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Did Linear Stochastic Processes predict accurately Short Term Interest Rate Intertemporal Behaviour?

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Abstract

This article attempts to identify the best model of the short term interest rates that can predict its stochastic process over time.
We studied nine different models of the short term interest rates. The choice of these models was the aim of analyzing the relevance of certain specifications of the short term interest rate stochastic process, the effect of mean reversion and the sensitivity of the volatility to the level of interest rate.
The yield on US three months treasury bills is used as a proxy for the short term interest rates. The parameters of the different stochastic process are estimated using the generalized method of moments. The results show that the effect of mean reversion is not statistically significant and that volatility is highly sensitive to the level of interest rates.
To further study the performance prediction of the intertemporal behavior of the short term interest rate of the various models; we simulated their stochastic process for different periods.
The results show that none of the studied models reproduce the actual path of the short term interest rates. The problem lies in the parametric specification of the mean and volatility of the diffusion process
To further study the accurate parametric specification of the interest rate stochastic process we use a nonparametric estimation of the drift and the diffusion functions. The results prove that both should be nonlinear.

JEL classification numbers: C13, C14, C15, C22, C32, C52, E43, E47
Keywords: short term interest rate, diffusion process, GMM, Monte Carlo simulation, nonparametric estimation.

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1 Introduction

The short term interest rate is one of the most important and fundamental prices determined in financial market. Several models have been developed to explain the short term interest rate stochastic process in a continuous-time framework. A list of these models include those by Merton (1973), Vasicek (1979), Brennan and Schwartz (1980a), Dothan (1978), Cox, Ingersoll and Ross (1980, 1985b), Rendleman and Bartter (1980), Cox and Ross (1976) and Chan, Karoly, Longstaff and Schwartz (1992). These models make the assumption that the spot short term interest rate follows a gauss-wiener process. The process of the short term interest rate, r, has the following formulation:

$$dr = \mu(r)\,dt + \sigma(r)\,dz$$

The drift rate, μ, and the instantaneous standard deviation, σ, are functions of r, but independent of time, and z is a wiener process.
The models mentioned above differ by their specifications of the drift and the diffusion function of the short term interest rate process.
The choice of these models is with the goal of studying the impact of some innovation in the short-term interest rate process, namely:

- Verify if the consideration of the mean reverting in the short-term interest rate process improve or not it prediction performance.
- Verify also if the consideration of the sensitivity of the volatility for the short-term interest improve or not the prediction performance of the short term interest rate process and determine the degree of the sensitivity do we should keep?

The relevance of the linear specification of the interest rate stochastic process is studied by estimating nonparametrically the drift and the diffusion functions of the stochastic process.

2 The Data

We estimate the interest rate models considering as proxy of the short-term interest rate, the US 3 month Treasury bill rate. The data are weekly and cover the period from January 1970 to December 2011, providing 2165 observations. The observations are taken from the Federal Reserve website of Saint Louis. The quoted rates are transformed to compound-yields to maturity as follow: $r_{YTM} = \frac{1}{M} In\left(1 + r_D * \frac{M}{100}\right)$ where, r_D is the observed rate and M is the maturity in years, for the 3 month Treasury bill rate, M = 0.25.
The time series of short-term interest rates shown in Figure 1 is suggestive of a change in the process during the late 1970 and early 1980. Both the level and the volatility appear elevated.
The table 1 shows the means, standards deviations and part of the 11 autocorrelation of the weekly rates and the weekly changes in the spot rate. The unconditional average level of the weekly rate is 5.46%, with a standard deviation of 3.13%. Although the autocorrelations in interest rate level decays very slowly, those of the week-to week changes are generally small and are not consistently positive or negative. This offers some evidence that the interest rates changes are stationary. The results of a formal augmented Dickey-fuller nonstationarity test are also reported in Table 1. The null hypothesis of

nonstationarity is accepted for the interest rate levels but is rejected for the interest rate changes at the 1% significance level.

Table 1: Summary statistics of the data and stationarity test

Summary statistics									
	N	Mean	Standard deviation	ρ_1	ρ_3	ρ_5	ρ_7	ρ_9	ρ_{11}
r_t	2165	5,46	3,13	0,995	0,983	0,969	0,955	0,941	0,928
r_{t+1}-r_t	2164	-0,003	0,22	0,262	0,056	0,052	-0,087	-0,027	-0,039

Augmented Dickey-Fuller stationarity test				
H_0		Test statistic	Critical values	
Nonstationarity	r_t	-1,8725	1%	-3,4343
	r_{t+1}-r_t	-35,5354	10%	-2,5708

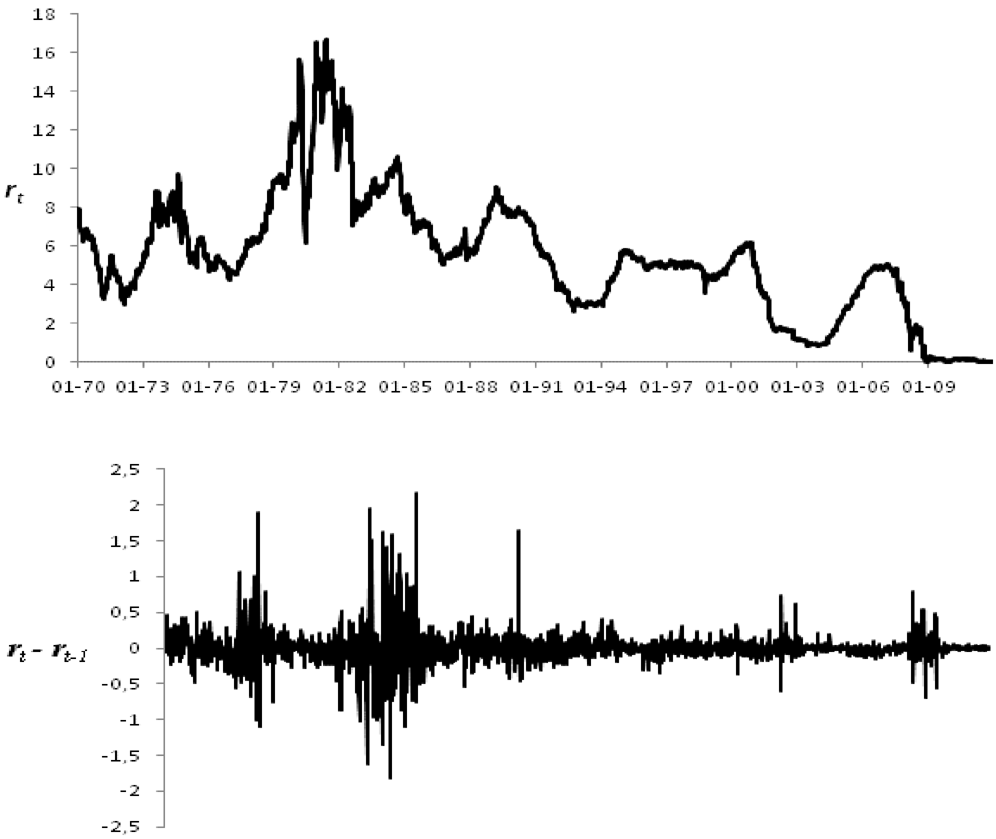


Figure 1: (a) The 3-month T-Bill rate; (b) absolute changes in the 3-month T-Bill rate

3 Research Design and Methodology

3.1 The Parametric Approach

We follow the Chan, Karolyi and Sanders (1992) econometric approach to compare the ability of nine models to capture the stochastic behaviour of the short term interest rate.
The authors present a common framework in which different models could be nested.
The following stochastic differential equation defines a broad class of interest rate processes,

$$dr_t = (a + br_t)dt + \sigma r_t^{\gamma} dz \tag{1}$$

The dynamics implies that the conditional mean and variance of changes in the short term interest rate depends on the levels of *r*. The model incorporates mean reversion on the interest rate; i.e, the interest rate is pulled back over time to some long-run average. When *r* is high, mean reversion tends to cause it to have a negative drift; when *r* is low mean reversion tends to cause it to have a positive drift. The mean reversion phenomenon is included in the stochastic process by the specification of the drift, $\beta(\alpha - r)$, where the speed of adjustment is given by the parameter β and the long-run average is given by the parameter α. The short rate is pulled to level α at rate β. So we have that $a = \alpha\beta$ and $b = -\beta$.
The parameters of the stochastic process given by (1) are estimated in discrete time using Generalized Method of Moments (GMM) technique of Hansen (1982). This technique has a number of advantages which makes it one of the best methods for estimating the short-term interest rate process. Indeed, GMM provides a unified approach to the econometric estimation of all different types of short-term interest rates. Moreover, to achieve the asymptotic convergence of the estimator, GMM does not require that the distribution of the interest rate changes is normal but only stationary and ergodic is to say that the instantaneous conditional residuals variance is proportional to the length of the sample. This feature is of particular importance for the estimation of the short-term interest rate models when each model implies a different distribution of interest rate changes. In fact, for the Merton (1973) and Vasicek (1977) models, changes in interest rates are normal, while for the model Cox Ingersoll and Ross (1985), they are proportional to a noncentral χ^2.
We test the restrictions imposed by the alternative short term interest rate models nested within equation (1).
Several models can be obtained from (1) placing the appropriate restrictions on the four parameters α, β, σ and γ. The specifications that we focus on are presented in Table 2:

Table 2: Alternative models of short-term interest rate and parameter restrictions imposed

Model	*Restrictions*			
	a	b	σ	γ
Chan, Karoly, Longstaff & Sanders (1992)	—	—	—	—
Merton (1973)	—	0	—	0
Vasicek (1977)	—	—	—	0
Cox, Ingersoll & Ross (1985)	—	—	—	0.5
Dothan (1978)	0	0	—	1
Rendleman & Bartter (1980)	0	—	—	1
Brennan & Schwartz (1980)	—	—	—	1
Cox, Ingersoll & Ross (1980)	0	0	—	1.5
Cox & Ross (1976)	0	—	—	—

We estimate the parameters of the continuous-time model using a discrete-time econometric specification

$$r_{t+1} - r_t = a + b\, r_t + \varepsilon_{t+1} \tag{2}$$

$$E\left[\varepsilon_{t+1}\right] = 0\,,\ E\left[\varepsilon_{t+1}^2\right] = \sigma^2\, r_t^{2\gamma} \tag{3}$$

This discrete-time model has the advantage of allowing the variance of interest rate changes to depend directly on the level of the interest rate in a way consistent with the continuous-time model.

Define θ to be the parameter vector with elements α, β, σ^2 and γ and given $\varepsilon_{t+1} = r_{t+1} - r_t - a - br_t$, estimators of these parameters are obtained from the first and second moments conditions. We define also two instrumental variables, a constant and r_t. we obtain then, four orthogonality restrictions.

$$\underset{4\times1}{f_t}\left(a, b, \sigma^2, \gamma\right) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ r_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ \left(\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}\right) r_t \end{bmatrix} \tag{4}$$

Under the null hypothesis that the restrictions implied by (2) and (3) are true, $E\left[f(\theta)\right] = 0$

The GMM procedure consists of replacing $E\left[f(\theta)\right]$ with its sample counterpart, g[θ], using T observations where

$$g(\theta) = \frac{1}{T}\sum_{t=1}^{T} f(\theta) \tag{5}$$

And then choosing parameters that minimize the quadratic form,

$$J_T(\theta) = g'_T(\theta)W_T(\theta)g_T(\theta) \tag{6}$$

Where $W_T(\theta)$ is a positive-definite symmetric weighting matrix.

The minimized value of the quadratic form in (6) is distributed χ^2 under the null hypothesis that the model is true with degree of freedom equal to the number of orthogonality conditions net of the number of the parameters to be estimated. This χ^2 measure provides a goodness-of-fit test for the model. A high value of this statistic means that the model is misspecified.

In addition, in order to gauge further the relative performance of the alternative nested models, we test their forecast power of interest rate changes. In addition, we test their forecast power for squared interest rate changes, which provide simple ex-post measures of interest rate volatility. This is done by first computing the time series of conditional expected-yield changes and conditional variances for each model using the fitted values of (2) and (3). We then compute the proportion of the total variation in the ex post yield changes or squared yield changes that can be explained by the conditional expected-yield changes and conditional volatility measures, respectively. We refer to this as the coefficient of determination, or R^2. These R^2 values provide information about how well each model is able to forecast the future level and volatility of the short term rate. The results are presented in the last two columns of Table 3.

Additionally to compare the performance of each model to capture the stochastic evolution of the spot interest rate, we simulate the path of the interest rate produced by each model and we compare it to the real short term interest rate stochastic path.

To generate data from the specification on interest rate model, we consider a first order Euler's approximation.

The discrete time version of the CKLS (1992) is as follow:

$$r_{t+\Delta} = r_t + \beta(\alpha - r_t)\Delta_t + \sigma r_t^{\gamma}\sqrt{\Delta t}\,\varepsilon_{t+\Delta} \text{ , where } \varepsilon_{t+\Delta} \approx N(0,1) \tag{7}$$

The discrete time version of the other eight models is obtained by imposing the appropriate restrictions on the four parameters α, β, σ and γ as in table 2.

Δt is the time step between two successive observations. Given the weekly frequency of the data the time interval, Δt, is equal to 1/52.

The study of the predictive performance of the different nine models will be on both sides, a study, of the predictive performance, "in the sample" and "out of the sample".

- The first period "in the sample" cover the period from 1979 to 1982 which is a high volatile period. The purpose of this choice is to study the predictive performance of the models in this exceptional period.
- The second period "in the sample" cover the period from 2007 to 2008 which is characterise by the subprime crisis and at the end of 2008, the Federal Reserve have decide to reduce interest rates at a range of 0% to 0.25%.
- Contrary to the two first periods, the third "in the sample" period from 1997 to 1998 is relatively a stable period.
- The "out of the sample" period cover the period from 2010 to 2011, characterized by low interest rates as decided by the Federal Reserve.

The period as chosen as we can study the predictive performance of each model in normal and turbulent periods.
The performance of each model to predict the real short term interest rate path is measured by the "Mean Squared Error":

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\sqrt{r_{io} - r_{is}} \tag{8}$$

Where N, is the observation number, r_{io}, is the i^th^ observed interest rate and r_{is}, is the i^th^ simulated interest rate.

3.2 The Nonparametric Approach

One potentially serious problem with any parametric model that prefers one functional form another is misspecification which can lead to serious pricing and hedging errors.
For further study the short interest rate stochastic process specification we use the nonparametric approach to estimate the functional form of the drift and diffusion functions of the interest rate stochastic process. The nonparametric approach does not impose any restrictions on their functional forms but leave them unspecified. The resulting functional forms should result in a process that follows interest rate closely.
The nonparametric approach presents the flexibility to fit the data allowing the identification of the appropriate specification of the interest rates stochastic process.
Florens-Zmirou (1993) and Ait-Sahalia (1996) pioneered the idea of modeling the diffusion function of for the stochastic interest rate process by the data themselves through a nonparametric approach. The idea has been extended to both the drift and the diffusion functions by Stanton (1997), Jiang and Knight (1997) and, more recently, by Bandi and Phillips (2003).
Renò, Roma and Schaefer (2006) prove that the Stanton and Bandi and Phillips estimators perform better than the Ait-Sahalia estimator.
In this study we follow the Stanton approach (1997). In contrary to the Ait-Sahalia that proposes a nonparametric diffusion function estimator based on the linear mean-reverting drift function for the stochastic process, the Stanton approach (1997) avoids making parametric assumptions about either the drift or the diffusion functions of the interest rate stochastic process; it estimates both functions nonparametrically from data observed only at discrete time intervals.
This approach consists of the construction of approximation of the true drift and the diffusion functions then these approximations are estimated nonparametrically from discretely sampled data. More specifically, Stanton (1997) uses the infinitesimal generator and a Taylor series expansion to give the first-order approximations to the drift and the diffusion functions.
Consider, the diffusion process of the interest rate, r_t, which satisfies the stochastic differential equation:

$$dr(t) = \mu(r_t)dt + \sigma(r_t)dz_t \tag{9}$$

The first order approximations of the drift and diffusion functions, under the Taylor series is respectively as follows:

$$\mu(r_t) = \frac{1}{\Delta} E \ (r_{t+\Delta} - r_t) + O(\Delta) \tag{10}$$

$$\sigma^2(r_t) = \frac{1}{\Delta} E \ (r_{t+\Delta} - r_t)^2 + O(\Delta) \tag{11}$$

Where Δ denotes a discrete time step in a sequence of observations of the process r_t and O(Δ), the asymptotic order symbol where $\lim_{\Delta \to 0} O(\Delta) = 0$.

The nonparametric estimation of the approximations of the drift and the diffusion function are based on the stationary density.

Let $\{r_t^{\Delta}\}_{t=1}^{T}$ be a sample of size T from the continuous time process r_t, observed at discrete interval Δ. Furthermore, let $\{u_i\}_{i=1}^{N}$ be a set of size N points defining an equally spaced partition of a subset of the support of the stationary density. If the stationary density of r_t is denoted *f(u)*, the Rosenblatt-Parzen kernel estimator is of the following form:

$$\hat{f}(u) = \frac{1}{Th} \sum_{t=1}^{T} K\left(\frac{u - r_t}{h}\right) \tag{12}$$

The kernel estimator is completely characterized by the choice of a particular kernel function and the appropriate bandwidth h.

The kernel function provides a method of weigthing “nearby” observations in order to construct a smoothed histogram of the density estimator. In our case we use the Gaussian kernel, $K(u) = \sqrt{2\pi}\, e^{-1/2u^2}$.

The parameter h is called the smoothing parameter; it determines the width of the kernel function around any partition point u_t, it specifies how (and how many) “neighboring” points of r_t^{Δ}, are to be considered in constructing the density estimator at r_t. in our case we choice $h = 4 \times \hat{\sigma}\, T^{-1/5}$

Now, the drift and diffusion function can estimated nonparametrically using the familiar Nadaraya-watson kernel regression estimator as follow:

$$\hat{\mu}(r) = \frac{\sum_{t=1}^{T-1} (r_{t+\Delta} - r_t) K\left(\frac{r - r_t}{h}\right)}{\sum_{t=1}^{T-1} K\left(\frac{r - r_t}{h}\right)},$$
$$\hat{\sigma}(r)^2 = \frac{\sum_{t=1}^{T-1} (r_{t+\Delta} - r_t)^2 \ K\left(\frac{r - r_t}{h}\right)}{\sum_{t=1}^{T-1} K\left(\frac{r - r_t}{h}\right)} \tag{13}$$

The Stanton approach by (1997) is convenient because it enables us to estimate μ(r) and σ^2(r) separately, whereas the approaches by Ait-Sahalia (1996), Jiang and Knight (1997) and Jiang (1998) require their sequential estimation.

4 Empirical Findings and Results Analysis

In this section, we present our empirical results. We begin by estimating the unrestricted and the eight restricted interest rate processes. We compare the models in terms of their explanatory power for an ex post measure of interest rate volatility. We compare after their ability to reproduce the stochastic path of the short term interest rate through a simulation study. Finally, we present the results of the nonparametric estimation of the drift and diffusion function of the short term interest rate process

4.1 Estimation Results and Models Comparison

Table 3 reports the parameters estimates, asymptotic t-statistics, and GMM minimized criterion (χ^2) values for the unrestricted model and for the each of the eight nested models. As shown, the models vary in their explanatory power on interest rate changes. The χ^2 tests for goodness-of-fit suggest that all the models that assume $\gamma \leq 1$ are misspecified. In fact Merton (1973), Vasicek (1977), Cox Ingersoll & Ross (1985), Dothan (1978), Rendleman & Bartter (1980), Brennan & Schwartz (1980) models, have χ^2 values, in excess of 5% and can be rejected at the 95% confidence level. Except for the Cox, Ingersoll & Ross (1980), and the Cox & Ross (1976) models, those assume $\gamma > 1$. These models present values of the χ^2 relatively low and cannot be rejected at the 5% significance level.

These results suggest that the relation between interest rate volatility and the level of r is the most important feature of the dynamic model of the short term interest rate. This is significant since the Vasicek (1977) and Merton (1973) models are often criticized for allowing negative interest rates. This result indicates that a far more serious drawback of these models is their implication that interest rate changes are homoskedastic.

The estimates of the models provide also a number of interesting insights about the dynamics of the short term interest rate. First, the weak evidence of the mean reversion in the short term interest rate; the parameter β is insignificant in the unrestricted model and also in all the restricted models. Second, the conditional volatility of the process is highly sensitive to the level of the short term interest rate. The unconstrained estimate of γ in the Cox & Ross (1976) and Chan, Karoly, Longstaff & Sanders (1992) models are respectively 1.5513 and 1.5424. This result is important since these values are higher than the values used in most of the models. In particular, six of the eight nested models imply $0\leq\gamma\leq1$. The t-statistic for γ is 9.20 and 8.05 respectively for the Cox & Ross (1976) and Chan, Karoly, Longstaff & Sanders (1992) models, which imply that the parameter γ is highly significant. These results prove the importance of the relation between volatility of the interest rate and the level of the short term interest rate in the dynamic of the interest rate and that the degree of the sensitivity is higher than 1.5.

These findings are similar to those of Ferreira (1998) that has followed the same approach for Portuguese interest rates. The results show a weak mean reverting effect and a high sensitivity of the volatility to the interest rate level equal to 1.13.

The last two columns of the Table 3 present the results of the forecast power of all the models for interest rate changes and the squared interest rate changes. The first R^2 measure describes the fit of the various models for the actual yield changes. Expect for the Merton (1973), Dothan (1978), Cox, Ingersoll & Ross (1980), and the Cox & Ross (1976) models which have no explanatory power for interest rate changes, the other models are similar in their forecast ability. They explain only 0.01% to 0.13% of the total variation in yield changes.

For the volatility of interest rate changes, the proportion of the total variation in volatility captured by the various models ranges from 0.44% from the Cox, Ingersoll & Ross (1985) model to 17.4% for the & Ross (1976) model. Note that the R^2 for the Merton (1973) and Vasicek (1977) models are zero since these models imply that the volatility of interest rate changes is constant. Remark that the higher predictive power for the volatility for interest rate changes is for the models that assume an estimated value of $\gamma \geq 1.5$. It is equal to 17.4% for the Cox & Ross (1976) model and 15.86% for the Chan, Karoly, Longstaff & Schwartz (1992) model.

These results are similar to those produced by the χ^2 test, which prove again the importance of the sensibility of the volatility of the interest rate to the level of short term interest rate in the dynamic of the spot interest rate. Figure 2 plot the absolute value of the interest rate changes and the estimated conditional volatility estimate from the Cox, Ingersoll & Ross (1985) and the Chan, Karoly, Longstaff & Schwartz (1992) models.

We remark that contrary to the Cox Ingersoll & Ross (1985) model, the Chan, Karoly, Longstaff & Schwartz (1992) model reproduce nearly exactly the shape of observed volatility of interest rate changes without adjusting the actual levels of interest rates .

Table 3: Estimates of alternative models for the short term interest rate

Models	*a*	*b*	σ^2	γ	χ^2 *test (p-value)*	*dl*	R_1^2	R_2^2
Chan Karoly Longstaff Sanders (1992)	0.004510 (0.45)	-0.113786 (-0.54)	0.915860 (1.55)	**1.542416** (8.05)	0.0000	0	0.0008	0.1586
Merton (1973)	-0.000776 (-0.36)	0.0000	0.000121 (6.05)	0.0000	15.5003 (0.0004)	2	0.0000	0.0000
Vasicek (1977)	0.002520 (0.27)	-0.071269 (-0.34)	0.000122 (5.98)	0.0000	15.5172 (0.0001)	1	0.0003	0.0000
Cox Ingersoll and Ross(1985)	0.003209 (0.34)	-0.092045 (-0.44)	0.002739 (6.91)	0.5000	14.5178 (0.0001)	1	0.0005	0.0044
Dothan (1978)	0.0000	0.0000	0.048578 (8.52)	1.0000	10.3857 (0.0156)	3	0.0000	0.0298
Rendleman and Bartter (1980)	0.0000	-0.028005 (-0.62)	0.049714 (8.31)	1.0000	10.0911 (0.0064)	2	0.0001	0.0312
Brennan and Schwartz (1980)	0.005209 (0.55)	-0.141874 (-0.67)	0.050287 (8.25)	1.0000	9.8074 (0.0017)	1	0.0013	0.0319
Cox Ingersoll and Ross(1980)	0.0000	0.0000	0.725366 (9.69)	1.5000	0.5475 (0.9083)	3	0.0000	0.1379
Cox and Ross (1976)	0.0000	-0.015935 (-0.35)	0.949175 (1.57)	**1.551295** (8.20)	0.2243 (0.6358)	1	0.0000	0.174

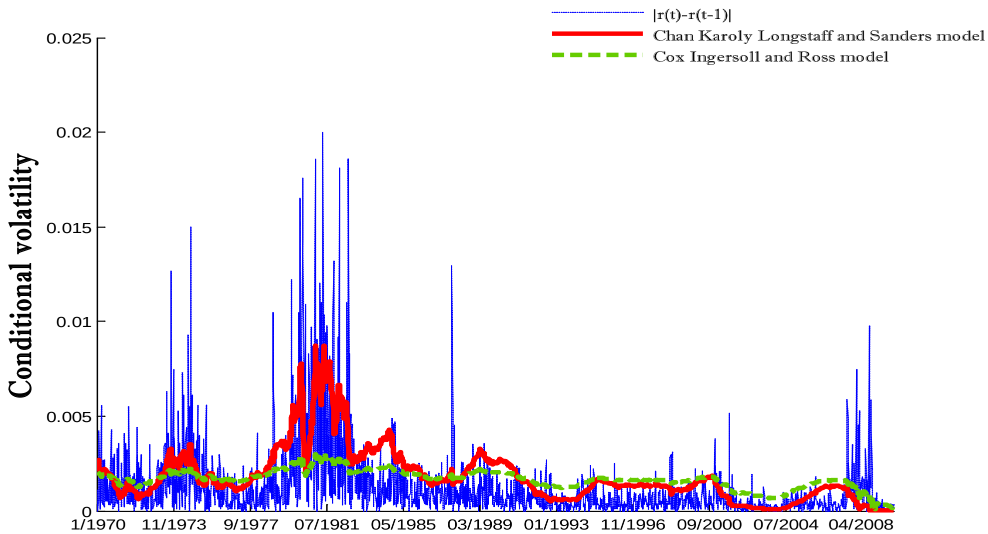


Figure 2: Forecast of weekly ex post volatility of short term interest rate using the Cox, Ingersoll & Ross (1985) and the Chan, Karoly, Longstaff & Schwartz (1992) models.

4.2 Monte Carlo Simulation Results

We analyse first, the results of the simulation “in the sample”.

For the first period “in the sample” from 1979 to 1982, which is characterized as a very volatile period, from the figure 3(a), we remark that no model was able to predict the highly variation in short term interest rate.

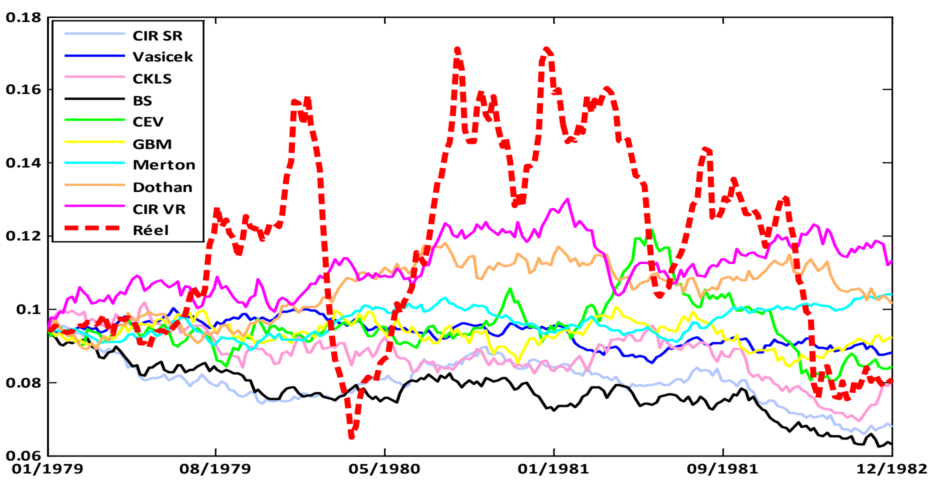
For the second period “in the sample” from 2007 to 2008, which is a characterized by a downward trend of the interest rates, from the figure 3(b), we remark that as for the first period, no model was able to predict the continuous decline of the interest rates.

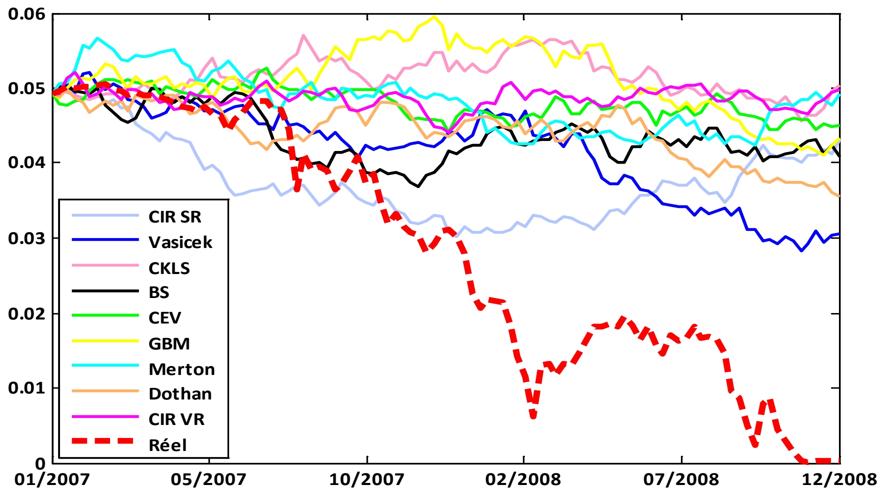
For the third period “in the sample” from 1997-1998, which is relatively the most stable, we remark from the figure 3(c), that the models that reproduce relatively the nearest path to the real interest rate evolution are the Chan, Karoly, Longstaff & Schwartz (1992), the Cox & Ross (1976) and the Rendleman & Bartter (1980) models. These models have the lowest “mean squared error”. These models present a high sensitivity of the volatility to the level of the interest rate which confirms the results of the GMM estimation.

For the “out of sample” period from 2010 to 2011, characterized by a very low level of the interest rate, the Cox & Ross (1976) and the Rendleman & Bartter (1980), the Cox, Ingersoll and Ross (1980) and Dothan (1978) models produce the nearest path to the real inter-temporal path of the interest rate (figure 3d). Their paths are confused. They present the lowest “mean squared error”. In fact they show level of interest rate similar to those really observed but they do not produce the real evolution shape. This can be explain by the fact that the interest rates are close to zero and the inter-temporal evolution is nearly constant, so the drift of the interest rate tends to zero almost, which is the case of the Cox & Ross (1976) model and the Rendleman & Bartter (1980) models that present a drift proportional to the interest rates level and Cox, Ingersoll and Ross (1980) and Dothan (1978) that have drift equal to zero.

Table 4: "Mean Squared Error" of the Monte Carlo simulation

Model	MSE			
	« In sample » 1979-1982	« In sample » 1997-1998	« In sample 2007-	« out of sample » 2010-2011
Chan Karoly Longstaff Sanders (1992)	*4.5662e-009*	*4.6321e-009*	2.3690e-005	*2.1168e-007*
Cox Ingersoll Ross (1985)	*6.0405e-007*	*2.3946e-008*	*1.6333e-005*	*1.7988e-007*
Vasicek (1977)	*3.1532e-007*	*1.4628e-006*	*8.8589e-006*	*1.8967e-006*
Merton (1973)	*2.8466e-006*	*2.5116e-007*	*2.1893e-005*	*7.4431e-006*
Cox & Ross (1976)	*6.4781e-008*	*4.4038e-009*	*1.9411e-005*	*2.4801e-009*
Rendleman and Bartter (1980	*6.8501e-007*	*1.5526e-009*	*1.7262e-005*	*2.1617e-009*
Brennan and Schwartz (1980)	*1.2494e-006*	*3.0099e-008*	*1.7230e-005*	*2.2512e-007*
Dothan (1978)	*2.5055e-006*	*4.6191e-007*	*1.2460e-005*	*1.1951e-009*
Cox Ingersoll and Ross(1980)	*4.9648e-006*	*4.5464e-007*	*2.3548e-005*	*2.4038e-009*





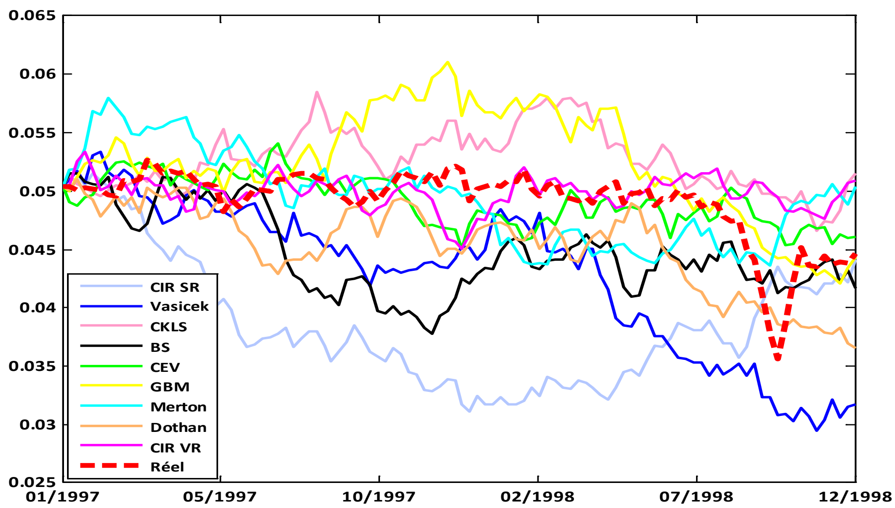


Figure 3: Simulation "in the sample"; (a) 1979-1982 period; (b) 2007-2008 period, (c) 1997-1998 period.

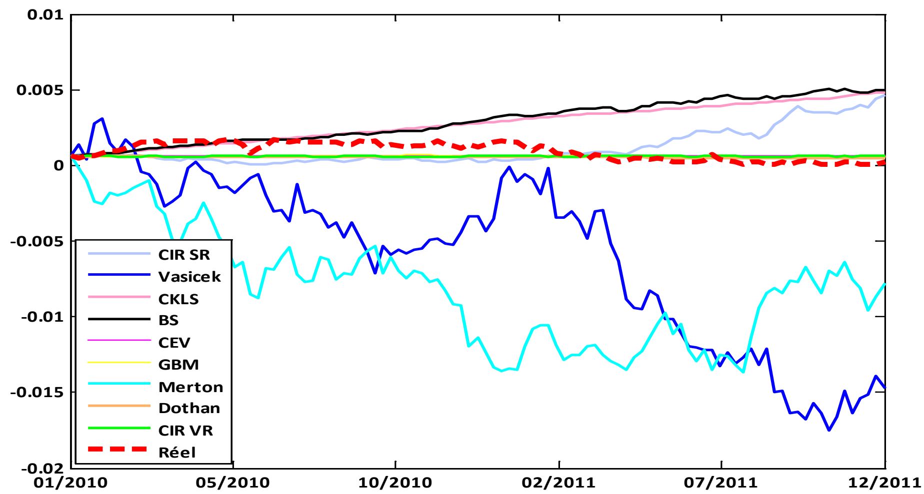


Figure 4: Simulation « out of the sample » 2010-2011 period.

4.3 Estimation Results of the Non-parametric Model

To estimate the drift function, the stationary density of the short term interest rate is estimated first and plotted in Figure 5. The non parametric stationary density is obviously not normal with a flatter right tail than the normal density. This tail corresponds to the interest rate above 10 par cent.

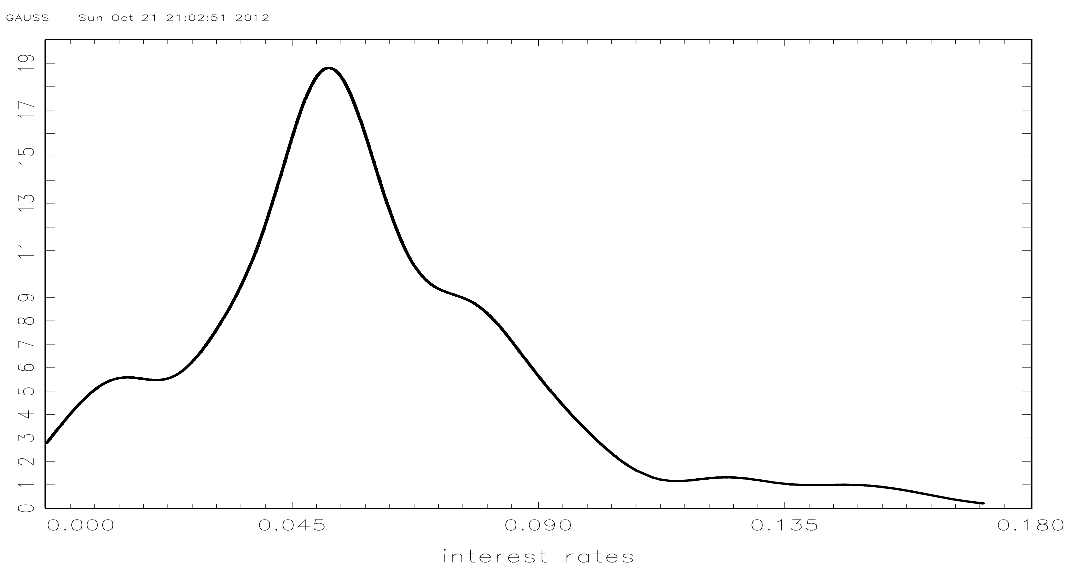


Figure 5: Nonparametric stationary density of short rate

From Figure 6, we note that the drift is constant and it is close to being zero for low and medium values of interest rate. But when the short rate is beyond 14%; the short rate drift decrease dramatically. It presents a negative linear trend.

This confirm the empirical finding of Ait-Sahalia (1996), Stanton (1997), Jiang and Knight (1997), Jiang (1998), Sam and Jiang (2009) and Gospodinov and Hirukawa (2011) that the drift term of the short term interest rate is zero for the most of interest rate ranges and overall nonlinearly mean reverting.

These results prove also the finding of Arapis and Gao (2006) that the nonparametric drift is unlike the linear mean reverting specification.

These findings suggest that interest rates follow a random walk at low and medium level, while being overall stationary. But because of the high level of interest rates (beyond 14%) observed mainly during the period of early 80's, the nonstationarity test based on the short-term interest rate level is rejected.

Compared to the linear mean-reverting drift function (Figure 6), the nonparametric drift function shows a much weaker mean-reverting property for low and medium interest rate but a much stronger mean-reverting property for high interest rates. This finding confirms that the mean reverting is not significant for low and medium interest rate levels.

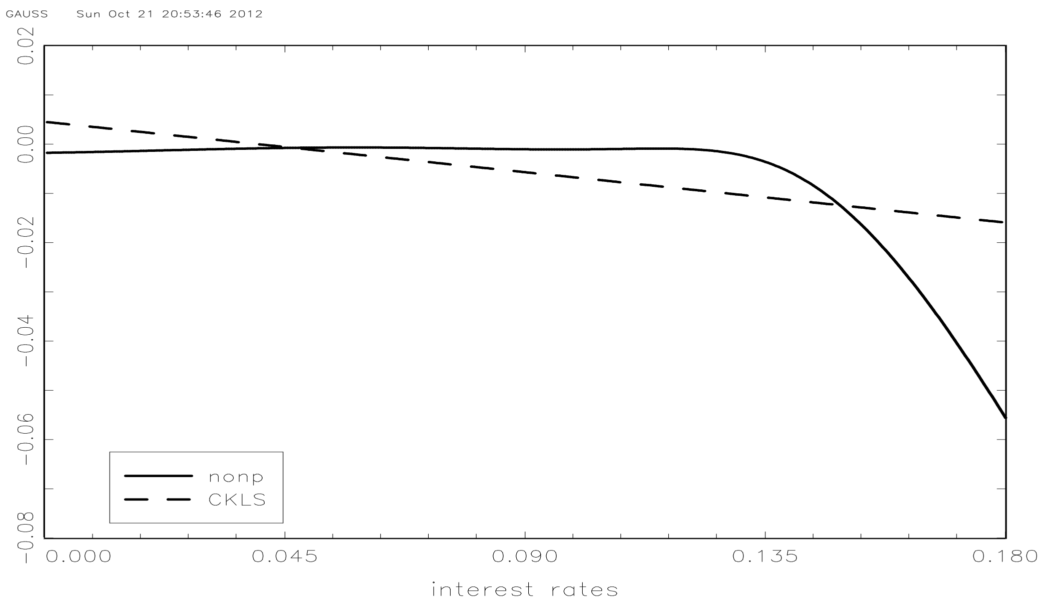


Figure 6: Non-parametric and Chan Karoly Longstaff and Sanders (1992) drift functions

The nonparametric diffusion function is plotted in Figure 7 and compared to those of the Chan, Karoly, Longstaff & Schwartz (1992) and Cox, Ingersoll and Ross (1985) models. Noticeable features of the nonparametric diffusion function estimator include: first, the diffusion function is a nonlinear but overall increasing function of the short rate, supporting the "level-effect" conjecture and rejecting the constant volatility model. That is, low interest rates are associated with low interest rates volatility and high interest rates are associated with high interest rates volatility. This result proves that the volatility is nonlinear contrary to the parametric models that suppose that the volatility is linear. These results are similar to those of Ait-Sahalia (1996), Stanton (1997), Jiang and Knight (1997), Jiang (1998), Sam and Jiang (2009) and Gospodinov and Hirukawa (2011).

Second, the nonparametric diffusion function, in general, agrees with that of Chan, Karoly, Longstaff & Schwartz (1992) models but shows more variation, but is different to the diffusion function of the Cox, Ingersoll and Ross (1985) model which presents a smaller volatility especially for high interest rate levels.

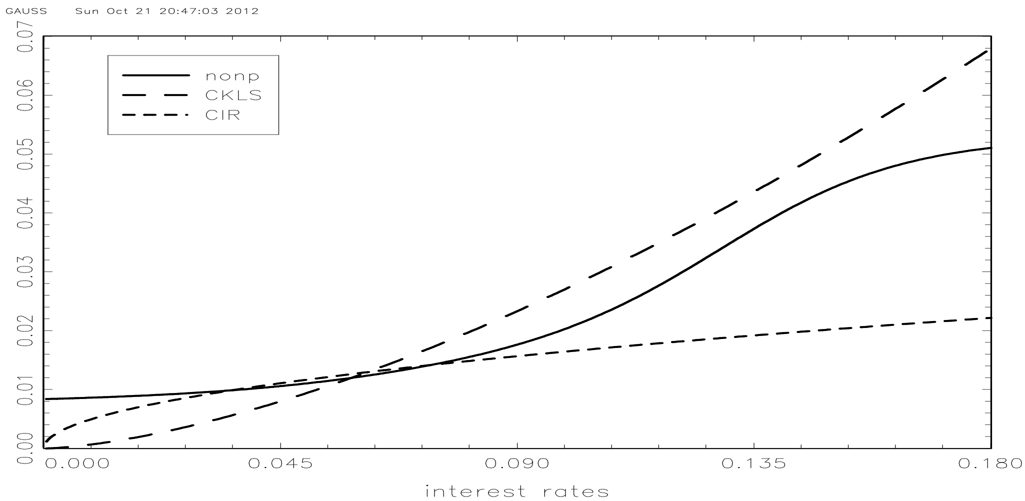


Figure 7: Non-parametric, Chan Karoly Longstaff Schwartz (1992) and Cox Ingersoll & Ross models diffusion functions

5 Conclusions

In this paper, we investigate the appropriate features of the short term interest rate. The results indicated that models that allowed the variability of interest rates to depend upon the level of interest rate captured the dynamic behaviour of short term interest rates more successfully.

The level effect is such that the interest rate volatility is positively and highly correlated with the level of interest rates. In addition the results prove also that the evidence on mean reversion in the short term to be not significant.

The results of the Monte Carlo simulations done on the nine different linear models prove that no model was able to predict the particular variation in short term interest rate observed in crisis period. For the relatively stable period, no model produces the real shape of the interest rate evolution. This is due to the misspecification of the interest rate stochastic process presented by these models. In fact all the models we studied assume a linear drift and diffusion functions but the nonparametric estimation results prove that both are nonlinear.

The nonparametric estimates reveal that the mean reversion in the nonlinear drift is weak for low and medium interest rates levels but becomes stronger at higher interest rates and the diffusion functions have the usual shape showing that volatility increases with the level of interest rates with a nonlinear shape

From these results, we can say that the linear stochastic process cannot predict accurately the short term interest rate intertemporal behavior and that the accurate parametric formulation of the interest rate stochastic process is a nonlinear formulation of the drift and diffusion functions.

References

[1] Ait-sahalia, Y, Nonparametric pricing of interest rate derivative securities, Econometrica **64**, (1996), 527-560.

[2] Arapis, M. Gao, J, Empirical comparisons in short-term interest rate models using nonparametric methods. Journal of Financial Econometrics, **4**, (2006), 310-345.

[3] Bandi, F. Phillips, P., Fully nonparametric estimation of scalar diffusion models. Econometrica **71**, (1), (2003), 241–283.

[4] Brennan, M., E. Schwartz, A Continuous Time Approach to the Pricing of Bonds, Journal of Banking and Finance, **3**, (1979), 135-155.

[5] Chan, K. C., G A Karolyi, F. A. Lonstaff, A. B. Sanders, An empirical comparison of alternative models of the short-term interest rate, Journal of Finance **47**, (1992), 1209-1227.

[6] Cox, J. C., J. E. Ingersoll, S. A. Ross, An Analysis of Variable Rate Loan Contracts. Journal of Finance **35**, (1980), 389–403.

[7] Cox, J. C., J. E. Ingersoll, S. A. Ross, An intertemporal general equilibrium model of asset prices, Econometrica, **53**, (1985a), 363–384

[8] Cox, J. C., J. E. Ingersoll, S. A. Ross, A Theory of the Term Structure of Interest Rates, Econometrica, **53**, (1985b), 385–407.

[9] Ferreira, A.M, An empirical test of short term interest rate models, Review of financial markets, **1**, (1998), 29- 48.

[10] Florens-Zmirou, D., On estimating the diffusion coefficient from discrete observations, Journal of Applied Probability **30**, (1993), 790–804.

[11] Gospodinov, N. Hirukawa, M, Nonparametric estimation of scalar diffusion processes of interest rates using asymmetric Kernels, (2011), working paper.

[12] Hansen, L., Large sample properties of generalized method of moment estimators, Econometrica, **50**, (1982), 1029-1056.

[13] Jiang, G. J. Knight, A nonparametric approach to the estimation of diffusion processes, with an application to a short-term interest rate model. Econometric Theory **13**, (1997), 615-645.

[14] Jiang, G. J, Nonparametric modeling of US interest rate term structure dynamics and the implications on the prices of derivatives securities, Journal of financial and quantitative analysis, **4**, (1998), 565-499.

[15] Merton, R., Theory of Rational Option Pricing, Bell Journal of Economics and Management Science, **4**, (1973), 637-654.

[16] Rendleman, R. and B, Bartter, The Pricing of Options on Debt Securities. Journal of Financial and Quantitative Analysis, **15**, (1980), 11–24.

[17] Renò, R. Roma, A. Stephen, S, A Comparison of Alternative Non-parametric estimators of the Short Rate Diffusion Coefficient, Economic Notes, **35**, (2006), 227-252.

[18] Sam, A G. Jiang, G, Nonparametric estimation of the short rate diffusion process from a panel of yields, Journal of Financial and Quantitative Analysis, **44**, (2009), 1197-1230.

[19] Stanton, R., A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk, Journal of Finance, **52**, (1997), 1973-2002.

[20] Vasicek, O. A., An equilibrium characterization of the term structure, Journal of Financial Economics **5**, (1977), 177-188.