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Measuring the Dependency between Securities via Factor-ICA Models

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Abstract

This paper proposes a new method for measuring the dependency between securities. Applying independent component analysis to the return data of the whole component securities in a universe, independent factors composing the returns are extracted. Reconstructing return data of individual component security based on such factors, we find that each security has a unique factor hierarchy. A comparative analysis of the hierarchies can find dependence structures between securities. Empirical studies show that the new method outperforms old measures based on correlation, and that it reveals very delicate dependence structures which otherwise remain hidden. Useful examples of applying it to the portfolio or risk management are also provided.

JEL Classification: C14, C45, C58, C63
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1 Introduction

When you select securities from a given universe to construct your own portfolio, it will pay off to take into consideration the dependence structure between securities. However, traditional tools in statistics measure only part of the dependence structure, and thus may be misleading in financial turmoils when frequent structural breaks in the data generating process would occur, or black-swan-like tail events could take place in the financial market. As noted in [1], the linear correlation is a measure of dependence for elliptically distributed variables,[3] and thus fallacies arise from the naive assumption that the dependence properties of elliptical world also hold in non-elliptical world.[4] What is worse, it only measures the central dependency, and hence is not able to explain the tail dependence. In finance, it is an prominent example of tail dependence that the stock returns are asymmetric in the sense that they are more highly dependent during market downturns than during market upturns [3]. Recently, [4] proposes a local correlation function to handle such asymmetry. However, it is only for bi-variate Gaussian distributions. Though various concepts of dependence are discussed in Chapter 5 of [5], their highly abstract theoretical nature prevents us from applying them to the real financial data.

The main objective of this paper is to devise a practical tool for measuring the interdependence between securities, which is easy to apply to the real financial data, not bound to a specific category of distributions, and can measure the whole aspects of dependency. When you say two persons from a family resemble each other, you are referring to the similarity between their entire physical features. They look alike because they share the same blood. Likewise, if we would extract, from the return data, the fundamental factors which compose each security return and find how the factors are structurally related to it, we could determine, by comparing the structures, how much two different securities are similar to each other.

ICA is a novel statistical signal processing technique to find independent sources given only observed data that are mixtures of unknown sources without

[3]Canonical examples are multivariate normally distributed variables.

[4]It is well known empirical facts that the distribution of security returns is in the non-elliptical world. Security returns are characterized not by normality but by the stylized facts such as fat tails, high peakedness (excess kurtosis) and skewness [2].

any prior knowledge of the mixing mechanism [6, 8]. It represents the original data with the components that are statistically independent, or as independent as possible. Such a representation captures the essential structure of the data in many applications [7]. [9], [7], and [11] provide excellent overviews on ICA. ICA has been successfully applied to financial time series and revealed some driving mechanisms that otherwise remain hidden [13, 19, 20, 17, 21, 16, 22, 25, 24, 23].

This paper proposes a new method for measuring the dependency between securities in a given universe. Applying independent component analysis to the return data of the whole component securities in the universe, we can extract independent factors which compose the returns of component securities. Reconstructing return data of each individual component security based on such factors, we can find hierarchy between the factors: we can order the factors according to their relative importance in reconstructing each individual security return. Each security has a unique factor hierarchy under certain conditions. Thus we can represent each security return with a linear combination of independent factors in the order specific to the security. Based on the fact that the security returns are non-Gaussian distributed and that the independent factors extracted by ICA are also non-Gaussian distributed (see Section 2), the linear combination can be considered as a proper representation of the security return.

A comparative analysis of the resulting hierarchies can find dependence structures between securities in a nonparametric and distribution-free context: since every return consists of the same independent factors, and has a unique hierarchy that determines the relative importance of each factor for the reconstruction of its return data, we can find the dependency between securities by comparing their factor hierarchies.

Due to the non-elliptical distributional attributes of security returns, those measures based on correlation, which is only a measure of dependence for elliptically distributed variables, cannot appropriately measure the dependency between securities. Whereas, we can find the dependency between securities appropriately by comparing their factor hierarchies. Empirical studies in this paper show that the new method outperforms the old measures in the sense that it can measure the whole aspects of dependency by comparing securities factor by factor of which the returns of the securities are composed. Furthermore, empirical studies show that the new method reveals very delicate dependence structures that otherwise remain hidden. We also provide useful examples of applying this new method to various areas in finance such as portfolio management or risk management.

This paper will proceed in the following order: Section 2 introduces Factor-ICA model; Section 3 explains the procedure of ordering the independent factors; Section 4 defines new measures of dependency; Section 5 reports empirical results of these new measures, and shows that they outperforms those measures based on correlation in many respects; Section 6 comments on several issues; Section 7 presents examples of applying the new measures to various areas in finance; Section 8 concludes this paper with a summary.

2 Factor-ICA Model

2.1 Independent Component Analysis (ICA)

ICA is a method for blind source separation developed in the area of signal processing [12]. Suppose that we can observe random variables $x_1, x_2, \cdots, x_N$ which are assumed to be linear combinations of unknown independent sources $s_1, s_2, \cdots, s_N$. Arranging the observed random variables and the sources into $\mathbf{x} = (x_1, x_2, \cdots, x_N)'$ and $\mathbf{s} = (s_1, s_2, \cdots, s_N)'$ respectively, a basic ICA model can express the linear relationship as $\mathbf{x} = \mathbf{A}\mathbf{s}$, where $\mathbf{A}$ represents a unknown $N \times N$ matrix of full rank, which is called mixing matrix. Given only the observed data that are mixtures of unknown sources, ICA can find the independent sources without any prior knowledge of the mixing mechanism [6, 8]. It represents the original data with the components that are statistically independent, or as independent as possible. Such a representation captures the essential structure of the data in many applications [7]. Various ICA algorithms are developed to find a de-mixing matrix $\mathbf{W}$ [7, 9, 10, 11]. The de-mixing matrix $\mathbf{W}$ transforms $\mathbf{x}$ into the independent components (ICs) $\mathbf{y}$. The ICs are used as the estimates of $\mathbf{s}$:

$$\begin{aligned} Mixing: &\quad \mathbf{x} = \mathbf{A}\mathbf{s} \\ De-mixing: &\quad \mathbf{y} = \mathbf{W}\mathbf{x} \end{aligned} \tag{1}$$

2.1.1 ICA Model for Time Series

Let $\mathbf{x}_i$ and $\mathbf{s}_i$ denote respectively each observed signal vector and each source signal vector, and $1 \leq i \leq N$. Both are assumed to be $T - step$ time series: $\mathbf{x}_i = [x_i(1), x_i(2), \cdots, x_i(T)]'$; $\mathbf{s}_i = [s_i(1), s_i(2), \cdots, s_i(T)]'$. Let $\mathbf{X}$ and $\mathbf{S}$ denote a $N \times T$ observation matrix and a $N \times T$ source matrix respectively: $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N]'$, $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N]'$. In the basic model of ICA, $\mathbf{X}$ is modeled as $\mathbf{X} = \mathbf{AS} = \sum_{i=1}^{N} \mathbf{a}_i \mathbf{s}_i'$, where $\mathbf{a}_i$ is the $i-th$ column of $\mathbf{A}$, and $\mathbf{s}_i'$ is the $i-th$ row of $\mathbf{S}$ [7]. The ICA model aims at estimating an unknown $N \times N$ de-mixing matrix $\mathbf{W}$ such that

$$\mathbf{Y} = [\mathbf{y}_i'] = \mathbf{WX}, \tag{2}$$

where $\mathbf{y}_i' = [y_i(1), y_i(2), \cdots, y_i(T)]$ is the $i - th$ row of $\mathbf{Y}$, and $1 \leq i \leq N$. In order to estimate the independent latent sources $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N$ using $\mathbf{y}_1$, $\mathbf{y}_2$,$\cdots$, $\mathbf{y}_N$ under the basic ICA model for time series, $\mathbf{y}_i$, $1 \leq i \leq N$ must be instantly mutually independent.[5] For the estimation of the sources, three other assumptions are required: at most, one of the sources is Gaussian distributed [7]; the mixing matrix is of full rank [9]; the observed signals are stationary [13].

2.1.2 Ambiguities in the ICA Model

If $\mathbf{W} = \mathbf{A}^{-1}$, then ICs are the same as source signals: $\mathbf{Y} = \mathbf{WX} = \mathbf{A}^{-1}\mathbf{AS} = \mathbf{S}$. However, this is not always satisfied. There are two inherent ambiguities in the ICA model [11]: magnitude and scaling ambiguity; permutation ambiguity. The first ambiguity means that the true variance of each source signal cannot be determined: since both $\mathbf{a}_i$ and $\mathbf{s}_i$ are unknown, $\mathbf{X}$ can be rewritten as $\mathbf{X} = \mathbf{AS} = \sum_{i=1}^{N} \left(\frac{1}{\alpha_i}\mathbf{a}_i\right)\left(\alpha_i \mathbf{s}_i'\right)$. The most simple solution to this ambiguity is to assume that each source signal has unit variance: $E[(s_i(t))^2] = 1$, $1 \leq i \leq N$, $1 \leq t \leq T$. Even after introducing this assumption, there still leaves the ambiguity of sign: the sign of each source signal cannot be determined.[6] The second ambiguity means that the order of

[5] It means that for any given t in $1 \leq t \leq T$, $y_1(t), y_2(t), \cdots, y_N(t)$ must be mutually independent.

[6] It may not be a serious problem because the sources can be multiplied by -1 without affecting the model and the estimation [7].

estimated independent components cannot be specified: introducing a permutation matrix $\mathbf{P}$ and its inverse, $\mathbf{X}$ can be rewritten as $\mathbf{X} = \mathbf{A}\mathbf{P}^{-1}\mathbf{P}\mathbf{S} = \mathbf{A}^*\mathbf{S}^*$. Since the elements of $\mathbf{S}^* = \mathbf{P}\mathbf{S}$ are the original sources in a different order and $\mathbf{A}^* = \mathbf{A}\mathbf{P}^{-1}$ is another unknown mixing matrix, we cannot distinguish $\mathbf{A}\mathbf{S}$ from $\mathbf{A}^*\mathbf{S}^*$ within the ICA model. Due to these ambiguities, we are only able to find $\mathbf{W}$ such that $\mathbf{W}\mathbf{A} = \mathbf{P}\mathbf{D}$ where $\mathbf{D}$ is a diagonal scaling matrix [14]. Thus, ICs are scaled source signals in a different order: $\mathbf{Y} = \mathbf{W}\mathbf{X} = \mathbf{W}\mathbf{A}\mathbf{S} = \mathbf{P}\mathbf{D}\mathbf{S}$.[7]

2.2 Implementation of ICA

2.2.1 Non-Gaussianity Maximization

According to central limit theorem, a sum of independent signals with arbitrary distributions tends toward a Gaussian distribution under certain conditions. This implies that independent variables are more non-Gaussian than their mixtures. Hence, non-Gaussianity is a measure of independence. This elucidate that the separation of independent signals from their mixtures can be accomplished by making the linear signal transformation as non-Gaussian as possible. The key to estimating ICA model is non-Gaussianity [7, 9, 11]. Therefore, we can implement the ICA model as an optimization problem by setting up a measure for the independence of ICs as an objective function. And then, we can use some optimization techniques to find the de-mixing matrix $\mathbf{W}$ [28]. Considering that what we are looking for in this paper is independent components which security returns consist of, and that the security returns are non-Gaussian distributed, non-Gaussianity-oriented ICA methods may be the most relevant for the aim of this paper. The non-Gaussianity of ICs can be measured by negentropy [27, 6]: $J(\mathbf{y}) = H(\mathbf{y}_{gauss}) - H(\mathbf{y})$, where $\mathbf{y}_{gauss}$ denotes a Gaussian random vector which has the same covariance matrix as $\mathbf{y} = [y_1, y_2, \cdots, y_N]'$. And $H(\mathbf{y})$ is the entropy of a random vector $\mathbf{y}$ with density $p(\mathbf{y})$, which is defined as $H(\mathbf{y}) = -\int p(\mathbf{y}) log\left(p(\mathbf{y})\right) d\mathbf{y}$. The negentropy is always non-negative, and is zero if and only if $\mathbf{y}$ has a Gaussian distribution. To overcome the computational difficulty, an approximation of negentropy[8] is

[7] This implies that the instant mutual independence of ICs is equivalent to that of source signals.

[8] In practice, we need the approximation of negentropy in $1 - dimensional$ only [6].

proposed in [27] as $J(y) \approx (E[G(y)] - E[G(v)])^2$, where v is a Gaussian variable of zero mean and unit variance, and $G(\cdot)$ is a non-quadratic function. In this paper, $G(\cdot)$ is given as $G(y) = -\exp(-y^2/2)$. For details on the selection of $G(\cdot)$, see [27].

2.2.2 Data Preprocessing

Before applying an ICA algorithm on the data, some preprocessing techniques that make the ICA estimation simpler and better conditioned are performed: centering and whitening the data [7]. The preprocessing step in ICA is the multivariate standardization of the data by using PCA. For details, see APPENDIX A.

2.2.3 FastICA Algorithm

FastICA algorithm proposed by [26] and [27] is a fast and efficient implementation of ICA, and adopted in this paper to find a de-mixing matrix $\mathbf{W}$. It has various appealing properties [7]:

1. It converges very fast. Under the assumptions of ICA model, the convergence is cubic or at least quadratic. The convergence of ordinary ICA algorithms based on stochastic gradient descent methods is only linear.

2. It is simple to implement. Contrary to gradient-based algorithms, there are no step size parameters to choose. Furthermore, it does not require any matrix inversions, which usually consume a lot of computing time.

3. It can estimate both sub-Gaussian and super-Gaussian ICs. Ordinary maximum likelihood algorithms only work for a given class of distributions.

4. With a kurtosis-based contrast functions, it can be shown to converge globally to the ICs [9].

FastICA beats almost all the other ICA methods in robustness, speed and simplicity. Those interested in the details on the comparison between FastICA and other algorithms are invited to [28] or [10]. The details on the procedure for implementing FastICA appears in APPENDIX B.

2.3 Factor Model for ICA

The returns of securities are assumed to be represented as linear combinations of some factors in many financial models [18]. Since factors are not necessary directly related to the observable economic variables, finding the factors for the model are not easy. [19] applied ICA to recover the hidden factors and the corresponding sensitivities. In the multifactor model, the return of the $k-th$ security, r_k, is represented as

$$r_k = \alpha_k + \sum_{m=1}^{M-1} \beta_{km} f_m + u_k, \tag{3}$$

where f_m and β_{km}, $1 \leq m \leq M-1$, are factors affecting the return and corresponding sensitivities, respectively. α_k is the zero factor of the $k-th$ security, which is invariant with time. And u_k is a zero mean random variable of the $k-th$ security, which is assumed that $cov(f_m, u_k) = 0,\ 1 \leq m \leq M-1$ and $cov(u_i, u_j) = 0,\ i \neq j$, where $cov(\cdot,\cdot)$ denotes the covariance. By subtracting mean, (3) can be rewritten as $r_k - E[r_k] = \sum_{m=1}^{M-1} \beta_{km}(f_m - E[f_m]) + u_k$. By treating the noise term u_k as an extra factor without loss of generality, i.e. putting $u_k = \beta_{kM} F_M$, [19] transformed the factor model in (3) into the product of a mixing matrix and factor time series as

$$R_k(t) = \sum_{m=1}^{M} \beta_{km} F_m(t),\ 1 \leq t \leq T,\ 1 \leq k \leq N, \tag{4}$$

where $R_k = r_k - E[r_k]$ and $F_m = f_m - E[f_m]$. (4) is the factor model for ICA. In this model, $F_1, F_2, \cdots, F_M$ are unknown independent source signals which are designated as source factors.

2.4 Applying ICA to the Factor Model

In order to separate independent factors, ICA is applied to the preprocessed return time series under the model in (4). The detailed procedure for this is as follows (see APPENDIX A and B):

1. Select a universe of N securities, and observe the $(T+1) - step$ time series of each component security: $p_k(t)$, $1 \leq k \leq N$, $0 \leq t \leq T$;

2. Calculate returns from the prices: $r_k(t) = (p_k(t) - p_k(t-1))/p_k(t-1)$, $1 \le k \le N$, $1 \le t \le T$;

3. Center each return time series: $R_k(t) = r_k(t) - E[r_k]$, $1 \le k \le N$, $1 \le t \le T$, where $E[r_k]$ is estimated as $\overline{r}_k = \sum_{t=1}^{T} r_k(t)/T$;

4. Whiten the centered return time series: $\tilde{\mathbf{R}}(t) = \mathbf{E}\mathbf{D}^{-1/2}\mathbf{E}'\mathbf{R}(t)$, where $\tilde{\mathbf{R}}(t) = [\tilde{R}_1(t), \tilde{R}_2(t), \cdots, \tilde{R}_N(t)]'$, $\mathbf{R}(t) = [R_1(t), R_2(t), \cdots, R_N(t)]'$, $1 \le t \le T$;

5. Apply the FastICA algorithm to the preprocessed return vector time series $\tilde{\mathbf{R}}(t)$, $1 \le t \le T$.

In the fourth step above, $\mathbf{E}$ is the orthogonal matrix of eigenvectors for the covariance matrix $E[\mathbf{R}\mathbf{R}']$. $\mathbf{D}$ is the diagonal matrix of its eigenvalues: $\mathbf{D} = diag(d_1, \cdots, d_N)$, and $\mathbf{D}^{-1/2} = diag(d_1^{-1/2}, \cdots, d_N^{-1/2})$. $E[\mathbf{R}\mathbf{R}']$ is estimated as $E[\mathbf{R}\mathbf{R}'] \approx \sum_{t=1}^{T} \mathbf{R}(t)\mathbf{R}(t)'/(T-1)$.

3 Hierarchy of Factors

Factor models can estimate the systematic risk, and there exist several methods to find out the number of factors in security returns [30, 31]. However, factors in the aforementioned articles are not independent, and at best uncorrelated. They estimate multifactor models by methods similar to "principal component analysis (PCA)." PCA transforms a data set in which there are a large number of interrelated variables into a new data set of variables, the principal components (PCs), which are uncorrelated. In PCA, the PCs are ordered according to the size of their eigenvalues so that the first few retain most of the variation present in all of the original variables [15]. In this article, the returns of component securities of a given universe are decomposed into independent factors, and thus both systematic and idiosyncratic risk factors can be included in the model as well as further decomposition of the uncorrelated factors is accomplished. This is quite different from the traditional factor model approaches that specify models (such as one factor model, two factor model, etc.) first and then use data to estimate them. The approach of this paper uses data first in order to identify risk factors, and then specify the model using the identified factors. Therefore, this approach can include all the risk factors contained in the data.

ICA cannot order ICs in the way as PCA orders PCs because it is assumed in ICA that each source signal has unit variance, and hence all the eigenvalues of ICs are normalized to unity through data preprocessing [16]. However, still can they be ordered according to their relative importance in data reconstruction [17]. [17] uses relative hamming distance (RHD) to construct the Q-measure which measures the data reconstruction error. Adopting RHD to measure data reconstruction error is based on the consideration that the trend of a time series may be mostly controlled by the underlying independent components. RHD compare the trend of original time series with that of reconstructed time series in a very simple way: if both time series are moving in the same direction at a given point of time, the value of RHD is 0; if both time series are moving in opposite directions, the value of RHD is 4; if one of the time series is moving in a direction while the other remains still, the value of RHD is 1. By minimizing the cumulative data reconstruction error, the ICs can be ordered according to their joint contribution in data reconstruction.

Other methods suggested for ordering ICs before [17] have decided the order based on each individual component without considering their interactions on the observed times series. For example, [13] decides the IC order according to the norm of each individual component; [29] suggests to select a subset of ICs based on the mutual information between the observation and the individual components; [28] sorts ICs to their non-Gaussianity. In these methods, the component order is determined based on each individual component only. However, the observed series are actually influenced by several components, whose individually decided optimum order is no longer optimum as a whole from the viewpoint of analyzing the observed series. Therefore, it may be more helpful to consider the joint contribution of the components to the time series in performing ordering [17].

3.1 Data Reconstruction

Let $x_1(t), x_2(t), \cdots, x_N(t)$ be the observed N signals at time t, which are instantaneous linear mixtures of unknown mutually independent sources $s_1(t)$, $s_2(t), \cdots, s_N(t)$ at time t. The observed signals can be modeled as $\mathbf{x}(t) =$

$\mathbf{As}(t)$, where $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_N(t)]'$, $\mathbf{s}(t) = [s_1(t), s_2(t), \cdots, s_N(t)]'$, and $\mathbf{A}$ is a $N \times N$ unknown mixing matrix. ICA can recover the source signal vector $\mathbf{s}(t)$ up to an unknown constant and a permutation of indices through a de-mixing matrix $\mathbf{W}$: $\mathbf{y}(t) = \mathbf{Wx}(t) = \mathbf{WAs}(t),\ 1 \le t \le T$, where $\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_N(t)]'$ is a IC vector at time t. Then the contribution of a independent component y_n to the reconstruction of the observed signal x_k can be denoted as

$$c_{kn}(t) = \mathbf{W}_{kn}^{-1} y_n(t),\ 1 \le t \le T, \tag{5}$$

where $\mathbf{W}_{kn}^{-1}$ denotes the $(k,n)-th$ element in the inverse matrix of $\mathbf{W}$, $\mathbf{W}^{-1}$.

3.2 Relative Hamming Distance (RHD)

Suppose that the N ICs $y_1(t), y_2(t), \cdots, y_N(t)$ are given, and that we determine a specific list L_k which shows the order of them. For example, if 5 ICs are given and $L_k = \{2, 1, 5, 3, 4\}$, then the ordering of ICs is y_2, y_1, y_5, y_3, y_4. Using the first m ICs under the list L_k, x_k is reconstructed as

$$\hat{x}_{L_k}^m(t) = \sum_{r=1}^{m} c_{kq(r)}(t), \tag{6}$$

where $q(r)$ denotes the $r-th$ element of L_k. The corresponding reconstruction error $Q(x_k, \hat{x}_{L_k}^m)$ is defined by the Relative Hamming Distance (RHD) function as

$$Q(x_k, \hat{x}_{L_k}^m) = RHD(x_k, \hat{x}_{L_k}^m) = \frac{1}{T-1} \sum_{t=1}^{T-1} [H_k(t) - \hat{H}_{L_k}^m(t)]^2, \tag{7}$$

where

$$H_k(t) = sign[x_k(t+1) - x_k(t)],\ \hat{H}_{L_k}^m(t) = sign[\hat{x}_{L_k}^m(t+1) - \hat{x}_{L_k}^m(t)], \tag{8}$$

and

$$sign(h) \quad = \quad \begin{cases} 1 & if\ h > 0 \\ 0 & if\ h = 0 \\ -1 & otherwise \end{cases} \tag{9}$$

$Q(x_k, \hat{x}^m_{L_k})$ measures how well the reconstructed time series mimics the original time series. The cumulative data reconstruction error J_{L_k} is given as

$$J_{L_k} = \sum_{m=1}^{N} Q(x_k, \hat{x}^m_{L_k}) \tag{10}$$

And hence, the optimum order list L_k^* under the Q measure criterion is given as

$$L_k^* = arg \min_{L_k} J_{L_k} \tag{11}$$

For each time series $\{x_k(t)\}_{t=1}^{T}$, $1 \le k \le N$, we can find a specific optimum order list. This method is termed "Exhaustive Search" in [17].

4 New Measures of Dependency

4.1 Independent Factor Order Distance ($IFOD$)

The factor model for ICA explained in Section 2.3 can be rewritten as $\mathbf{R}(t) = \boldsymbol{\beta}\mathbf{F}(t),\ 1 \le t \le T$, where $\boldsymbol{\beta}$ is a $N \times M$ sensitivity matrix and $\mathbf{F}(t) = [F_1(t), F_2(t), \cdots, F_m(t)]'$. Pre-multiplying the whitening matrix $\mathbf{K} = \mathbf{E}\mathbf{D}^{-1/2}\mathbf{E}'$ to the both sides, we arrive at $\tilde{\mathbf{R}}(t) = \tilde{\boldsymbol{\beta}}\mathbf{F}(t),\ 1 \le t \le T$, where $\tilde{\mathbf{R}}(t) = \mathbf{K}\mathbf{R}(t)$ and $\tilde{\boldsymbol{\beta}} = \mathbf{K}\boldsymbol{\beta}$. Once the de-mixing matrix $\mathbf{W}$ is estimated as $\hat{\mathbf{W}}$ by the procedure in APPENDIX B, then we can obtain the estimate of independent factor vector $\hat{\boldsymbol{F}}$[9] as $\hat{\boldsymbol{F}}(t) = \hat{\mathbf{W}}\tilde{\mathbf{R}}(t),\ 1 \le t \le T$. Since $\tilde{\mathbf{R}}(t) = \mathbf{K}\mathbf{R}(t)$, we have $\hat{\mathbf{W}}\tilde{\mathbf{R}}(t) = \hat{\mathbf{W}}\mathbf{K}\mathbf{R}(t) = \hat{\boldsymbol{F}}(t),\ 1 \le t \le T$. Thus, we finally arrive at

$$\mathbf{R}(t) = \left(\hat{\mathbf{W}}\mathbf{K}\right)^{-1} \hat{\boldsymbol{F}}(t),\ 1 \le t \le T. \tag{12}$$

We can rewrite (12) component-wisely as[10]

$$R_k(t) = \sum_{n=1}^{N} \left(\hat{\mathbf{W}}\mathbf{K}\right)^{-1}_{kn} \hat{\mathcal{F}}_n(t),\ 1 \le k \le N,\ 1 \le t \le T, \tag{13}$$

[9] $\hat{\boldsymbol{F}}$ corresponds to $\hat{\mathbf{Y}}$, the estimate of independent component vector $\mathbf{Y}$ in the usual ICA model: $\hat{\mathbf{Y}} = \hat{\mathbf{W}}\mathbf{X}$. In the factor model for ICA, we designate independent components as independent factors.

[10] It is assumed that the number of ICs (independent factors) equals to that of observed signals (security returns), i.e. $M = N$ [7].

where $\left(\hat{\mathbf{W}}\mathbf{K}\right)^{-1}_{kn}$ denotes the $(k,n)-th$ element of $\left(\hat{\mathbf{W}}\mathbf{K}\right)^{-1}$, and $\hat{\mathcal{F}}_n$ is the $n-th$ element of $\hat{\boldsymbol{F}}$. Using (13) and the definition of optimum order list L_k^* (see Section 3.2), we can represent the return time series of each security as

$$R_k(t) = \sum_{r=1}^{N} \left(\hat{\mathbf{W}}\mathbf{K}\right)^{-1}_{kq_k^*(r)} \hat{\mathcal{F}}_{q_k^*(r)}(t),\ 1 \leq k \leq N,\ 1 \leq t \leq T, \tag{14}$$

where L_k^* denotes the optimum order list for the return time series of Security k, and $q_k^*(r)$ denotes the $r-th$ element of L_k^*. In other words, $q_k^*(r)$ is the factor index of the $r-th$ contribution, under the Q-measure criterion, to the reconstruction of return time series of Security k using the estimates of independent factors, $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \cdots, \hat{\mathcal{F}}_N$. Thus, we can translate the return of each security into a linear combination of the independent factors in the order specific to the security.

Considering that the return of every security is composed of the same independent factors as in (13) and that every security has a unique optimum order list which determines the contribution ranking of each independent factor in reconstructing its return data as in (14), it may be quite natural to conjecture that similar securities have similar optimum order lists. And hence, it may be a natural conclusion that we can measure the dependency between the securities by comparing their optimum order lists. Thus, we propose "Independent Factor Order Distance ($IFOD$)" as a new measure of dependency between securities, which is defined as

$$IFOD(i,j) = \frac{1}{N}\sum_{n=1}^{N}\left(L_i^*(n) - L_j^*(n)\right)^2,\ 1 \leq i,\ j \leq N, \tag{15}$$

where $L_i^*(n)$ and $L_j^*(n)$ denote the location of Factor n in the optimum order list for the return of Security i and Security j, respectively. $IFOD$ measures the average of all the squared factor distances between two securities.[11] The smaller the value of $IFOD$ is, the larger the dependency between the two securities is.

4.2 Changes of $IFOD$

We assign rankings to the values of $IFOD(k,i)$, $i = 1, \cdots, N$ for a fixed

[11] A factor distance means the location difference of a factor between two securities.

Stock k so that the smaller value has the higher ranking. Thus the higher ranking of $IFOD(k,i)$ implies the more dependency between Stock k and Stock i. Based on the comparison of rankings of $IFOD(k,i)$, $1 \leq i \leq N$ in two different periods, $p1$ and $p2$, for the fixed Stock k, we propose an index that can measure the changes in the attributes of Stock k as follows:

$$CHANGE(k,p1,p2) = \frac{1}{N}\sum_{i=1}^{N}\left(RIFOD(k,i)_{p1} - RIFOD(k,i)_{p2}\right)^2, \quad (16)$$

where $RIFOD(k,i)_{p1}$ and $RIFOD(k,i)_{p2}$ denote the ranking of $IFOD(k,i)$ in the period of $p1$ and $p2$, respectively. The smaller the value of $CHANGE$ is, the lesser the attributes of Stock k has changed between the two periods. We also assign rankings to the values of $CHANGE(k,p1,p1)$, $k = 1,\cdots,N$ in a descending order, i.e. $CHANGE$ with the smaller value takes the higher ranking. Thus the higher ranking of $CHANGE(k,p1,p2)$ implies the lesser changes in the attributes of Stock k between $p1$ and $p2$.

5 Empirical Results

5.1 Data

Using the daily return times series of current component stocks of which Dow Jones Industrial Average (DJIA) is comprised, we test the performance of $IFOD$. The list of component stocks appears in Table 1. To analyze the changes of $IFOD$ during the recent financial crisis, the daily market closing prices of component stocks from 9/2/2005 to 8/31/2006 and those from 9/3/2008 to 8/31/2009 are used in the calculation of stock returns. The former period, which is designated as Period 1, represents a relatively quite period, while the latter, which is designated as Period 2, represents a turbulent period. Data source is *http://finance.yahoo.com*. All the prices are adjusted for dividends and splits as of 7/25/2013. The number of price observations for each stock is 251 in each Period. Thus, the number T of return observations of each stock is 250 in each Period. Figure 1 shows the graph of DJIA from 9/2/2005 to 8/31/2009, from which we can find the difference between the two periods by simple visual inspection.

5.2 Independent Factors of Dow Jones Market

Though the number of DJIA component stocks is 30 in Table 1, we use only 5 stocks of them in the first empirical study of this paper to save computing time.[12] They are Stock 6 (CSCO), Stock 12 (HPQ), Stock 13 (IBM), Stock 14 (INTC), and Stock 21 (MSFT). All of them are related to the IT sector, which is sensitive to the economic turbulence. Figure 2 shows, $\hat{\mathcal{F}}_n(t)$, $1 \leq n \leq 5$, $1 \leq t \leq T$, the estimates of independent factors both in Period 1 and Period 2, which are extracted by FastICA from the return data. There is one important thing to which you have to pay attention in reading the figure: we name each independent factor according to the order in which FastICA estimates it. Because FastICA is initialized randomly each time it runs, it may estimate the same factor in different order each time it runs. As explained in Section 2.1.2, the order of estimated independent components cannot be specified. Therefore Factor k in Period 1 may be completely different from Factor k in Period 2.

Table 2 shows the optimum order lists (L_k^*, $k = 6, 12, 13, 14, 21$) and the cumulative data reconstruction errors ($J_{L_k^*}$, $k = 6, 12, 13, 14, 21$) of the 5 stocks both in Period 1 and in Period 2. In Period 1, Stock 12 (HPQ) has 2 different factor hierarchies. And in Period 2, Stock 21(MSFT) has two factor hierarchies. These multiple optimum order lists result from the fact that we apply FastICA to a universe composed of a small number of stocks which come from a single industry sector, and hence some factors extracted by FastICA seem to be similar to each other. If we apply FastICA to a universe with a large number of stocks which come from diverse industry sectors, this problem will disappear (see Section 6.2 of this paper).

5.3 *IFOD* of Dow Jones Market

From Sub-table (a) in Table 2, we can observe that Factor 3 in Period 1 takes the first position in the optimum order list of Stock 21 (MSFT), while it takes the last position in that of Stock 6 (CSCO) and second in that of Stock 13 (IBM). Therefore, we can say that the distance between Stock 21 and Stock 6 are as far as 4(=5-1) with respect to Factor 3, while the distance between

[12]The exhaustive search for ordering N independent factors requires $(N + 1)!$ times of data reconstruction. For large N, we will use TnA algorithm (see Section 6.1 of this paper).

Stock 21 and Stock 13 is as close as 1(=2-1) with respect to Factor 3. The $IFOD$ measures the average of squared factor distances between two stocks.

Table 3 compiles $IFODs$ both in Period 1 and in Period 2. From Sub-table (a) in Table 3, we can find 2 different $IFOD$ ranking matrices corresponding to the multiple optimum order lists of Stock 12 (HPQ) in Period 1. And, from Sub-table (b) in Table 3, we can also find 2 different $IFOD$ ranking matrices corresponding to the multiple optimum order lists of Stock 21 (MSFT) in Period 2. The $i-th$ row of each matrix in Table 3 shows the values of $IFOD$ between Stock i and the other Stocks: $IFOD(i,j)$, $j = 6, 12, 13, 14, 21$. Note that $IFOD(i,i) = 0$. The smaller the value of $IFOD(i,j)$ is, the larger the dependency between Stock i and Stock j is.

To read the information from these matrices more easily, we assign rankings to the elements of each row according to their values in descending order: the smaller value an element in the row has, the higher ranking is assigned to it. Thus, the higher ranking implies the larger dependency. Each ranking is designated as $RIFOD(i,j)$, $i = 6, 12, 13, 14, 21$, $j = 6, 12, 13, 14, 21$, and appear in parenthesis beside the corresponding $IFOD$. Note that $RIFOD(i,i) = 0$. When $L^*_{12} = (1, 2, 3, 5, 4)$, $RIFOD(6, 12)$ and $RIFOD(6, 14)$ are respectively 1.5 and 1.5, i.e. $IFOD(6, 12)$ ties with $IFOD(6, 14)$ (see the first matrix in Sub-table (a) of Table 3). When $L^*_{12} = (1, 2, 5, 3, 4)$, $RIFOD(6, 12)$ and $RIFOD(6, 14)$ are respectively 1 and 2 (see the second matrix in Sub-table (a) of Table 3).

5.4 Effects of Recent Financial Crisis on $IFOD$

Suppose that for a fixed Stock k we make a comparison of $IFOD(k,j)$, $j = 6, 12, 13, 14, 21$ in Period 1 with those in Period 2. Then we can find how much the attributes of Stock k has changed during the recent financial crisis. For example, from Table 3 we can find that Stock 6 (CSCO) was the most dissimilar to Stock 21 (MSFT) before the crisis (see the two matrices in Sub-table (a) of Table 3, where you can find that $RIFOD(6, 21) = 4$), whereas it is the most dissimilar to Stock 12 (HPQ) after the crisis (see the two matrices in Sub-table (b) in Table 3, where you can find that $RIFOD(6, 12) = 4$). $CHANGE(k,$Period 1, Period 2) shows how much Stock k has experienced changes in its attributes between Period 1 and Period 2 by comparing the rankings of $IFOD(k, j)$, $j = 6, 12, 13, 14, 21$ in Period 1 with those in Period 2. Because Stock 12 (HPQ) has two optimum order lists in Period 1 and Stock 21 (MSFT) also has two optimum order lists in Period 2, there are all 4 pairs of ranking matrices which we have to consider for the calculation of $CHANGE$. For a fixed Stock k, we calculate all the values of $CHANGE(k, Period1, Period2)$ using the 4 pairs of ranking matrices and then average them out. Table 4 reports the result. We can observe that Stock 12 (HPQ) has experienced the smallest changes of attributes during the crisis, while Stock 13 (IBM) the largest changes.

5.5 Correlation versus $IFOD$

From Table 5, we can observe all the correlations between the returns of 5 IT stocks from DJIA components, and their rankings. The larger the correlation of a pair of stocks is, the higher ranking it takes in the row of the corresponding correlation matrix. The larger correlation means the more dependency between stocks, whereas the smaller value of $IFOD$ implies the more dependency between stocks. For the purpose of comparison, we also define another index that can measure the change of characteristics of the stock, $CHANGE^*(k, Period1, Period2)$, which is based on correlation rankings:

$$CHANGE^*(k, p1, p2) = \frac{1}{N} \sum_{i=1}^{N} \left(Rcorr(k, i)_{p1} - Rcorr(k, i)_{p2} \right)^2, \tag{17}$$

where $Rcorr(k, i)_{p1}$ and $Rcorr(k, i)_{p2}$ denote the ranking of correlation between the return of Stock k and Stock i in the period of $p1$ and $p2$, respectively.

According to the rankings of $CHANGE^*(k, Period1, Period2)$,[13] Stock 13(IBM) seems to have experienced the least changes in attributes between Period 1 and Period 2 (see Sub-table (c) in Table 5). This is a prominent contrast to the result of $CHANGE$, which shows that Stock 13 (IBM) has experience the largest changes in attributes between Period 1 and Period 2 (see Table 4). Considering that the distribution of stock returns is non-elliptical whereas the linear correlation is a measure of dependence for elliptically distributed variables, we can understand why there is such a large difference between $CHANGE$ and $CHANGE^*$: $CHANGE$ is based on the $IFOD$,

[13] The higher ranking implies the lesser changes.

which is not dependent on the distribution of stock returns, while $CHANGE^*$ is based on the correlation which is confined to the elliptically distributed variables. And, using another canonical difference between $IFOD$ and correlation we can also explain why there exists such a large difference difference between $CHANGE$ and $CHANGE^*$: $IFOD$ measures the whole aspects of dependency through factor-by-factor comparison of security returns, whereas correlation measures only the central dependency between security returns. In other words, $CHANGE$ can reflect appropriately tail events that may happen during the economic turbulence owing to the holistic nature of $IFOD$, whereas $CHANGE^*$ cannot due to the limited ability of correlation.

Not only through the difference between $CHANGE$ and $CHANGE^*$ but also through the difference between measurements on the variation of dependency according to the economic conditions, can we find another important distinction between $IFOD$ and correlation. From Table 5, we can observe that every correlation in Period 2 is larger than that in Period 1, which implies that stock returns are more dependent during market downturns (Period 2) than during market upturns (Period 1). From the viewpoint of $IFOD$, however, we can find that the dependency of stock returns does not always behave in this pattern. Table 6 shows average $IFOD$s in Period 1 and those in Period 2, which are respectively the result of averaging out the $IFOD$ ranking matrices in Sub-table (a) and in Sub-table (b) of Table 3. From Table 6, we can observe that some stock returns are less dependent during market downturns: the values of $IFOD(6, 12)$, $IFOD(12, 6)$, $IFOD(12, 13)$, $IFOD(13, 12)$, $IFOD(13, 21)$, and $IFOD(21, 13)$ in Period 2 is larger than those in Period 1, which implies that the dependency in Period 2 between Stock 6 and 12, between Stock 12 and 13, and between Stock 13 and 21 are less than those in Period 1, respectively.

From the observations above in this subsection, we can conclude that owing to the factor-wise comparison $IFOD$ can provide the more fundamental concept of relationship between securities than the traditional statistical tools based on correlation can do: $IFOD$ measures the dependency between stock returns by comparing the relationship of independent factors which comprise the returns, while correlation measures the dependency by comparing the relationship of the returns themselves. Furthermore, we can say that $IFOD$ extracts information from the return data exhaustively in the sense that the

Q-measure (see Section 3.2) uses the cumulative data reconstruction error to determines the optimum order list for a given stock: when a permutation of N factors is given for the data reconstruction, the Q-measure accumulates information by increasing the number of factors that are used in the calculation of the error one by one according to the order specified by the permutation until it uses up all the factors. It keeps to perform the same procedure with a new permutation of N factors until it uses up all the permutations. Thus the resulting optimum order list for the given stock carries all the information that can be extracted from the independent factors and the relationship between them. Since each stock return can be expressed in a linear combination of these factors in the order which the optimum order list specifies, the $IFOD$ ranking matrices can represent the more delicate relationship between securities than the correlation ranking matrices do.

6 Discussion

6.1 TnA Algorithm

Even for a minor increase of N, the exhaustive search explained in Section 3.2 requires a rapid increase of computing time because it needs to reconstruct the data $(N+1)!$ times for ordering N independent factors. Therefore, calculating the $IFODs$ with respect to all the components of DJIA is almost impossible for a humble desktop computer due to the astronomical number of iterations, which amounts to 31! Luckily, [17] also proposes another ordering method called "Test-and-Acceptance (TnA)", which requires only $N(N+1)/2-1$ times of data reconstruction, and thus does not consume such huge computing time even for a relatively large number of N.[14]

Using the algorithm in APPENDIX C, TnA produces a sub-optimum order list $\hat{L}_k^*$ as an estimate of the optimum order list L_k^*. Because it estimates the optimum order list for a given stock by deleting factors as explained in the appendix, it can not always find the optimum order list for the stock: there

[14] To order the 30 independent factors with respect to all the component securities using the TnA algorithm, it takes a desktop computer, which is equipped with an Intel CPU of i7-3930K and the 32G main memories, less than 2 minutes.

is a trade-off between the speed and the accuracy. And hence, we recommend you to use the exhaustive search when N is less than 10; otherwise to use TnA.

Table 7 reports the optimum order lists by TnA for the whole component stocks of DJIA in Period 1.[15] The bold numbers in the headline column of the table denote component stocks, and those in the headline row denote the factor order. Thus the $i-th$ row other than the headline row of the table represents the optimum factor order list corresponding to Stock i in Period 1. For example, the first row except the headline row in Table 7, which is read as "29, 6, 23, ..., 4", is the optimum order list of Stock 1 (AA) in Period 1: in this optimum order list, Factor 29 takes the first place, Factor 6 takes the second place, Factor 23 takes the third place,..., and Factor 4 takes the last place.

6.2 Solutions to the Multiple Hierarchies

Suppose that a universe contains a small number of stocks or it is composed of homogeneous stocks, for example, stocks from a single industry sector. Then it happens that a single component stock of the universe may have multiple optimum factor lists because the resulting independent factors may not be distinctive enough to describe the universe. There is a simple solution to this problem: adding to the universe some stocks from diverse industry sectors. The additional stocks will act as dummy variables to guarantee the uniqueness of factor hierarchy for every component stock.

We add two stocks, Stock 15 (JNJ) and Stock 26 (UNH), to the universe composed of the 5 IT stocks in Section 5.2. And hence, we can show that, in the new universe, each of the 5 IT stocks has a unique factor hierarchy (see Table 8).

6.3 Asymmetry in the $IFOD$ Rankings

In realty, we sometimes experience asymmetric relationship with others: for example, person A considers person B as his best friend, while person B considers person C other than person A as his best friend. Amazingly, $IFOD$

[15] When a stock has multiple optimum order lists, we pick up the first one to compile this table.

ranking matrices mimic this human behavior: the linear correlation can measure only the symmetric relationship between two stocks in the sense that $corr(Stock\, i,\ Stock\, j) = corr(Stock\, j,\ Stock\, i)$, whereas the $IFOD$ ranking matrices can detect whether the relationship is symmetric or asymmetric. For example, from the first $IFOD$ ranking matrix in Sub-table (a) of Table 3 we can observe that Stock 21 (MSFT) considers Stock 13 (IBM) as his best friend,[16] whereas Stock 13 (IBM) considers Stock 12 (HPQ) as his best friend instead of Stock 21 (MSFT). This asymmetry does not imply any miscalculation of $IFOD$: we can find that all the $IFOD$ matrices in this paper are symmetric as the definition of $IFOD$ in Equation (15) enforces. We can also find symmetric relationship between securities as well: for example, from the first $IFOD$ ranking matrix in Sub-table (a) of Table 3 we can observe that Stock 12 (HPQ) considers Stock 13 (IBM) as his best friend at the same time Stock 13 (IBM) also considers Stock 12 (HPQ) as his best friend.

7 Applications of $IFOD$

In this section, we present some useful examples of applying $IFOD$ to various areas in finance. Section 7.1 and 7.4 show how we can apply $IFOD$ to the area of portfolio management, and Section 7.2 and 7.3 to the area of risk management.

7.1 Selecting Alternatives

Suppose that regulations or some other reasons prevent you from investing in a specific security which you have found very attractive and promising. One of the relevant alternatives you can choose to cope with this situation may be to select alternative securities, which are similar to the non-allowable security, from your universe. Assume that you are a fund manager only allowed to invest your budget in DJIA components, while you have found that "EBAY",

[16] In this context, "the best friend" means the security which is closest to Stock 21 (MSFT) by the criterion of $IFOD$, i.e. which has the highest $IFOD$ ranking within $\{IFOD(21, j) : j = 6, 12, 13, 14\}$.

which is not in your universe, would be promising. If you add EBAY to your universe and then apply ICA as well as TnA to the new universe, you can find best friends of EBAY from your universe. Let us designate EBAY as Stock 0 (EBAY) for the sake of convenience. Then, we can calculate $IFOD(0, j)$, $1 \leq j \leq N$, and hence find Stock j^*, which has the smallest value of $IFOD$ among the 30 component stocks of DJIA. Stock j^* is the best friend of EBAY. Now, you can invest some of your budget in the best friend instead of EBAY to which you want to take some exposure. What is the difference between selecting the best friend by $IFOD$ and that by correlation? The correlation measures only the central dependency, whereas the $IFOD$ takes into account the whole aspects of dependency in the sense that it compares Security 0 with Security j factor by factor. Therefore, we can expect that the best friend selected by $IFOD$ may be quite different from that selected by correlation.

Table 9 shows $IFOD(0, j)$, $1 \leq j \leq 30$ and their rankings both in Period 1 and Period 2. The smaller $IFOD$ takes the higher ranking. And, Table 10 shows $corr(0, j)$, $1 \leq j \leq 30$ and their rankings both in Period 1 and Period 2. The larger correlation has the higher ranking. In period 1, best 5 friends of EBAY selected by $IFOD$ are XOM, UTX, DIS, DD, and AA (see Sub-table (a) of Table 11), whereas those selected by correlation are WMT, CAT, INTC, AXP, and GE (see Sub-table (b) of Table 11). We can see the best 5 friends selected by $IFOD$ are quite different from those selected by correlation. Furthermore, two of the best 5 friends selected by correlation (INTC and AXP) are in the list of worst friends selected by $IFOD$. In Period 2, the best 5 friends of EBAY selected by $IFOD$ are INTC, DD, IBM, HPQ, MRK, while the best 5 friends of EBAY selected by correlation are INTC, CSCO, DD, UTX, DIS. Both criteria select INTC as the best friend of EBAY unanimously. And, both criteria also select DD as one of the best 5 friends. However, two of the best 5 friends selected by correlation (CSCO, UTX) are in the list of worst friends selected by $IFOD$.

7.2 Detecting Structural Breaks

Based on the conjecture that the attributes of security would not change significantly in stable periods, we can detect structural breaks using Equation (16). First, we calculate $RIFOD(k, i)$, $1 \leq i \leq N$ in Period 1 for a given

Stock k. We define a new data set, which we designate as t_1, by attaching to Period 1 the return data of next month, and then calculate $RIFOD(k,i)$, $1 \le i \le N$ in t_1. And, we define another new data set, which we designate as t_2, by attaching to t_1 the return data of the next month, and then calculate $RIFOD(k,i)$, $1 \le i \le N$ in t_2. We continue the same procedure until t_T. By using (16), we measure how much the attributes of Stock k has changed between Period 1 and t_1. And then we measure it between t_1 and t_2. We continue the same procedure until t_T. Now, we can find structural breaks by observing the graph of $CHANGE(k, t_{i-1}, t_i)$, $1 \le i \le T$, where t_0 means Period 1. If a structural break has occurred, we can find a large spike in the graph. Because each stock has different attributes, the patterns of the graphs are quite different across stocks: some stocks are very sensitive to the structural breaks and hence they respond instantly to them, while others are insensitive and hence respond slowly or do not respond at all. We also calculate the average of $CHANGE(k, t_{i-1}, t_i), 1 \le k \le N$ for each i and designate it as $AveCAHNGE(t_{i-1}, t_i)$:

$$AveCAHNGE(t_{i-1}, t_i) = \frac{1}{N}\sum_{k=1}^{N} CHANGE(k, t_{i-1}, t_i) \tag{18}$$

Figure 3 shows the graph of $AveCAHNGE(t_{i-1}, t_i), 1 \le i \le 36$.

From this graph, we can observe a large spike at $i = 18$ and at $i = 28$, respectively. In other words, one structural break occurred between January and February of 2008, and the other between November and December of 2008. Table 12 shows the values of $AveCAHNGE(t_{i-1}, t_i), 1 \le i \le 36$ and their change ratios in percentage.

7.3 Measuring Diversification

Suppose that the universe is composed of N securities and your portfolio consists of two stocks among them, Stock i and Stock j. Then we can measure the degree of diversification of your portfolio from the viewpoint of $IFOD$. Since the largest value of $RIFOD(i,\cdot)$ is $N-1$,[17] $RIFOD(i,j)/(N-1)$ can measure the relative distance between Stock i and Stock j. Taking into account that $RIFOD(i,j)$ does not always equal to $RIFOD(j,i)$ due to the asymmetry

[17] Note that $RIFOD(i,i) = 0$

discussed in Section 6.3, we define the whole relative distance between Stock i and Stock j ($WRD(i,j)$) as follows:

$$WRD(i,j) = \frac{1}{2}\left(RIFOD(i,j)/(N-1) + RIFOD(j,i)/(N-1)\right) \tag{19}$$

We designate the generalized version of (19) for a portfolio composed of k stocks, Stock n_1, Stock n_2,$\cdots$, Stock n_k, as $DIVF(n_1, n_2, \cdots, n_k)$ and define it as follows:

$$DIVF(n_1, n_2, \cdots, n_k) = \frac{1}{{}_kC_2} \sum_{i \in \{n_1, \cdots, n_k\}} \sum_{j \in \{n_1, \cdots, n_k\}, j > i} WRD(i,j), \tag{20}$$

where ${}_kC_2 = k(k-1)/2$. Because $0 \leqq WRD(i,j) \leqq 1$ and the number of $WRDs$ in (20) is ${}_kC_2$, $DIVF$ ranges from 0 to 1. The closer to 1 the $DIVF$ of the portfolio is, the more diversified it is from the viewpoint of $IFOD$. $DIVF$ does not depend on the weights of individual component stocks of the portfolio. Using Table 7, we can calculate every $RIFOD(i,j)$ in Period 1. Table 13 shows $RIFODs$ in Period 1. Suppose that your portfolio is composed of Stock 1(AA), Stock 2(AXP), and Stock 3(BA). From Table 13, we can read the values of 6 $RIFODs$ in Period 1: $RIFOD(1,2) = 3$, $RIFOD(2,1) = 3$, $RIFOD(1,3) = 5$, $RIFOD(3,1) = 2$, $RIFOD(2,3) = 2$, and $RIFOD(3,2) = 1$. Thus we can calculate $WRDs$ related to these values: $WRD(1,2) = 0.103448$, $WRD(1,3) = 0.12069$, $WRD(2,3) = 0.0.051724$. Now, we obtain the value of $DIVF(1,2,3)$ in Period 1: it is 0.091954.

7.4 *IFOD* CAPM

In this subsection, we discuss $IFOD$ version of capital asset pricing model (CAPM). In the CAPM, each security beta is defined as the ratio of the return covariance between the security and the market portfolio to the return variance of the market portfolio. Because we are working with the 30 component stocks of DJIA, we substitute DJIA for the market portfolio. For the sake of convenience we will designate DJIA as Stock 0. If we can calculate $RIFOD(0,i)$ and $RIFOD(i,0)$, then we can use $WRD(0,i)$ as the $IFOD$ version of beta for Security i. Now, in order to calculate $RIFOD(0,i)$ and $RIFOD(i,0)$, we have to find the hierarchy of factors with respect to DJIA. According to the ICA framework, the return time series of each component stock is a linear combination of independent factors. Because DJIA is a linear combination of price time series of the component stocks, the return time series of DJIA can also be a linear combination of the same independent factors.[18] Ordering the independent factors according to their ability to mimic the trend of return time series of DJIA, can we find the hierarchy of factors with respect to DJIA. The similar procedure has been done with respect to the return time series of individual component stocks in order to find their factor hierarchies. Borrowing the idea of TnA algorithm, the ability of each factor to mimic the trend of return time series of DJIA is measured by RHD, which is discussed in Section 3.2. There is one important thing we have to pay attention to: as discussed in Section 2.1.2, the ambiguity of sign still remains even after data preprocessing. In other words, the sign of each source signal cannot be determined. Therefore, we have to calculate two different RHD, RHD^+ and RHD^-. RHD^+ is defined as RHD between a factor and the centered return of DJIA, while RHD^- as RHD between the inverse signed factor and the centered return of DJIA:

$$RHD_n^+ = \frac{1}{T-1}\sum_{t=1}^{T-1}\{sign\,[CRDJIA(t+1)-CRDJIA(t)] - sign\,[F_n(t+1)-F_n(t)]\}^2$$

$$RHD_n^- = \frac{1}{T-1}\sum_{t=1}^{T-1}\left\{sign\,[CRDJIA(t+1)-CRDJIA(t)] - sign\left[F_n^-(t+1)-F_n^-(t)\right]\right\}^2,$$

where $CRDJIA(t)$ is the centered return of DJIA at time t, and $F_n^-(t) = -F_n(t)$, $1 \leq n \leq N$. The RHD_n (RHD for Factor n) is defined as the smaller one between RHD_n^+ and RHD_n^-.

Table 14 shows $RHDs$ and their rankings both in Period 1 and Period 2. The factor with the smaller value of RHD takes the higher position in the factor hierarchy of Stock 0 (DJIA). The factor hierarchy of Stock 0 (DJIA) in Period 1 and Period 2, respectively designated as $\hat{L}_0^{*Period1}$ and $\hat{L}_0^{*Period2}$, are as follow:

[18] Because the return time series of Stock 0 (DJIA) is a linear combination of return time series of 30 DJIA component stocks, the rank of return matrix of the 31 stocks will be smaller than 31. Therefore we can not apply ICA directly to it.

$$
\begin{aligned}
\hat{L}_0^{*Period1} &= \{18, 5, 28, 23, (10, 27), 30, (6, 26), 17, 24, 14, (2, 4), \\
&\quad 13, 16, 25, 29, 7, 9, 3, 12, 21, 8, 15, (1, 22), (11, 19, 20)\} \\
\hat{L}_0^{*Period2} &= \{9, 5, 17, 11, (20, 26), 22, 10, (13, 25, 29, 30), 6, \\
&\quad (2, 18, 21), (12, 23), (8, 14), 1, 15, 7, (4, 28), 24, 27, (3, 16, 19)\}
\end{aligned}
$$

The factors in parenthesis are interchangeable in the corresponding list of factor ordering because they tie in their RHD values. For example, Factor 10 and 27 have the same RHD value of 1.638554 in Period 1, and hence Factor 10 or 27 can take the fifth position in the factor ordering list of Stock 0 (DJIA) in Period 1.

Table 15 shows $RIFOD(0, i)$, $RIFOD(i, 0)$, and $WRD(0, i)$, $1 \leq i \leq N$ both in Period 1 and Period 2. Because DJIA is considered as Stock 0, $RIFOD(0, i)$ and $RIFOD(i, 0)$ are divided by 30 instead of 29 in the calculation of $WRD(0, i)$. To ease the calculation of $IFODs$ in Period 1, we use $L_0^*(10) = L_0^*(10) = 5.5$; $L_0^*(6) = L_0^*(26) = 8.5$; $L_0^*(2) = L_0^*(4) = 13.5$; $L_0^*(1) = L_0^*(22) = 26.5$; and $L_0^*(11) = L_0^*(19) = L_0^*(20) = 29$ in Equation (15) instead of using multiple hierarchies of Stock 0 (DJIA). And in Period 2, we use $L_0^*(20) = L_0^*(26) = 5.5$; $L_0^*(13) = L_0^*(25) = L_0^*(29) = L_0^*(30) = 10.5$; $L_0^*(2) = L_0^*(18) = L_0^*(21) = 15$; $L_0^*(12) = L_0^*(13) = 17.5$; $L_0^*(8) = L_0^*(14) = 19.5$; $L_0^*(4) = L_0^*(28) = 24.5$; and $L_0^*(3) = L_0^*(16) = L_0^*(19) = 29$ in Equation (15).

Table 15 also shows the CAPM version of betas for DJIA component stocks both in Period 1 and Period 2. The $IFOD$ version of beta for Stock i, $WRD(0, i)$, shows how different the behavior of Stock i is from that of the market: the closer to 1 $WRD(0, i)$ is, the more differently from the market Stock i behaves. Comparing both betas (WRD and CAPM beta) in Period 1 and those in Period 2, we can find several interesting facts:

1. Those stocks which have large $WRDs$ in Period 1, i.e. which behave quite differently from the market in Period 1, reduce their $WRDs$ significantly in Period 2. Stock 7, 8, 22, 28, 30 have large $WRDs$, which are over 0.8 in Period 1. In Period 2, they reduce their $WRDs$ by 20%~70%.

2. Those stocks which have small $WRDs$ in Period 1, i.e. which behave in accordance with the market in Period 1, increase their $WRDs$ sig-

nificantly in Period 2. Stock 2, 6, 21, 23, 27 have small $WRDs$, which are below 0.2 in Period 1. In Period 2, they increase their $WRDs$ by 60%~1200%. There is one exception: Stock 9 has a WRD as small as 0.1667 in Period1, and it reduces its WRD further by 20% in Period 2.

3. Those stocks with large CAPM betas have relatively small $WRDs$ in Period 1, whereas they have relatively large $WRDs$ in Period 2: in Period 1, Stock 1, 3, 5, 14, 16 have CAPM betas which are over 1.2, and they all have $WRDs$ which are below 0.5; in Period 2, Stock 1, 2, 4 have CAPM betas which are over 1.8 and they all have $WRDs$ which are over 0.6. There is one exception: in Period 2, Stock 16 has a CAPM beta as large as 1.869, while its WRD is as small as 0.2667. The economy in Period 1 is in boom, and hence large CAPM betas mean to behave in accordance with the market, which explains why those stock with large CAPM betas in Period 1 also have small $WRDs$. The economy in Period 2 is in recession, and hence large CAPM betas mean to behave differently from the market, which explains why those stock with large CAPM betas in Period 2 also have large $WRDs$.

8 Conclusion

This paper proposes a new method for measuring the dependency between securities in a given universe. Applying independent component analysis to the return data of the whole component securities in the universe, we can extract independent factors which compose the returns of component securities. Reconstructing return data of individual component security based on such factors, we find that each security has a unique factor hierarchy under certain conditions: we can order the factors according to their relative importance in reconstructing each individual security return. Thus, we can express its return in a linear combination of independent factors in the order specific to the security. Based on the fact that the security returns are non-Gaussian distributed and that the independent factors extracted by ICA are also non-Gaussian distributed (see Section 2.2.1), the linear combination can be considered as a proper representation of the security return.

A comparative analysis of the resulting hierarchies can find the dependence structure between securities in a nonparametric and distribution-free context. Due to the non-elliptical distributional attributes of security returns, those measures based on correlation which is only a measure of dependence for elliptically distributed variables cannot appropriately measure the dependency between securities. However, by comparing their factor hierarchies we can appropriately measure the dependency. To compare their factor hierarchies systematically, we define a new measure called "$IFOD$". The $IFOD$, which is defined in Equation (15), calculates the average of all the squared factor distances between two securities. The smaller the value of $IFOD$ is, the larger the dependency between the two securities is. $IFOD$ reflects the whole aspects of dependency between securities through the factor-by-factor comparison of their returns, whereas correlation measures only the central dependency between the security returns. And, we also define another measure, which is designated as "$CHANGE$" in Equation (16). It can measure how much the attributes of a stock has changed between two different periods by comparing $RIFODs$ of the security in the two periods. We also compare the performance of $CHANGE$ with that of $CHANGE^*$, which is based on correlation: $CHANGE$ can reflect appropriately tail events that may happen during the economic turbulence owing to the holistic nature of $IFOD$, whereas $CHANGE^*$ cannot due to the limited ability of correlation.

Empirical studies of this paper show that the new method outperforms the old measures which are based on correlation: owing to the factor-wise comparison $IFOD$ can provide the more fundamental concept of relationship between securities than the traditional statistical tools based on correlation can do. $IFOD$ measures the dependency between stocks by analyzing the relationship of independent factors which comprise the returns, while correlation measures it by comparing the relationship of the returns themselves. Furthermore, $IFOD$ can reveals very delicate dependence structures between securities that otherwise remain hidden.

We have discussed TnA algorithm to cope with large N (the number of securities in a universe), and also discussed how to deal with multiple factor hierarchies of a single stock. We have also found that the $IFOD$ ranking matrices show asymmetry, i.e. that every security selects his best friend in a human-like attitude: Stock A chooses Stock B as his best friend, whereas

Stock B happens to choose Stock C instead of Stock A as his best friend.

Finally, we provide several useful examples of applying $IFOD$ to various areas in finance such as portfolio management (see Section 7.1 and 7.4) or risk management (see Section 7.2 and 7.3).

Considering that important variables in finance such as security returns are non-elliptically distributed and that the correlation is only relevant for elliptically distributed variables, $IFOD$ will be a viable alternative to the correlation-based measures of dependency between securities. It provides a nonparametric distribution-free approach for measuring the dependency.

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Appendix A. Data Preprocessing

Centering the random vector $\mathbf{x} = [x_1, x_2, \cdots, x_N]'$[19] means to subtract its mean vector $\mathbf{m} = E[\mathbf{x}]$ from $\mathbf{x}$ so to make it a zero-mean variable. This implies that $\mathbf{s}$ is a zero-mean variable as well. After being centered, $\mathbf{x}$ is linearly transformed into a white random vector $\tilde{\mathbf{x}}$. The components of $\tilde{\mathbf{x}}$ are uncorrelated and their variances are unity: $E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}'] = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix. In order to whiten $\mathbf{x}$, the eigenvalue decomposition is applied to the covariance matrix $E[\mathbf{x}\mathbf{x}']$ as follows: $E[\mathbf{x}\mathbf{x}'] = \mathbf{EDE}'$, where $\mathbf{E}$ is the orthogonal matrix of eigenvectors of $E[\mathbf{x}\mathbf{x}']$, and $\mathbf{D}$ is the diagonal matrix of its eigenvalues: $\mathbf{D} = diag(d_1, d_2, \cdots, d_N)$, and d_i, $1 \leq i \leq N$ are eigenvalues. $E[\mathbf{x}\mathbf{x}']$ is estimated in a standard way from the available sample $\mathbf{x}(1), \mathbf{x}(2), \cdots, \mathbf{x}(T) : E[\mathbf{x}\mathbf{x}'] \approx \frac{1}{T-1}\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}(t)'$. Now, whitening is done as follows: $\widetilde{\mathbf{x}} = \mathbf{E}\mathbf{D}^{-1/2}\mathbf{E}'\mathbf{x}$, where $\mathbf{D}^{-1/2} = diag(d_1^{-1/2}, d_2^{-1/2}, \cdots, d_N^{-1/2})$.

Appendix B. FastICA Algorithm

In the FastICA algorithm, the approximation of negentropy gives an objective function for estimating $\mathbf{W}$. By maximizing the function given as

$$J_G(\mathbf{w}) = [E[G(\mathbf{w}'\mathbf{x})] - E[G(v)]]^2$$

we can find one independent component as $y_i = \mathbf{w}'\mathbf{x}$. $\mathbf{w}$ is a $N - dimensional$ weight vector constrained so that $E[(\mathbf{w}'\mathbf{x})^2] = 1$. For whitened data, this constraint implies that the norm of $\mathbf{w}$ to be unity: $E[(\mathbf{w}'\mathbf{x})^2] = E[\mathbf{w}'\mathbf{x}\mathbf{x}'\mathbf{w}] = \mathbf{w}'E[\mathbf{x}\mathbf{x}']\mathbf{w} = \mathbf{w}'\mathbf{I}\mathbf{w} = \mathbf{w}'\mathbf{w} = 1$.

The one-unit objective function can be extended to compute the whole matrix $\mathbf{W}$ as follows:

$$\underset{\{\mathbf{w}_i\}_{i=1}^{N}}{Max} \sum_{i=1}^{N} J_G(\mathbf{w}_i)$$
$$s.t. \quad E[(\mathbf{w}_i'\mathbf{x})(\mathbf{w}_j'\mathbf{x})] = \delta_{jk}$$

[19] $\mathbf{x}$ means a random vector corresponding to cross-sectional data: for given t, $\mathbf{x}(t)$ denotes $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_N(t)]'$. Meanwhile, $\mathbf{x}_i$ means the time series of $i - th$ signal: $\mathbf{x}_i = [x_i(1), x_i(2), \cdots, x_i(T)]'$.

, where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ if $j \neq k$. This extension results from maximizing the sum of N one-unit objective functions and taking into account the constraint of de-correlation. At the maximum, every vector $\mathbf{w}_i^{'}, i = 1, 2, \cdots, N$ gives one of the rows in the de-mixing matrix $\mathbf{W}$.

Algorithm for One Unit

Estimation of $\mathbf{w}$ proceeds iteratively with the following steps, until convergence is achieved. Convergence means that the old and new value of $\mathbf{w}$ point to the same direction, i.e. their dot-product is almost equal to 1.

1. Choose an initial random vector $\mathbf{w}$ with $||\mathbf{w}|| = 1$.
2. $\mathbf{w} \leftarrow E[\mathbf{x}g(\mathbf{w}^{'}\mathbf{x})] - E[g^{'}(\mathbf{w}^{'}\mathbf{x})]\mathbf{w}$, where $g(z) = dG/dz$, and $g^{'}(z) = dg(z)/dz$.
3. $\mathbf{w} \leftarrow \mathbf{w}/||\mathbf{w}||$.
4. If $|\mathbf{w}_{old}^{'}\mathbf{w}_{new} - 1| \leq \varepsilon$ then stop; otherwise go back to step 2.

Algorithm for Multiple Units

Estimating several independent components needs to run the one-unit FastICA algorithm using several units with weight vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\cdots$, $\mathbf{w}_N$. To prevent different vectors from converging to the same maximum, the outputs $\mathbf{w}_1^{'}\mathbf{x}$, $\mathbf{w}_2^{'}\mathbf{x}$, $\cdots$,$\mathbf{w}_N^{'}\mathbf{x}$ must be de-correlated at every iteration. For whitened $\mathbf{x}$, such a de-correlation is equivalent to orthogonalization. Step 4 below is for this operation.

1. Estimate $\mathbf{w}_{1,}\mathbf{w}_2, \cdots, \mathbf{w}_p$.
2. Choose an initial random vector $\mathbf{w}_{p+1}$ with $||\mathbf{w}_{p+1}|| = 1$.
3. $\mathbf{w}_{p+1} \leftarrow E[\mathbf{x}g(\mathbf{w}_{p+1}^{'}\mathbf{x})] - E[g^{'}(\mathbf{w}_{p+1}^{'}\mathbf{x})]\mathbf{w}$, where $g(z) = dG/dz$, and $g^{'}(z) = dg(z)/dz$.
4. $\mathbf{w}_{p+1} \leftarrow \mathbf{w}_{p+1} - \sum_{j=1}^{p} \mathbf{w}_{p+1}^{'}\mathbf{w}_j\mathbf{w}_j$.

5. $\mathbf{w}_{p+1} \leftarrow \mathbf{w}_{p+1}/||\mathbf{w}_{p+1}||$.

6. If $|\mathbf{w}'_{p+1old}\mathbf{w}_{p+1new} - 1| \leq \varepsilon$ then stop; otherwise go back to step 3.

Appendix C. TnA Algorithm

The basic procedure of TnA algorithm is given as follow: from the set of N independent components, pick y_r as the last one in the ordering, which makes the RHD error between x_k and the corresponding reconstruction from those $\{y_i\}_{i=1,i\neq r}^{N}$ minimized; then, remove this independent component from the component set; next, repeat the same operation on the remaining component set $\{y_i\}_{i=1,i\neq r}^{N}$ and select the second-last component, ..., and so forth.

1. Let $Z = \{i|1 \leq i \leq N\}$, $l = 1$, $L_k = ()$.

2. For each $i \in Z$, let $v_{ki}(t) = \sum_{p\neq i, p\in Z} c_{kp}(t)$, $1 \leq t \leq T$.

3. Select $\beta = \underset{i\in Z}{argminRHD}(x_k, v_{ki})$ as the $l-th$ element of L_k.

4. Let $Z = Z - \{\beta\}$

5. If $Z \neq \{\}$, let $l = l + 1$ and go to step 2; otherwise go to step 6

6. Let $\hat{L}_k^* = L_k^{-1}$, where $\hat{L}_k^*$ is an estimate of L_k^*, and L_k^{-1} denotes the inverse order of L_k

7. Stop.

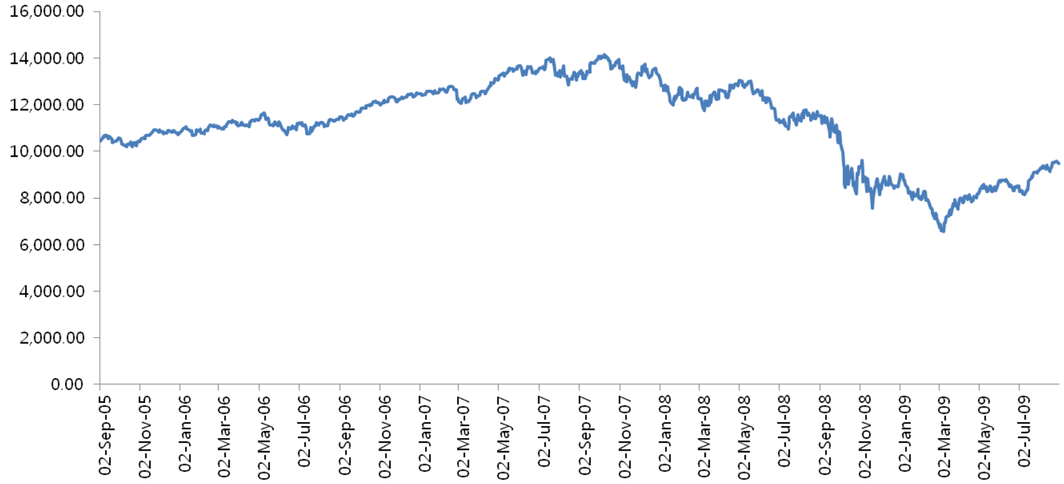
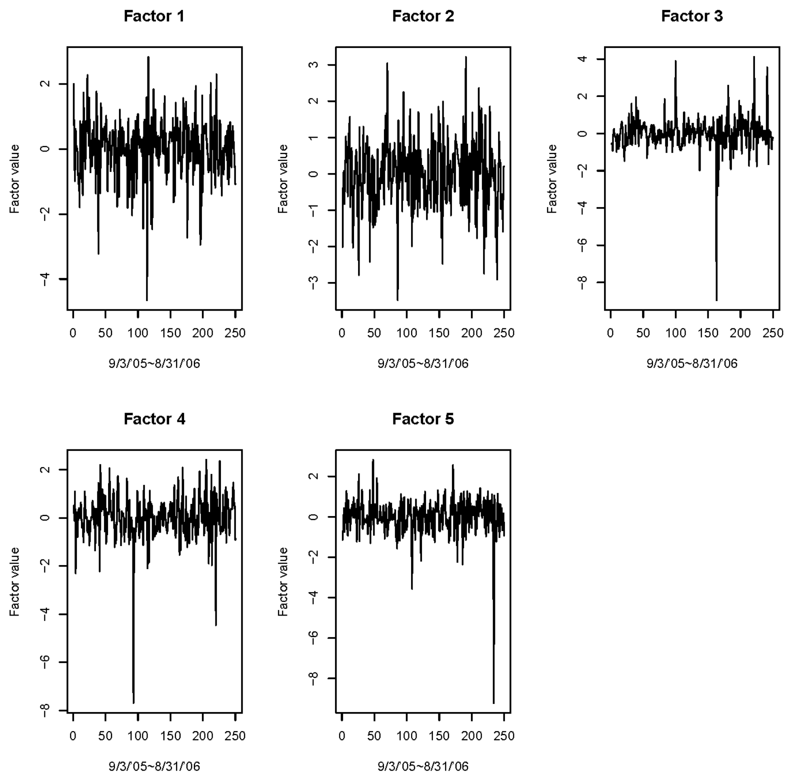


Figure 1: Dow Jones Industrial Average from 9/2/2005 to 8/31/2009

This graph shows market closing prices of Dow Jones Industrial Average from 9/2/2005 to 8/31/2009. All the prices are adjusted for dividends and splits. Data source: http://finance.yahoo.com.

[Independent Factors in Period 1]



[Independent Factors in Period 2]

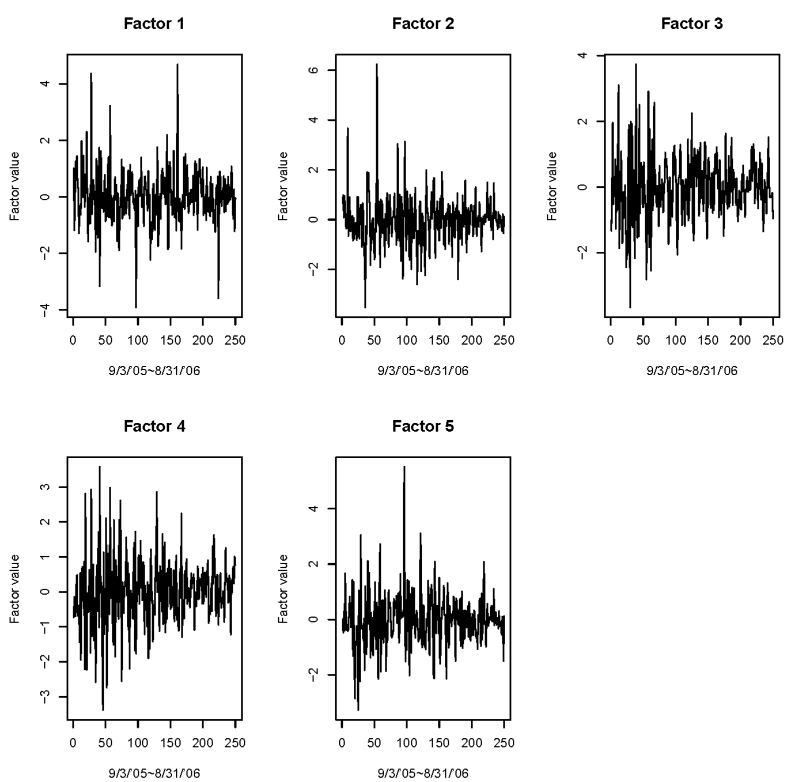


Figure 2: Independent Factors of 5 IT Stocks from DJIA Components

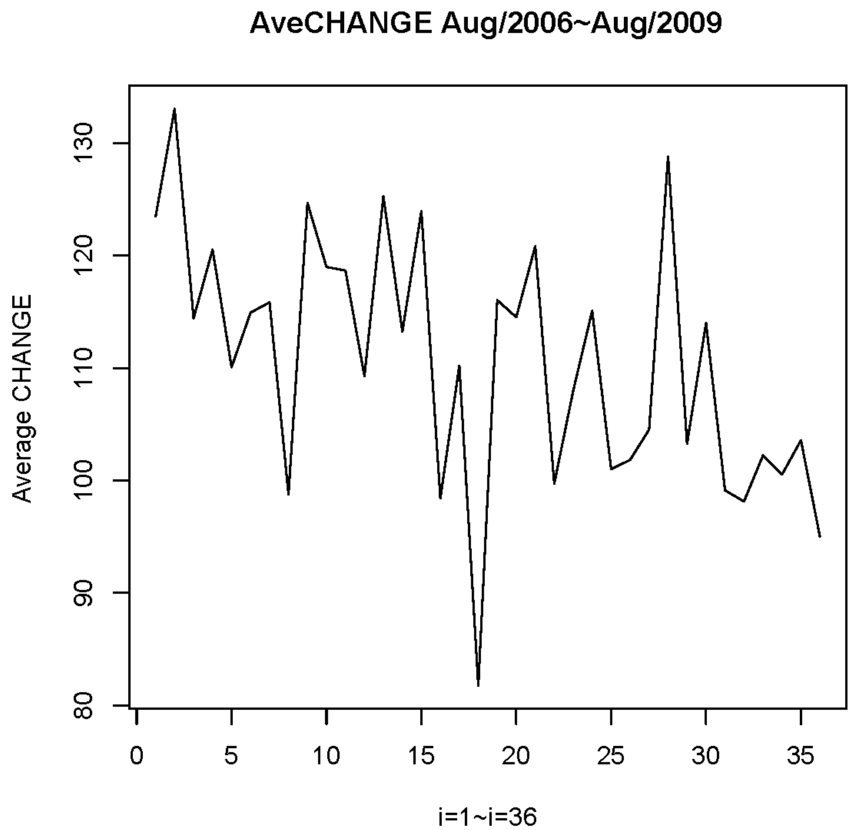


Figure 3: $AveCHANGE$

This graph show $AveCHANGE(t_{i-1}, t_i)$, $1 \leq i \leq 36$. $AveCHANGE(t_{i-1}, t_i)$ is the average of $CHANGE(k, t_{i-1}, t_i)$ across k, $1 \leq k \leq 30$. There are two large spikes at $i = 18$ and $i = 28$, which imply that two structural breaks occurred: the one between 1/2008 and 2/2008; the other between 11/2008 and 12/2008.

Table 1: Component Stocks of Dow Jones Industrial Average

This table shows the list of current component stocks of which Dow Jones Industrial Average is comprised. Data source: *http://en.wikipedia.org.*

Stock #	Symbol	Company	Industry	Data Added
1	AA	Alcoa	Aluminum	6/01/1959
2	AXP	American Express	Consumer finance	8/30/1982
3	BA	Boeing	Aerospace & defense	3/12/1987
4	BAC	Bank of America	Banking	2/19/2008
5	CAT	Caterpillar	Constr. & mining equip.	5/06/1991
6	CSCO	Cisco Systems	Computer networking	7/08/2009
7	CVX	Chevron Corp.	Oil & gas	2/19/2008
8	DD	DuPoint*	Chemical industry	11/20/1935
9	DIS	Walt Disney	Broadcast. & entertain.	5/06/1991
10	GE	General Electric	Conglomerate	11/07/1907
11	HD	The Home Depot	Home improv. retailer	11/01/1999
12	HPQ	Hewlett-Packard	Technology	3/17/1997
13	IBM	IBM	Computers & tech.	6/29/1979
14	INTC	Intel	Semiconductors	11/01/1999
15	JNJ	Johnson & Johnson	Pharmaceuticals	3/17/1997
16	JPM	JP Morgan Chase	Banking	5/06/1991
17	KO	Coca-Cola	Beverages	3/12/1987
18	MCD	McDonald's	Fast food	10/30/1985
19	MMM	3M	Conglomerate	1/09/1976
20	MRK	Merck	Pharmaceuticals	6/29/1979
21	MSFT	Microsoft	Software	11/01/1999
22	PFE	Pfizer	Pharmaceuticals	4/08/2004
23	PG	Procter & Gamble	Consumer goods	5/26/1932
24	T	AT&T	Telecommunication	11/11/2001
25	TRV	Travelers	Insurance	6/08/2009
26	UNH	UnitedHealth Group**	Managed health care	9/24/2012
27	UTX	United Tech. Corp.	Conglomerate	3/14/1939
28	VZ	Verizon	Telecommunication	4/08/2004
29	WMT	Wal-Mart	Retail	3/17/1997
30	XOM	Exxon Mobile	Oil & gas	10/01/1928

*DuPoint was also included for 1/22/1924 - 8/31/1925.

**UnitedHealth Group replaced Kraft Foods (KTF) on 9/24/2012.

Table 2: Optimum Order Lists of the 5 IT Stocks from DJIA Components

These tables show factor hierarchies of 5 IT stock both in each Period and Period 2. The exhaustive search orders independent factors with respect to each stock so that the cumulative reconstruction error is minimized.

(a)[Optimum Order Lists in Period 1]

Stock # (k)	Symbol	Optimum factor order(L_k^*)	Cumulative reconstruction error($J_{L_k^*}$)
6	CSCO	5, 2, 1, 4, 3	1.156627
12	HPQ	1, 2, 3, 5, 4 1, 2, 5, 3, 4	0.915663
13	IBM	2, 3, 1, 5, 4	0.273092
14	INTC	4, 2, 1, 5, 3	0.899598
21	MSFT	3, 2, 4, 1, 5	1.429719

(b)[Optimum Order Lists in Period 2]

Stock # (k)	Symbol	Optimum factor order(L_k^*)	Cumulative reconstruction error($J_{L_k^*}$)
6	CSCO	3, 4, 1, 5, 2	0.963855
12	HPQ	4, 2, 3, 1, 5	2.024096
13	IBM	3, 4, 5, 2, 1	2.088353
14	INTC	4, 3, 5, 1, 2	1.156627
21	MSFT	1, 4, 3, 5, 2 4, 1, 3, 5, 2	2.152610

Table 3: $IFODs$ Of the 5 IT Stocks

These tables show $IFODs$ of the 5 IT stocks both in Period 1 and Period 2. Because Stock 12 (HPQ) has two factor hierarchies in Period 1, Sub-table (a) shows $IFOD$ matrices corresponding to the factor hierarchies of Stock 12. And, because Stock 21 (MSFT) also has two factor hierarchies in Period 2, Sub-table (b) shows $IFOD$ matrices corresponding to the factor hierarchies of Stock 21. Each number in () represents the ranking of $IFOD$ in the row of the corresponding matrix.

(a)[$IFODs$ and Their Rankings in Period 1]

	$L^*_{12} = (1,2,3,5,4)$						$L^*_{12} = (1,2,5,3,4)$				
	Stock 6	Stock 12	Stock 13	Stock 14	Stock 21		Stock 6	Stock 12	Stock 13	Stock 14	Stock 21
Stock 6 (CSCO)	0.0 (0)	3.6 (1.5)	4.0 (3)	3.6 (1.5)	6.8 (4)	Stock 6 (CSCO)	0.0 (0)	2.0 (1)	4.0 (3)	3.6 (2)	6.8 (4)
Stock 12 (HPQ)	3.6 (2.5)	0.0 (0)	1.2 (1)	4.8 (4)	3.6 (2.5)	Stock 12 (HPQ)	2.0 (1.5)	0.0 (0)	2.0 (1.5)	4.4 (3)	5.2 (4)
Stock 13 (IBM)	4.0 (3)	1.2 (1)	0.0 (0)	5.2 (4)	1.6 (2)	Stock 13 (IBM)	4.0 (3)	2.0 (2)	0.0 (0)	5.2 (4)	1.6 (1)
Stock 14 (INTC)	3.6 (1)	4.8 (3)	5.2 (4)	0.0 (0)	4.4 (2)	Stock 14 (INTC)	3.6 (1)	4.4 (2.5)	5.2 (4)	0.0 (0)	4.4 (2.5)
Stock 21(MSFT)	6.8 (4)	3.6 (2)	1.6 (1)	4.4 (3)	0.0 (0)	Stock 21(MSFT)	6.8 (4)	5.2 (3)	1.6 (1)	4.4 (2)	0.0 (0)

(b)[$IFODs$ and Their Rankings in Period 2]

	$L^*_{21} = (1,4,3,5,2)$						$L^*_{21} = (4,1,3,5,2)$				
	Stock 6	Stock 12	Stock 13	Stock 14	Stock 21		Stock 6	Stock 12	Stock 13	Stock 14	Stock 21
Stock 6 (CSCO)	0.0 (0)	3.2 (4)	1.2 (2)	0.8 (1)	1.6 (3)	Stock 6 (CSCO)	0.0 (0)	3.2 (4)	1.2 (2.5)	0.8 (1)	1.2 (2.5)
Stock 12 (HPQ)	3.2 (3)	0.0 (0)	2.8 (1.5)	2.8 (1.5)	4.0 (4)	Stock 12 (HPQ)	3.2 (4)	0.0 (0)	2.8 (2)	2.8 (2)	2.8 (2)
Stock 13 (IBM)	1.2 (2)	2.8 (3)	0.0 (0)	0.8 (1)	4.4 (4)	Stock 13 (IBM)	1.2 (2)	2.8 (3)	0.0 (0)	0.8 (1)	3.2 (4)
Stock 14 (INTC)	0.8 (1.5)	2.8 (4)	0.8 (1.5)	0.0 (0)	2.4 (3)	Stock 14 (INTC)	0.8 (1.5)	2.8 (4)	0.8 (1.5)	0.0 (0)	1.2 (3)
Stock 21(MSFT)	1.6 (1)	4.0 (3)	4.4 (4)	2.4 (2)	0.0 (0)	Stock 21(MSFT)	1.2 (1.5)	2.8 (3)	3.2 (4)	1.2 (1.5)	0.0 (0)

Table 4: Average $CHANGE(k, p1, p2)$

Because Stock 12 (HPQ) has two optimum order lists in Period 1 and Stock 21 (MSFT) also has two optimum order lists in Period 2, there are all 4 pairs of ranking matrices which we have to consider for the calculation of $CHANGE$. For a fixed Stock k, we calculate all the values of $CHANGE(k, Period1, Period2)$ using the 4 pairs of ranking matrices and then average them out. This table reports the result.

Stock k	Symbol	$CHANGE(k, p1, p2)$	Ranking of CHANGE
Stock 6	CSCO	2.100	3
Stock 12	HPQ	1.625	1
Stock 13	IBM	3.800	5
Stock 14	INTC	1.750	2
Stock 21	MSFT	3.600	4

Table 5: Correlations between the 5 IT Stocks from DJIA Components

This table shows correlations and their rankings of 5 IT stocks in each Period. The larger correlation takes the higher ranking. This table also shows $CHANGE^*$ of the 5 IT stocks. $CHANGE^*$ is a correlation version of $CHANGE$.

(a)[Correlations and Their Rankings in Period 1]

	Correlations in Period 1						Correlation Rankings in Period 1				
	CSCO	HPQ	IBM	INTC	MSFT		CSCO	HPQ	IBM	INTC	MSFT
CSCO	1.0	0.3315794	0.3412751	0.3075256	0.2174684	CSCO	0	2	1	3	4
HPQ	0.3315794	1.0	0.3610597	0.2521328	0.2707628	HPQ	2	0	1	4	3
IBM	0.3412751	0.3610597	1.0	0.3314999	0.3293906	IBM	2	1	0	3	4
INTC	0.3075256	0.2521328	0.3314999	1.0	0.2752798	INTC	2	4	1	0	3
MSFT	0.2174684	0.2707628	0.3293906	0.2752798	1.0	MSFT	4	3	1	2	0

(b)[Correlations and Their Rankings in Period 2]

	Correlations in Period 2						Correlation Rankings in Period 2				
	CSCO	HPQ	IBM	INTC	MSFT		CSCO	HPQ	IBM	INTC	MSFT
CSCO	1.0	0.7301774	0.7763356	0.8125835	0.7813958	CSCO	0	4	3	1	2
HPQ	0.7301774	1.0	0.7140514	0.7130443	0.6774483	HPQ	1	0	2	3	4
IBM	0.7763356	0.7140514	1.0	0.6997273	0.6756504	IBM	1	2	0	3	4
INTC	0.8125835	0.7130443	0.6997273	1.0	0.7509389	INTC	1	3	4	0	2
MSFT	0.7813958	0.6774483	0.6756504	0.7509389	1.0	MSFT	1	3	4	2	0

(c)[$CHANGE^*(k,\ Period\,1,\ Period\,2)$]

Stock k	Symbol	$CHANGE*$(k, Period 1, Period 2)	Ranking of $CHANGE*$
Stock 6	CSCO	4.0	4
Stock 12	HPQ	1.0	2
Stock 13	IBM	0.5	1
Stock 14	INTC	3.0	3
Stock 21	MSFT	4.5	5

Table 6: Average *IFOD* in Period 1 and Period 2

This table shows average *IFOD*s in Period 1 and those in Period 2, which are respectively the result of averaging out the *IFOD* ranking matrices in Sub-table (a) and in Sub-table (b) of Table 3.

(a)[Average *IFOD* in Period 1]

	CSCO	HPQ	IBM	INTC	MSFT
CSCO	0	2.8	4	3.6	6.8
HPQ	2.8	0	1.6	4.6	4.4
IBM	4	1.6	0	5.2	1.6
INTC	3.6	4.6	1.6	4.4	0
MSFT	6.8	4.4	1.6	4.4	0

(b)[Average *IFOD* in Period 2]

	CSCO	HPQ	IBM	INTC	MSFT
CSCO	0	3.2	1.2	0.8	1.4
HPQ	3.2	0	2.8	2.8	3.4
IBM	1.2	2.8	0	0.8	3.8
INTC	0.8	2.8	0.8	0	1.8
MSFT	1.4	3.4	3.8	1.8	0

Table 7: The Optimum Order Lists for the Whole DJIA Components in Period 1 by TnA

This table shows factor hierarchies of the 30 components of DJIA in Period 1, which are derived by TnA instead of Exhaustive Search. The bold numbers in the headline column of each table denote component stocks, and those in the headline row denote the factor order. The $i-th$ row represents the optimum factor order list corresponding to Stock i.

Stock/Order	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
1	29	6	23	22	7	18	8	26	10	15	2	14	27	25	28	11	3	9	1	5	17	24	20	12	30	13	16	21	19	4
2	6	18	25	23	26	30	28	14	9	20	7	10	11	4	27	1	29	22	15	17	5	16	12	3	13	2	21	19	8	24
3	27	6	26	13	25	11	18	10	29	30	9	2	3	21	15	23	28	20	14	7	19	22	1	8	16	12	4	24	5	17
4	18	7	6	26	23	19	15	3	30	9	2	21	10	5	17	13	12	11	25	14	27	24	16	4	8	29	1	22	28	20
5	22	18	7	27	21	28	2	13	17	16	15	5	30	20	10	3	4	12	19	1	25	26	23	9	24	8	14	29	6	11
6	10	18	17	6	13	26	20	5	30	23	8	9	3	28	25	4	19	7	15	1	29	11	27	2	16	22	24	21	14	12
7	11	3	23	18	8	13	9	14	7	28	25	19	21	27	5	22	10	4	2	17	15	6	1	24	26	29	16	12	20	30
8	22	18	3	23	19	15	9	28	17	11	10	1	20	13	6	25	16	29	14	24	21	27	8	30	5	4	26	12	7	2
9	26	29	9	12	30	28	7	18	5	2	25	6	23	27	1	14	20	15	21	19	11	24	4	17	16	13	10	3	22	8
10	5	9	6	27	12	18	25	23	11	8	28	7	14	30	13	15	2	19	21	4	1	20	17	29	10	24	26	22	16	3
11	17	30	18	19	6	15	7	1	4	11	9	29	23	26	2	28	20	14	5	25	8	24	10	22	13	27	21	16	3	12
12	10	18	17	5	28	9	13	8	21	22	23	30	16	19	11	20	29	2	15	27	25	1	12	14	26	7	6	24	4	3
13	11	5	27	23	25	9	8	21	18	4	15	20	14	6	17	16	29	13	19	10	22	24	1	12	2	28	30	7	3	26
14	16	27	5	15	2	18	22	8	6	17	26	10	29	9	24	3	12	19	23	11	1	7	13	28	25	20	14	30	21	4
15	5	18	26	23	11	16	12	1	28	9	30	2	13	3	15	6	10	21	4	22	20	7	19	29	27	24	25	17	14	8
16	18	23	26	21	22	5	19	15	7	30	25	10	14	28	3	4	12	11	13	27	2	29	24	20	17	6	16	9	8	1
17	5	8	18	6	20	12	4	26	22	30	13	10	21	17	3	16	2	29	7	25	24	11	15	9	28	27	23	1	19	14
18	27	18	25	23	26	11	8	19	16	3	9	15	28	10	12	6	17	21	5	30	1	24	29	22	14	20	13	7	4	2
19	24	5	6	18	30	28	22	23	27	7	15	12	13	25	17	14	4	9	26	21	10	8	3	19	1	20	2	16	11	29
20	20	1	18	9	29	16	13	23	11	12	5	2	21	15	25	26	22	10	19	30	28	24	17	7	4	14	6	3	27	8
21	14	5	18	27	15	6	28	26	22	9	10	23	16	30	8	17	21	24	19	11	2	7	4	25	29	20	12	3	1	13
22	1	10	19	25	20	8	18	5	6	23	11	26	28	22	2	17	27	3	13	9	16	12	14	29	30	24	21	15	4	7
23	5	4	23	27	18	3	12	6	22	16	9	11	25	28	21	1	10	30	17	24	15	20	29	13	26	19	2	14	8	7
24	8	5	21	17	23	7	3	6	2	4	24	25	16	18	28	19	14	10	9	30	22	12	11	15	29	13	26	27	20	1
25	28	13	22	9	8	18	6	21	30	20	11	5	15	12	1	14	23	29	26	17	3	27	25	19	2	7	16	4	24	10
26	8	5	21	2	23	11	29	25	14	3	30	24	6	16	20	17	18	13	10	27	28	12	4	1	19	22	7	15	26	9
27	23	13	12	18	25	26	3	2	6	28	17	30	16	5	8	14	11	29	7	15	21	19	27	24	20	10	4	1	22	9
28	7	6	8	23	10	29	18	19	17	28	22	2	11	21	12	9	15	13	26	1	27	24	4	3	30	25	20	14	5	16
29	17	18	6	2	25	19	21	7	13	12	23	30	10	9	29	22	3	16	27	26	14	24	5	20	15	11	28	8	4	1
30	8	9	25	23	15	26	18	5	11	7	29	13	28	3	19	10	6	1	4	17	27	22	12	16	2	20	14	21	30	24

Table 8: Optimum Order Lists and *IFODs* of the 5 IT Stocks in the New Universe

In order to resolve the problem of multiple factor hierarchies of the universe composed of the 5 IT stocks, two stock from different sectors are added to the universe. In the new universe, each of the 5 IT stocks has a unique factor hierarchy.

(a)[Optimum Order Lists in the New Universe]

		Optimum order list in Period 1		Optimum order list in Period 2	
Stock #	Symbol	L_k^*	$J_{L_k^*}$	L_k^*	$J_{L_k^*}$
6	CSCO	1, 5, 6, 3, 2, 4, 7	1.991968	3, 6, 2, 5, 4, 7, 1	2.104418
12	HPQ	1, 5, 3, 6, 7, 2, 4	1.863454	2, 4, 5, 3, 7, 6, 1	2.827309
13	IBM	3, 6, 5, 1, 7, 4, 2	1.526104	5, 3, 4, 2, 7, 6, 1	2.329317
14	INTC	4, 2, 6, 3, 1, 5, 7	3.228916	4, 3, 6, 7, 5, 2, 1	2.393574
21	MSFT	7, 6, 1, 3, 4, 5, 2	2.714859	1, 3, 4, 6, 5, 7, 2	4.385542

(b)[*IFODs* in the New Universe]

	IFOD in Period 1						*IFOD* in Period 2				
	CSCO	HPQ	IBM	INTC	MSFT		CSCO	HPQ	IBM	INTC	MSFT
CSCO	0.00	1.14	4.00	9.43	8.86	CSCO	0.00	5.71	4.57	4.57	8.86
HPQ	1.14	0.00	2.86	12.86	6.57	HPQ	5.71	0.00	2.57	6.29	12.29
IBM	4.00	2.86	0.00	10.57	5.14	IBM	4.57	2.57	0.00	4.86	9.43
INTC	9.43	12.86	10.57	0.00	11.71	INTC	4.57	6.29	4.86	0.00	6.57
MSFT	8.86	6.57	5.14	11.71	0.00	MSFT	8.86	12.29	9.43	6.57	0.00

Table 9: $IFODs$ between EBAY and DJIA Components and Their Rankings

EBAY is added to the universe composed of 30 DJIA components. Applying ICA and TnA to this new universe, $IFOD$s between EBAY and the other stocks to find the best friend of EBAY. In Period 1, Stock 30 (XOM) is the best friend of EBAY; in Period 2, Stock 14 (INTC) is the best friend.

(a)[$IFODs$ and Their Rankings in Period 1]

j	**1 (AA)**	**2 (AXP)**	**3 (BA)**	**4 (BAC)**	**5 (CAT)**	**6 (CSCO)**	**7 (CVX)**	**8 (DD)**	**9 (DIS)**	**10 (GE)**
$IFOD(0,j)$	122.12903	218.77419	157.22581	156.70968	126.38710	151.16129	128.00000	120.32258	112.70968	148.70968
$IFOD$ Ranking	5	29	16	15	6	13	7	4	3	11
j	**11 (HD)**	**12 (HPQ)**	**13 (IBM)**	**14 (INTC)**	**15 (JNJ)**	**16 (JPM)**	**17 (KO)**	**18 (MCD)**	**19 (MMM)**	**20 (MRK)**
$IFOD(0,j)$	165.22581	141.93548	151.09677	228.45161	168.38710	136.12903	175.80645	188.64516	157.61290	167.54839
$IFOD$ Ranking	18	10	12	30	20	8	22	26	17	19
j	**21 (MSFT)**	**22 (PFE)**	**23 (PG)**	**24 (T)**	**25 (TRV)**	**26 (UNH)**	**27 (UTX)**	**28 (VZ)**	**29 (WMT)**	**30 (XOM)**
$IFOD(0,j)$	186.70968	176.00000	177.16129	201.03226	175.41935	200.83871	101.1613	138.64516	153.29032	90.19355
$IFOD$ Ranking	25	23	24	28	21	27	2	9	14	1

(b)[$IFODs$ and Their Rankings in Period 2]

j	**1 (AA)**	**2 (AXP)**	**3 (BA)**	**4 (BAC)**	**5 (CAT)**	**6 (CSCO)**	**7 (CVX)**	**8 (DD)**	**9 (DIS)**	**10 (GE)**
$IFOD(0,j)$	117.22581	125.74194	134.70968	145.48387	140.12903	142.38710	109.61290	92.70968	110.19355	113.87097
$IFOD$ Ranking	11	13	19	27	24	26	7	2	8	9
j	**11 (HD)**	**12 (HPQ)**	**13 (IBM)**	**14 (INTC)**	**15 (JNJ)**	**16 (JPM)**	**17 (KO)**	**18 (MCD)**	**19 (MMM)**	**20 (MRK)**
$IFOD(0,j)$	127.93548	96.51613	94.83871	87.22581	119.41935	127.87097	138.58065	135.54839	138.58065	105.29032
$IFOD$ Ranking	16	4	3	1	12	15	21.5	20	21.5	5
j	**21 (MSFT)**	**22 (PFE)**	**23 (PG)**	**24 (T)**	**25 (TRV)**	**26 (UNH)**	**27 (UTX)**	**28 (VZ)**	**29 (WMT)**	**30 (XOM)**
$IFOD(0,j)$	130.19355	126.32258	139.48387	153.35484	116.83871	150.06452	162.96774	128.45161	142.06452	105.48387
$IFOD$ Ranking	18	14	23	29	10	28	30	17	25	6

Table 10: *Correlation* between EBAY and DJIA Components and Their Rankings

In order to compare the best friend selected by $IFOD$ with that selected by correlation, correlations between EBAY and the other stocks and their rankings are calculated. The best friend selected by correlation is Stock 29 (WMT) in Period 1, and Stock 14 (INTC) in Period 2.

(a)[*Correlation* in Period 1]

Period 1										
j	**1 (AA)**	**2 (AXP)**	**3 (BA)**	**4 (BAC)**	**5 (CAT)**	**6 (CSCO)**	**7 (CVX)**	**8 (DD)**	**9 (DIS)**	**10 (GE)**
$corr(0,j)$	0.17369987	0.3047355	0.2177693	0.2167696	0.3338698	0.16037104	0.20953906	0.2476379	02648477	0.3023642
Correlation Ranking	23	4	14	15	2	26	16	12	8	5
j	**11 (HD)**	**12 (HPQ)**	**13 (IBM)**	**14 (INTC)**	**15 (JNJ)**	**16 (JPM)**	**17 (KO)**	**18 (MCD)**	**19 (MMM)**	**20 (MRK)**
$corr(0,j)$	0.2870991	0.1596327	0.2260565	0.30690096	0.03927539	0.2555800	0.1970498	0.2739106	0.16847422	0.10938793
Correlation Ranking	6	27	13	3	30	10	18	7	25	29
j	**21 (MSFT)**	**22 (PFE)**	**23 (PG)**	**24 (T)**	**25 (TRV)**	**26 (UNH)**	**27 (UTX)**	**28 (VZ)**	**29 (WMT)**	**30 (XOM)**
$corr(0,j)$	0.19341091	0.19719973	0.1794065	0.1722712	0.2560197	0.15656987	0.2530603	0.1955051	0.34020232	0.18651537
Correlation Ranking	20	17	22	24	9	28	11	19	1	21

(b)[*Correlation* in Period 2]

Period 2										
j	**1 (AA)**	**2 (AXP)**	**3 (BA)**	**4 (BAC)**	**5 (CAT)**	**6 (CSCO)**	**7 (CVX)**	**8 (DD)**	**9 (DIS)**	**10 (GE)**
$corr(0,j)$	0.5890158	0.6641604	0.5801081	0.5512827	0.6780231	0.7363518	0.6711793	0.7300711	0.6860543	0.6044966
Correlation Ranking	21	11	22	26	8	2	9	3	5	19
j	**11 (HD)**	**12 (HPQ)**	**13 (IBM)**	**14 (INTC)**	**15 (JNJ)**	**16 (JPM)**	**17 (KO)**	**18 (MCD)**	**19 (MMM)**	**20 (MRK)**
$corr(0,j)$	0.6601145	0.6147778	0.6705232	0.7377577	0.5964717	0.6100989	0.4762749	0.5505375	0.6846528	0.5518962
Correlation Ranking	13	17	10	1	20	18	30	27	6	25
j	**21 (MSFT)**	**22 (PFE)**	**23 (PG)**	**24 (T)**	**25 (TRV)**	**26 (UNH)**	**27 (UTX)**	**28 (VZ)**	**29 (WMT)**	**30 (XOM)**
$corr(0,j)$	0.6804649	0.5680280	0.5693309	0.6284004	0.6621852	0.5009387	0.6888461	0.6435880	0.4766297	0.6275043
Correlation Ranking	7	24	23	15	12	28	4	14	29	16

Table 11: Best and Worst Friends of EBAY Selected by *IFOD* and by *Correlation*

Sub-table (a) shows best 5 friends and worst 5 friends of EBAY in each Period, which are selected by *IFOD*, and Sub-table (b) shows those selected by *Correlation*.

(a)[Best and Worst Friends by *IFOD*]

	Ranking	Period 1	Period 2
Best	1	Stock 30 (XOM)	Stock 14 (INTC)
	2	Stock 27 (UTX)	Stock 8 (DD)
	3	Stock 9 (DIS)	Stock 13 (IBM)
	4	Stock 8 (DD)	Stock 12 (HPQ)
	5	Stock 1 (AA)	Stock 20 (MRK)
Worst	30	Stock 14 (INTC)	Stock 27 (UTX)
	29	Stock 2 (AXP)	Stock 24 (T)
	28	Stock 24 (T)	Stock 26 (UNH)
	27	Stock 26 (UNH)	Stock 4 (BAC)
	26	Stock 18 (MCD)	Stock 6 (CSCO)

(b)[Best and Worst Friends by *Correlation*]

	Ranking	Period 1	Period 2
Best	1	Stock 29 (WMT)	Stock 14 (INTC)
	2	Stock 5 (CAT)	Stock 6 (CSCO)
	3	Stock 14 (INTC)	Stock 8 (DD)
	4	Stock 2 (AXP)	Stock 27 (UTX)
	5	Stock 10 (GE)	Stock 9 (DIS)
Worst	30	Stock 15 (JNJ)	Stock 17 (KO)
	29	Stock 20 (MRK)	Stock 29 (WMT)
	28	Stock 26 (UNH)	Stock 26 (UNH)
	27	Stock 12 (HPQ)	Stock 18 (MCD)
	26	Stock 6 (CSCO)	Stock 4 (BAC)

Table 12: $AveCHANGE$

This table shows $AveCHANGE(t_{i-1}, t_i)$, $1 \leq i \leq 36$. The percentage of variation is calculated as $100 \times (AveCHANGE(t_i, t_{i+1}) - AveCHANGE(t_{i-1}, t_i))/AveCHANGE(t_{i-1}, t_i)$.

From	To	i	$AveCHANGE(t_{i-1}, t_i)$	change ratio(%)
8/2006	9/2006	1	123.4778	
9/2006	10/2006	2	133.0589	7.759371
10/2006	11/2006	3	114.4289	-14.00131821
11/2006	12/2006	4	120.5311	5.332743739
12/2006	1/2007	5	110.0772	-8.673197208
1/2007	2/2007	6	114.9233	4.402455731
2/2007	3/2007	7	115.8378	0.7795758121
3/2007	4/2007	8	98.765	-14.73853958
4/2007	5/2007	9	124.6661	26.22497848
5/2007	6/2007	10	118.9839	-4.557935156
6/2007	7/2007	11	118.6672	-0.266170465
7/2007	8/2007	12	109.285	-7.906312781
8/2007	9/2007	13	125.2644	14.62176877
9/2007	10/2007	14	113.2217	-9.613824838
10/2007	11/2007	15	123.9594	9.483782702
11/2007	12/2007	16	98.40833	-20.61245053
12/2007	1/2008	17	110.1939	11.97619145
1/2008	2/2008	18	81.7444	-25.8176723
2/2008	3/2008	19	116.0289	41.94109933
3/2008	4/2008	20	114.5183	-1.301917022
4/2008	5/2008	21	120.8239	5.506194207
5/2008	6/2008	22	99.705	-17.47907492
6/2008	7/2008	23	107.9728	8.292262173
7/2008	8/2008	24	115.0767	6.579342205
8/2008	9/2008	25	101.005	-12.22810526
9/2008	10/2008	26	101.8333	0.820058413
10/2008	11/2008	27	104.5228	2.641081061
11/2008	12/2008	28	128.8156	23.24162766
12/2008	1/2009	29	103.2506	-19.84619875
1/2009	2/2009	30	114.0039	10.41475788
2/2009	3/2009	31	99.12222	-13.05365869
3/2009	4/2009	32	98.13944	-0.99148304
4/2009	5/2009	33	102.2328	4.170963274
5/2009	6/2009	34	100.5156	-1.679695753
6/2009	7/2009	35	103.5756	3.044303571
7/2009	8/2009	36	95.01333	-8.266686362

Table 13: $RIFODs$ in Period 1

The $i-th$ row of this table shows the rankings of $IFOD(i,j)$, $1 \leq j \leq 30$ in Period 1. The smaller $IFOD$ takes the higher rankings, which implies the more dependency.

Stock i / Stock j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	0	3	5	8	29	12	9	18	7	16	14	25	23	4	26	13	24	11	15	28	6	10	27	22	19	21	17	1	20	2
2	3	0	2	14	26	5	23	18	1	7	4	24	19	29	11	10	25	12	8	21	6	15	13	28	17	27	16	22	20	9
3	2	1	0	3.5	28	6	11	18	5	10	21	25	19	22	16	14	27	3.5	26	20	17	13	24	29	12	23	9	15	7	8
4	18	17	6	0	20	5	16	22	11	14	4	24	28	19	8	2	21	12	13	26	15	25	23	10	27	29	3	7	1	9
5	23	24	19	11	0	15	17	13	20	21	22	1	27	5	9	2	6	25	3	14	7	26	8	12	10	29	16	18	4	28
6	17	4	8	7	28	0	20	10	26	19	6	5	25	23	14	22	3	13	16	27	18	1	24	21	15	29	9	11	12	2
7	12	21	13	9	24	17	0	2	28	4	25	10	3	26	23	8	29	5	22	27	19	16	18	6	11	14	15	7	20	1
8	14	12	17	21	22	7	1	0	29	25	11	2	13	19	18	16	28	3	23	9	15	4	8	26	5	27	24	10	20	6
9	9	1	6	7	22	18	28	29	0	2	4	23	25	20	3	12	21	16	10	5	14	24	19	27	15	26	8	17	13	11
10	17	7	13	11	28	23	6	29	2	0	15	16	1	25	14	24	27	10	3	26	8	21	9	20	4	22	12	18	19	5
11	9	1	18	3	27	2	23	8	5	7	0	17	20	25	17	14	26	22	12	19	11	13	28	21	15	29	24	4	10	6
12	27	25	26	28	4	2	10	3	29	16	23	0	8	13	12	18	17	15	22	6	7	5	21	20	1	19	24	9	14	11
13	20	14	16	25	29	17	3	13	28	1	19	7	0	15	23	22	21	4	18	11	8	10	2	12	9	5	27	24	26	6
14	2	29	21	12	7	17	27	14	24	20	26	5	13	0	9	28	6	3	16	22	1	8	10	18	25	23	19	11	15	4
15	24	8	17	5	19	12	23	20	3	13	22	11	25	15	0	6	14	9	21	1	18	16	2	29	7	28	4	28	26	10
16	13	7	14	1	3	20	6	17	11	22	21	16	26	29	5	0	23	10	2	25	4	27	15	18	24	28	8	19	9	12
17	20	23	24	14	11	1	28	29	25	17	26	9	22	6	8	15	0	27	7	18	19	13	5	3	2	10	4	16	12	21
18	18	13	5	12	29	15	7	4	20	10	27	17	3	9	14	16	28	0	23	26	6	2	8	25	22	24	11	19	21	1
19	15	5	26	7	6	12	21	24	10	2	16	19	23	18	17	3	13	22	0	29	1	28	4	9	8	27	11	20	14	25
20	24	10	11	23	16	15	26	5	3	17	12	2	7	18	1	20	19	21	28	0	27	6	9	29	4	22	14	25	13	8
21	3	4	21	10	20	16	22	17	13	7	14	8	9	2	16	6	25	5	1	29	0	24	12	18	11	28	26	23	27	19
22	8	6	13	25	29	1	11	3	27	15	21	4	9	10	14	24	18	2	28	7	22	0	20	26	16	17	19	12	23	5
23	27	7	25	24	14	21	13	6	23	5	28	19	2	10	1	11	9	4	3	15	8	17	0	12	16	22	18	29	26	20
24	19	26	28	4	18	15	2	23	27	10	21	14	9	17	24	13	8	20	7	29	11	22	12	0	25	1	5	6	3	16
25	19	14	13	28	21	11	8	3	16	2	22	1	13	27	6	24	4	23	5	7	10	18	20	29	0	25	17	15	26	9
26	10	26	14	23	29	18	4	27	25	6	28	7	2	15	21	19	5	13	24	16	17	8	11	1	12	0	3	20	9	22
27	17	15.5	7	2	25	6	16	28	9	8	27	24	29	22	3	11	13	5	14	23	26	19	21	10	18	4	0	21	1	12
28	1	19	14	5	24	9	6	10	22	13	4	7	25	12	27	17	23	16	21	28	20	11	29	8	15	26	18	0	2	3
29	14	16	5	1	6	8	20	19	10	13	12	9	29	17	27	7	15	21	11	22	25	23	24	4	28	18	2	3	0	26
30	3	10	12	9	29	4	1	11	19	5	15	16	7	18	14	20	26	2	25	21	22	8	24	23	17	28	13	6	27	0

Table 14: *RHDs* between the Return of DJIA and Those of Independent Factors

This table shows the *RHDs* between the independent factors and DJIA in Period 1. RHD_n measure how well Factor n mimics the trends of DJIA.

	Period 1				Period 2			
Factor(n)	RHD Ranking	RHD_n	RHD_n^+	RHD_n^-	RHD Ranking	RHD_n	RHD_n^+	RHD_n^-
1	26.5	1.959839	1.959839	2.040161	21.0	1.847390	1.847390	2.152610
2	13.5	1.799197	2.200803	1.799197	15.0	1.799197	2.200803	1.799197
3	22.0	1.911647	1.911647	2.088353	29.0	1.975904	1.975904	2.024096
4	13.5	1.799197	1.799197	2.200803	24.5	1.911647	2.088353	1.911647
5	2.0	1.429719	1.429719	2.570281	2.0	1.477912	2.522088	1.477912
6	8.5	1.670683	2.329317	1.670683	13.0	1.783133	2.216867	1.783133
7	19.0	1.879518	1.879518	2.120482	23.0	1.895582	1.895582	2.104418
8	24.0	1.927711	2.072289	1.927711	19.5	1.831325	2.168675	1.831325
9	20.0	1.895582	2.104418	1.895582	1.0	1.365462	2.634538	1.365462
10	5.5	1.638554	1.638554	2.361446	8.0	1.702811	1.702811	2.297189
11	29.0	1.975904	1.975904	2.024096	4.0	1.590361	2.409639	1.590361
12	22.0	1.911647	2.088353	1.911647	17.5	1.815261	1.815261	2.184739
13	16.0	1.831325	1.831325	2.168675	10.5	1.767068	2.232932	1.767068
14	12.0	1.783133	1.783133	2.216867	19.5	1.831325	2.168675	1.831325
15	25.0	1.943775	2.056225	1.943775	22.0	1.863454	2.136546	1.863454
16	16.0	1.831325	2.168675	1.831325	29.0	1.975904	2.024096	1.975904
17	10.0	1.686747	1.686747	2.313253	3.0	1.542169	2.457831	1.542169
18	1.0	1.381526	1.381526	2.618474	15.0	1.799197	1.799197	2.200803
19	29	1.975904	2.024096	1.975904	29.0	1.975904	2.024096	1.975904
20	29	1.975904	2.024096	1.975904	5.5	1.654618	1.654618	2.345382
21	22.0	1.911647	1.911647	2.088353	15.0	1.799197	1.799197	2.200803
22	26.5	1.959839	2.040161	1.959839	7.0	1.670683	2.329317	1.670683
23	4.0	1.574297	2.425703	1.574297	17.5	1.815261	1.815261	2.184739
24	11.0	1.751004	1.751004	2.248996	26.0	1.927711	1.927711	2.072289
25	16.0	1.831325	2.168675	1.831325	10.5	1.767068	1.767068	2.232932
26	8.5	1.670683	1.670683	2.329317	5.5	1.654618	1.654618	2.345382
27	5.5	1.638554	1.638554	2.361446	27.0	1.943775	1.943775	2.056225
28	3.0	1.542169	2.457831	1.542169	24.5	1.911647	1.911647	2.088353
29	18.0	1.847390	2.152610	1.847390	10.5	1.767068	1.767068	2.232932
30	7.0	1.654618	1.654618	2.345382	10.5	1.767068	2.232932	1.767068

Table 15: $IFOD$ Version of CAPM

This table shows $IFOD$ version of beta for Stock i, $WRD(0,i)$ both in Period 1 and Period 2. It shows how different the behavior of Stock i is from that of the market. The closer to 1 $WRD(0,i)$ is, the more differently from the market Stock i behaves.

	Period 1				Period 2			
Stock(i)	$RIFOD(0,i)$	$RIFOD(i,0)$	$WRD(0,i)$	β_i	$RIFOD(0,i)$	$RIFOD(i,0)$	$WRD(0,i)$	β_i
Stock 1	14	8	0.3667	1.3178118	23	23	0.7667	1.9101529
Stock 2	3	4	0.1167	1.1964542	22	17	0.6500	1.8075688
Stock 3	17	12	0.4833	1.2872906	29	26	0.9167	1.0666729
Stock 4	10	13	0.3833	0.8733478	21	16	0.6167	2.4108033
Stock 5	18	6	0.4000	1.6704378	6	10	0.2667	1.2823385
Stock 6	4	5.5	0.1583	1.0308212	9	7	0.2667	1.1368270
Stock 7	27	24	0.8500	0.8685007	15	25	0.6667	1.2668234
Stock 8	29	27	0.9333	0.9642655	7	8	0.2500	1.2429143
Stock 9	6	4	0.1667	0.8748191	2	6	0.1333	1.2663147
Stock 10	8	11	0.3167	0.8155936	17	19	0.6000	1.2651749
Stock 11	22	13	0.5833	1.1018816	3	3	0.1000	1.0435409
Stock 12	13	10	0.3833	1.1771519	30	26	0.9333	0.9530258
Stock 13	23	18	0.6833	0.7417735	1	4	0.0833	0.8104965
Stock 14	16	6	0.3667	1.2673706	4	1	0.0833	1.0883026
Stock 15	12	8	0.3333	0.5134052	26	26	0.8667	0.6303141
Stock 16	11	6	0.2833	1.2018840	11	5	0.2667	1.8690681
Stock 17	21	12	0.5500	0.6205777	28	23	0.8500	0.6536436
Stock 18	19	19	0.6333	1.1754949	25	25	0.8333	0.6351993
Stock 19	1	1	0.0333	0.9133816	5	7	0.2000	0.8734384
Stock 20	30	27	0.9500	0.9119413	8	6	0.2333	0.9248901
Stock 21	2	1	0.0500	0.8469132	20	20	0.6667	1.0605571
Stock 22	26	24	0.8333	1.0189088	16	11	0.4500	0.8224013
Stock 23	7	5	0.2000	0.7682673	19	15	0.5667	0.6979561
Stock 24	9	5	0.2333	0.7583039	24	27	0.8500	0.9492408
Stock 25	28	29	0.9500	1.0355466	18	12	0.5000	1.2936727
Stock 26	20	6	0.4333	0.6863094	27	24	0.8500	1.2302972
Stock 27	5	2	0.1167	1.0967719	12	19	0.5167	1.0406635
Stock 28	25	24	0.8167	0.9050683	13	17	0.5000	0.8594748
Stock 29	15	9	0.4000	0.9190650	14	18	0.5333	0.5727569
Stock 30	24	25	0.8167	0.9597460	10	17	0.4500	1.1187344