf{x}]^{T}[\mathbf{1}] = 1 \\
&[\mathbf{x}]^{T}[\mathbf{k}] = \mathrm{r_p}
\end{aligned} \tag{10}$$

Multiplying both side of Equation (10) by $[\boldsymbol{\Sigma}]^{-1}$, we get:

$$[\mathbf{x}] = \lambda_1[\boldsymbol{\Sigma}]^{-1}[1] + \lambda_2[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}] \tag{11}$$

Multiply and divide the first term of the Equation (11) by $[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]$, and multiply and divide the second term of the Equation (11) by $[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]$ to get:

$$[\mathbf{x}]_{\mathrm{q}} = \lambda_1\{[\mathbf{1}]^{\mathrm{T}}[\Sigma]^{-1}[\mathbf{1}]\} \times \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]}{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]} + \lambda_2\{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]\} \times \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]}{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathrm{k}]}$$

From Equations (3) and (7) we can write:

$$[\mathbf{x}]_{\mathrm{q}} = \lambda_1\{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]\} \times [\mathbf{x}]_{\mathrm{MVP}} + \lambda_2\{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]\} \times [\mathbf{x}]_{\mathrm{TP}}$$

Where, $[\mathbf{x}]_q$ is the proportions invested within any desired portfolio on the efficient frontier. $[\mathbf{x}]_{MVP}$ and $[\mathbf{x}]_{TP}$ are the proportions invested within the MVP and the Tangent Portfolio, respectively. By using the $[\mathbf{z}]$ and $[\mathbf{w}]$ notation we can write:

$$[\mathbf{x}]_q = \lambda_1 \left(\sum_{i=1}^{n} z_i \right) [\mathbf{x}]_{MVP} + \lambda_2 \left(\sum_{i=1}^{n} w_i \right) [\mathbf{x}]_{TP} \tag{12}$$

Verifications in [10] shows that any portfolio on the efficient frontier can be generated from only two efficient portfolios. That is, portfolios on the efficient frontier are a linear combination of two distinct portfolios on the curve. Let's consider the minimum variance portfolio and the tangent portfolio as the two distinct portfolios in order to find the proportions of any portfolio on the efficient portfolio curve as:

$$[\mathbf{x}]_q = (1 - \delta)[\mathbf{x}]_{MVP} + \delta[\mathbf{x}]_{TP} \tag{13}$$

Where, δ is the coefficient of such linear combination. Equation (13) can also be written as:

$$[\mathbf{x}]_q = [\mathbf{x}]_{MVP} + \delta[[\mathbf{x}]_{TP} - [\mathbf{x}]_{MVP}]$$

By comparing Equation (12) with Equation (13) we can determine the value of λ_1 and λ_2 as:

$$(1 - \delta) = \lambda_1 \left(\sum\nolimits_{i=1}^{n} z_i \right)$$

$$\delta = \lambda_2 \left(\sum\nolimits_{i=1}^{n} w_i \right)$$

$$\lambda_1 = \frac{(1 - \delta)}{\sum_{i=1}^{n} z_i} \qquad \lambda_1 = \frac{(1 - \delta)}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{1}]} \tag{14}$$

$$\lambda_2 = \frac{\delta}{\sum_{i=1}^{n} w_i} \qquad \lambda_2 = \frac{\delta}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{k}]} \tag{15}$$

λ_1 and λ_2 are the Lagrangian multipliers and they represent the sensitivity of the variance of the portfolio with respect to the constraints. Note that the value of λ_1 and λ_2 expressed in Equations (14) and (15) are direct function of the δ coefficient. Equation (13) shows that if δ is zero, then minimum variance portfolio is the answer, and if δ coefficient is 1, the tangent portfolio is the answer. Therefore, δ can be interpreted as an appraisal of investor's desire to hold risky assets and reflecting the degree of investor's hesitation or inclination toward risk. Thus, δ displays the investor's risk-returns preference and reveals his/her degree of risk tolerance. δ can take values greater than 1 if a client has an exceptional information, but ordinarily it ranges from 0 to 1. [Appendix B shows the similarity of δ, the "Risk Tolerance", and the Arrow-Pratt "Risk Aversion"].

Similarly any portfolio on the Capital Market Line is a linear combination of a risk free bond ($R_f \cdot$ Bond) and the tangent portfolio from r_f labeled as $[\mathbf{x}]_{MAX}$ in Equation (9). Thus, the allocations within any portfolio on the Capital Market Line can be computed by Equation (16).

$$[\mathbf{x}]_q = (1-\delta)(R_f.\text{Bond}) + \delta[\mathbf{x}]_{Max} \tag{16}$$

We now simplify the computations of all the prior derivations and show that all can be achieved by one simple matrix multiplication presented in Equation (17).

$$[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}] \tag{17}$$

Where $[\boldsymbol{\Sigma}]$ is an $n \times n$ variance-covariance matrix, and $[\mathbf{D}]$ is an $n \times 3$ matrix consists of a column vector of 1's, a column vector of asset-returns, and a column vector of asset-returns minus the risk-free rate. The result of the multiplication of Equation (17) is an $n \times 3$ matrix that provides the needed values to calculate the allocations within MVP, Tangent portfolio, and the max-Sharpe ratio portfolio. The rest is just a few simple additions and divisions. If one prefers not to use arithmetic, the denominator of Equation (18) would perform the necessary additions and divisions.

$$[\mathbf{x}] = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}]}{[\mathbf{I}]\otimes\{[\mathbf{1}]^{T}[\Sigma]^{-1}[\mathbf{D}]\}^{T}} \tag{18}$$

Where, $[\mathbf{x}]$ is an $n \times 3$ matrix of weights for MVP, Tangent portfolio from origin, and Tangent portfolio from the Risk Free Rate, respectively. $[\mathbf{I}]$ is an 3×3 identity matrix, and $\otimes$ represents a tensor multiplication.
Thus, by having the results of Equation (18) one can determine the proportions invested in any desired portfolio with or without bond by using Equations (13) and (16). That is, for any given risk tolerance 'δ', the only task we need to complete is to multiply $[\boldsymbol{\Sigma}]^{-1}$ times $[\mathbf{D}]$. Thus, one matrix multiplication finds all the characteristics of all efficient portfolios, including proportions invested in each asset, Lagrange Multipliers, and risk and return of these optimal portfolios.

3 Numerical Examples

Let's suppose an investor considers 5 risky assets. The covariance matrix, and the average returns of these assets are: k_1 = 1.90%, k_2 = 1.30%, k_3 = 1.00%, k_4 = 1.52%, and k_5 = 1.30%. Let's also assume R_f = 0.5%.

$$[\mathbf{k}]^{T} = [0.0190 \quad 0.0130 \quad 0.0100 \quad 0.0152 \quad 0.0130]$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.056 & 0.017 & 0.008 & 0.023 & 0.007 \\ 0.017 & 0.031 & 0.009 & 0.013 & 0.004 \\ 0.008 & 0.009 & 0.038 & 0.018 & 0.002 \\ 0.023 & 0.013 & 0.018 & 0.032 & 0.006 \\ 0.007 & 0.004 & 0.002 & 0.006 & 0.090 \end{bmatrix}$$

A matrix multiplication expressed in Equation (17) provides the necessary values to find the following solutions.

Table 1: Proportions invested, return and variance of different portfolios

Asset	Minimum Variance	Tangent from Origin	Max-Sharpe Ratio
1	7.81%	22.15%	30.81%
2	34.38%	25.72%	20.49%
3	26.83%	14.34%	6.80%
4	15.80%	24.01%	28.97%
5	15.17%	13.77%	12.93%
Return	1.33%	1.46%	1.54%
Variance	0.0171	0.0187	0.0213

Below illustrates the stepwise application of Equation (18).

$$[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}] = \begin{bmatrix} 4.58 & 0.17 & 0.15 \\ 20.15 & 0.20 & 0.10 \\ 15.72 & 0.11 & 0.03 \\ 9.26 & 0.19 & 0.14 \\ 8.89 & 0.11 & 0.06 \end{bmatrix}$$

$$[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}] = [58.61 \quad 0.78 \quad 0.49]$$

$$[\mathbf{x}] = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}]}{[\mathbf{I}]\otimes\{[\mathbf{1}]^{\mathrm{T}}[\boldsymbol{\Sigma}]^{-1}[\mathbf{D}]\}^{\mathrm{T}}}$$

$$[\mathbf{x}] = \begin{bmatrix} 4.58 & 0.17 & 0.15 \\ 20.15 & 0.20 & 0.10 \\ 15.72 & 0.11 & 0.03 \\ 9.26 & 0.19 & 0.14 \\ 8.89 & 0.11 & 0.06 \end{bmatrix} \begin{bmatrix} 0.017 & 0 & 0 \\ 0 & 1.285 & 0 \\ 0 & 0 & 2.061 \end{bmatrix} = \begin{bmatrix} 7.81\% & 22.15\% & 30.81\% \\ 34.38\% & 25.72\% & 20.49\% \\ 26.83\% & 14.34\% & 6.80\% \\ 15.80\% & 24.01\% & 28.97\% \\ 15.17\% & 13.77\% & 12.93\% \end{bmatrix}$$

Furthermore, let's consider an investor with risk tolerance of 0.75, ($\delta = 0.75$). Equations (13) and (16) calculate the proportions in a portfolio with no bond and a portfolio of stocks and bond, respectively.

Table 2: Proportions invested, return and variance of respective portfolios

Asset	Equity Only	25% Bond plus Equity
1	18.6%	23.1%
2	27.9%	15.4%
3	17.5%	5.1%
4	22.0%	21.7%
5	14.1%	9.7%
Return	1.42%	1.28%
Variance	0.018	0.012

If the required return, $\mathrm{r_q}$ is given, we can easily find the investor's "risk tolerance index" as:

$$\delta = \frac{r_q - r_{MVP}}{r_{TP} - r_{MVP}} \quad \text{or} \quad \delta = \frac{r_q - r_f}{r_{MAX} - r_f}$$

We find λ_1, λ_2 for the equity portfolio on the efficient frontier by using Equations (14) and (15) as:

$$\lambda_1 = \frac{(1-\delta)}{\sum_{i=1}^{5} z_i} = \frac{(1-0.75)}{58.61} = 0.0043$$
$$\lambda_2 = \frac{\delta}{\sum_{i=1}^{5} w_i} = \frac{0.75}{0.78} = 0.9637$$

4 Conclusion

In this work we offered a simple formula that simplified and shortened the process of finding the proportions invested within:

a) The Minimum Variance Portfolio,
b) The Tangent Portfolio from origin,
c) Optimal Portfolios on the Efficient Frontier,
d) The Tangent Portfolio from the Risk Free Rate,
e) Optimal portfolios on the Capital Market Line

In fact one matrix multiplication produced all the information needed to find all the characteristics of every efficient portfolio on the efficient frontier or the Capital Market Line. Thus, one can determine the proportions invested in any desired portfolio with or without bond effortlessly. Additionally we introduced a 'risk tolerance' factor that not only helps an investor to choose an optimal portfolio based on his/her risk preference, but also it reveals the Lagrangian multiplies of those portfolios

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Appendix

Appendix A

To show the solution expressed in Equation (9), we first write the slope of the Capital Market Line. Equations (1A) and (2A) show the slope of the CML. Since CML in risk-return space has the highest slope, we maximize the slope of CML.

$$\text{Maximize:} \qquad \theta = \frac{r_p - r_f}{\sigma_p} \tag{1A}$$

$$\text{Maximize:} \qquad \theta = \frac{[\mathbf{x}]^T[\mathbf{c}]}{\{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{\frac{1}{2}}} \tag{2A}$$

$$\theta = [\mathbf{x}]^T[\mathbf{c}] \times \{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{-\frac{1}{2}}$$

We maximize the slope function by taking the total differentiation of this expression with respect to the weights.

$$\frac{d\theta}{dx} = [\mathbf{c}] \times \{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{-\frac{1}{2}} + [\mathbf{x}]^T[\mathbf{c}] \times -\frac{1}{2}\left[2 \times [\Sigma][\mathbf{x}] \times \{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{-\frac{3}{2}}\right] = 0$$

$$[\mathbf{c}]\{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{-\frac{1}{2}} - [\mathbf{x}]^T[\mathbf{c}][\Sigma][\mathbf{x}].\{[\mathbf{x}]^T[\Sigma][\mathbf{x}]\}^{-\frac{3}{2}} = 0$$

$$[\mathbf{c}] = \frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\Sigma][\mathbf{x}]} \times [\Sigma][\mathbf{x}]$$

$$[\mathbf{c}] = [\Sigma]\left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\Sigma][\mathbf{x}]} \times [\mathbf{x}]\right\} \tag{3A}$$

Let $[\mathbf{v}]$ a column vector of v_i stands for $\left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\Sigma][\mathbf{x}]} \times [\mathbf{x}]\right\}$.

$$[\mathbf{v}] = \left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\Sigma][\mathbf{x}]} \times [\mathbf{x}]\right\} \tag{4A}$$

Thus, Equation (3A) can be written as:

$$\begin{aligned} &[\mathbf{c}] = [\Sigma][\mathbf{v}] \\ &[\mathbf{v}] = [\Sigma]^{-1}[\mathbf{c}] \\ &\sum_{i=1}^{n} v_i = [\mathbf{1}]^T[\mathbf{v}] \\ &\sum_{i=1}^{n} v_i = [\mathbf{1}]^T[\mathbf{v}] = [\mathbf{1}]^T\left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\Sigma][\mathbf{x}]} \times [\mathbf{x}]\right\} \end{aligned} \tag{5A}$$

Since $[\mathbf{1}]^T[\mathbf{x}] = 1$, then summation of the elements in the vector $[\mathbf{v}]$ is the following scalar.

$$\sum_{i=1}^{n} v_i = \left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}]}\right\} \tag{6A}$$

Therefore, the following ratio results to a column vector $[\mathbf{x}]_{MAX}$, which is the proportions invested in the max-slope portfolio.

$$\frac{[\mathbf{v}]}{\sum_{i=1}^{n} v_i} = \frac{\left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}]}\right\} \times [\mathbf{x}]}{\left\{\frac{[\mathbf{x}]^T[\mathbf{c}]}{[\mathbf{x}]^T[\boldsymbol{\Sigma}][\mathbf{x}]}\right\}} = [\mathbf{x}]$$

$$[\mathbf{x}]_{MAX} = \frac{[\mathbf{v}]}{\sum_{i=1}^{n} v_i} = \frac{[\mathbf{v}]}{[\mathbf{1}]^T[\mathbf{v}]} \tag{7A}$$

We use the Equations (4A) to Substitute in Equation (7A) to get the solution expressed in Equation (9).

$$[\mathbf{x}]_{MAX} = \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]}$$

Realistically, there is always an opportunity to invest in risk-free bonds. Any portfolio on the CML is a combination of risky assets and risk-free bond. Therefore, the optimal portfolios are on a line connecting the risk-free asset to a particular portfolio of the risky assets. This is also known as the 'one-fund theorem' [13]. Thus, the proportions invested in risky assets in an optimal portfolio can be presented as:

$$[\mathbf{x}]_q = \delta \times \frac{[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]}{[\mathbf{1}]^T[\boldsymbol{\Sigma}]^{-1}[\mathbf{c}]}$$

Where, $[\mathbf{x}]_q$ is the proportions invested within an arbitrary portfolio on the Capital Market Line. Once again, δ is an appraisal of investor's desire to hold risky assets and reflects the degree of investor's risk tolerance. Thus, using CML as the locus of efficient portfolios, δ asserts the investor's degree of tolerance in risky assets.

Appendix B

It is important to mention the sameness of the aforementioned "Risk Tolerance" and the Arrow-Pratt "Risk Aversion". Essentially, Arrow-Pratt approach sets a certain level of acceptable risk, say $\sigma_p{}^2$, and then maximize the expected return of the portfolio.

Maximize: $[\mathbf{x}]^T\,[\mathbf{k}] = r_p$

Subject to: $[\mathbf{1}]^T\,[\mathbf{x}] = 1$

$$\sigma_p{}^2 = \frac{1}{2}[\mathbf{x}]^T[\mathbf{\Sigma}][\mathbf{x}]$$

The Lagrange function is expressed as:

$$L = [\mathbf{x}]^T[\mathbf{k}] - \gamma_1\big([\mathbf{1}]^T[\mathbf{x}] - 1\big) - \gamma_2(\frac{1}{2}[\mathbf{x}]^T[\mathbf{\Sigma}][\mathbf{x}] - \sigma_p{}^2)$$

Since the efficient frontier curve is convex, then due to the duality principle [14] we get identical results from the maximization and the minimization process. For convenience, both optimization functions are reproduced below.

Maximize Return: $L = [\mathbf{x}]^T[\mathbf{k}] - \gamma_1\big([\mathbf{1}]^T[\mathbf{x}] - 1\big) - \gamma_2(\frac{1}{2}[\mathbf{x}]^T[\mathbf{\Sigma}][\mathbf{x}] - \sigma_p{}^2)$

Minimize Risk: $L = \frac{1}{2}[\mathbf{x}]^T[\mathbf{\Sigma}][\mathbf{x}] - \lambda_1\big([\mathbf{1}]^T[\mathbf{x}] - 1\big) - \lambda_2\big([\mathbf{x}]^T[\mathbf{k}] - r_p\big)$

The Lagrangian multipliers λ_2 and γ_2 are expressing the same concept despite the fact that λ_2 is the reciprocal of γ_2. Equation (15) in the text shows that λ_2 has a direct relationship with δ, the risk tolerance. Consequently, γ_2 (the Arrow-Pratt risk aversion index) has an inverse relationship with δ.

For instance, when δ is small, which means the tolerance for risk is low, a less risky portfolio is preferred. We get the same result when the Arrow-Pratt 'aversion to risk' is high. Conversely, when δ is large, which means the tolerance for risk is high, a more risky portfolio is preferred. This is the same as when the Arrow-Pratt 'risk-aversion index' is low and an investor is willing to consider risky portfolios. Therefore, δ indicates the investor's risk tolerance and in this work we referred to δ as the "risk tolerance index".

In the presence of risk-free bond, the definition of δ as the 'Risk Tolerance' becomes more apparent. That is, when δ is zero, the investor has no tolerance for risk and all will be invested in risk-free bonds. Conversely, when δ is 1, the tolerance for risk is high to justify all to be invested in a portfolio of risky assets. Once again, given 'δ', we can easily calculate λ_1 and λ_2 by using Equations (14) and (15) presented in the text, and the following equations find the proportions and the return of the desired portfolios.

$$\mathbf{x}_q = \mathbf{x}_{MVP} + \delta(\mathbf{x}_{TP} - \mathbf{x}_{MVP})$$
$$r_q = r_{MVP} + \delta(r_{TP} - r_{MVP})$$