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An Application of MCMC Methods in Stochastic Volatility Model

Reza Habibi[1]

Abstract

In this paper, using the MCMC method, we derive the conditional distribution of "mean variance" variable. This random variable appears in the option pricing problem under the stochastic volatility assumption. Two real data sets are considered.

JEL classification numbers : C3, C32
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1 Introduction

In finance theory, the Black and Scholes's (1973) option pricing formula is used to model the dynamic of a specified stock price, $\{s_t,\ t \in [0,T]\}$. It implies that the increment of the $\log(s_t)$ over an infinitesimal interval of time is governed by the increment of Weiner process. However, the Black-Scholes model assumes that the volatility parameter is constant over time. Obviously, this model is not able to describe the observed market patterns. In practice, for goodness of fit purposes, we shall assume that the volatility parameter is

[1] Department of Statistics, Central bank of Iran, e-mail: habibi1356@yahoo.com

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a time varying stochastic function (say σ_t). Because of this, the stochastic volatility models (going back to the Hull and White (1987)) are introduced to be extensions for the traditional Black-Scholes model. In the stochastic volatility models, it is assumed that the volatility itself satisfies in the other stochastic differential equation. Let $v_t = \sigma_t^2$ be the variance function and

$$R = \log(s_T/s_0) = \log(s_T) - \log(s_0)$$

be the total variation of log-price process over time interval $[0, T]$.

Nowadays, stochastic volatility models are necessary instruments for analyzing asset prices (see Ghysels *et al.* (1996) and Shephard (2005)). These models work very well, in practice. Characteristics of financial markets are measured by defining quantities on these models. For example, Hull and White (1987) introduced the "mean variance" variable ($\overline{v}$) in stochastic volatility series as the integral of variance function of a fixed derivative security over its life time (i.e. T) over T, that is,

$$\overline{v} = (1/T)\int_0^T v_\tau d\tau.$$

They showed that, in a risk-neutral world, the distribution of R conditional upon $\overline{v}$ is normal distribution. They also stressed that there is no analytical form for the distribution of $\overline{v}$.

In this paper, following Chib *et al.* (2006), the price equation is given by

$$d\log(s_t) = \{\theta_1 + \theta_2\sigma_t^2\}dt + \sigma_t dB_t,$$

$t \in [0, T]$. Following Hull and White (1987), when $\rho = 0$, we can show that the conditional distribution of R given $\overline{v}$ is $N(\eta^*, \lambda^{*2})$ where

$$\eta = (\theta_1 + \theta_2\overline{v})T \text{ and } \lambda^2 = T\overline{v}.$$

To see this, first assume that the variance function σ_t^2 is deterministic and time varying. Then

$$R = \int_0^T \{\theta_1 + \theta_2\sigma_t^2\}dt + \int_0^T \sigma_t dB_t.$$

In this case, it is easy to see that the distribution of R is normal with mean and variance η^*, λ^{*2}, respectively. Note that the parameters of this normal distribution depend on v_t throughout $\overline{v}$. When σ_t^2 is stochastic, there are an infinite number of paths with the same $\overline{v}$, but all of paths present the same distribution for conditional distribution of R. This completes the proof. There is another proof. In this case, we assume the variance changes at only n equally spaced times in the interval $[0,T]$. Let s_i and v_i denote the price and volatility during the i-th interval. We can see that

$$\log(\frac{s_i}{s_{i-1}}) = \{\theta_1 + \theta_2 v_i\}(T/n) + \sqrt{v_i}N_i,$$

where N_i has $N(0,T/n)$ distribution. Therefore, $\log(\frac{s_i}{s_{i-1}})$ given v_i is normal distribution with mean

$$\{\theta_1 + \theta_2 v_i\}(T/n),$$

and variance $(T/n)v_i$. Therefore, R given $v = (v_1, ..., v_n)$ is normal random variable with mean

$$\theta_1 T + \theta_2 (T/n)(\sum_{i=1}^{n} v_i)$$

and variance $(T/n)(\sum_{i=1}^{n} v_i)$. By letting, $n \to \infty$, the proof is completed. This method of proof helps us in the case of $\rho \neq 0$.

One can note that $\overline{v}$ is a useful criteria to compare the situation of two stocks. However, it is reasonable to compare the performance of two stocks, based their mean variances, at the same levels of total variations. This fact motivates us to derive the density of $\overline{v}$ given $R = r$. The MCMC method can be applied to generate samples from this posterior distribution as follows. Note that this type of MCMC method is referred as "conditional diffusion process simulation" which is appeared in Baltazar-Larios and Sørensen (2009).

1. Generate an initial unrestricted stationary sample path $\{v_t^{(0)}, t \in [0,T]\}$ of the diffusion of v_t using for instance the Milstein scheme or one of the other methods in Kloeden & Platen (1999). Then calculate $\overline{v_0} = (1/T)\int_0^T v_\tau^{(0)} d\tau$.

2. Set $l = 1$.

3. Propose a new sample paths by simulating $\{v_t^{(l)}, t \in [0,T]\}$ and as well as calculate $\overline{v_l}$.

4. Accept the proposed diffusion with probability

$$\min(1, \frac{\phi(r, \eta_l, \lambda_l^2)}{\phi(r, \eta_{l-1}, \lambda_{l-1}^2)}),$$

otherwise set $v^{(l)} = v^{(l-1)}$. Here, $\phi(r, \eta_l, \lambda_l^2)$ is the density of normal distribution with mean η_l and variance λ_l^2 computed at r.

5. Set $l = l + 1$ and go to 2.

Remark 1.1. Following Smith and Gelfand (1992), there is an alternative method against MCMC approach for generating samples from posterior distribution $\overline{v}$ given $R = r$ as follows. Generate L sample paths $\{v^{(l)}\}_{l=1}^{L}$ and calculate $\{\overline{v}^{(l)}\}_{l=1}^{L}$, $\{\eta_l\}_{l=1}^{L}$ and $\{\lambda_l^2\}_{l=1}^{L}$. Therefore, compute importance weights $\{w_l\}_{l=1}^{L}$ defined by

$$w_l = \frac{\phi(r, \eta_l, \lambda_l^2)}{\sum_{l=1}^{L} \phi(r, \eta_l, \lambda_l^2)}.$$

Finally, by re-sampling $\{v^{(l)}\}_{l=1}^{L}$ according to w_l and calculating $\overline{v}^{(l)}$, samples form posterior distribution is extracted.

Next, suppose that $\rho \neq 0$ and let v_t obey the following stochastic process

$$dv_t = \mu v_t dt + \xi v_t dw_t.$$

Then, $\log(\frac{v_i}{v_{i-1}})$ is normal with mean

$$\frac{\mu T}{n} - \frac{\xi^2 T}{2n},$$

and variance $\frac{\xi^2 T}{n}$. It can be seen that the conditional distribution of $\log(\frac{s_i}{s_{i-1}})$ given v_i is normal with mean

$$(\theta_1 + \theta_2 v_i)(T/n) + \rho\frac{\sqrt{v_i}}{\xi}\{\log(\frac{v_i}{v_{i-1}}) - \frac{\mu T}{n} + \frac{\xi^2 T}{2n}\},$$

and variance

$$v_i(1 - \rho^2)T/n.$$

Therefore, R conditional on the path followed by $\{v_1, ..., v_n\}$ has a normal distribution with mean

$$\eta^* = (\theta_1 + \theta_2 \overline{v})T + \frac{\rho}{\xi}\sum_{i=1}^{n} \sqrt{v_i}\{\log(\frac{v_i}{v_{i-1}}) - \frac{\mu T}{n} + \frac{\xi^2 T}{2n}\},$$

and variance

$$\lambda^{*2} = (1-\rho^2)T\overline{v}.$$

Letting, $n \to \infty$, the conditional distribution of R given $\{v_t\}_{t\in[0,T]}$ is derived. Note that this conditional distribution depends on the path v_t rather that $\overline{v}$. Also, note that the above mentioned MCMC method can be applied by substituting η, λ^2 with η^*, λ^{*2}. To this end, it is necessary to generate stationary sample paths

$$\{v_t^{(l)}, t \in [0,T]\}, l \geq 1,$$

of the diffusion of v_t at time points $t_i = iT/n, i = 1, 2, ..., n$ and, using the above formulas, calculate η^* and λ^{*2}, respectively. The variable $\overline{v}$ is approximated by

$$(1/nT)\sum_{i=1}^{n} v_i.$$

This paper is organized as follows. Two real data sets S&P 500 and U.S. equity returns are considered in section 2.

2 Real data sets

In this section, we apply the mentioned MCMC to two real data sets. They are (1) Standard and Poor's 500 index (S&P 500) and (2) U.S. equity returns.

Example 2.1. The daily returns data on closing prices of the S&P 500 are accessible and valuable time series. The prices of 500 large-cap common stocks (actively traded in the U.S.) are aggregated in this weighted index. We consider this time series from 5/5/1995 to 14/4/2003. Following Chib *et al.* (2006), we consider the GARCH form for the variance process v_t defined by

$$dv_t = \theta_4(\theta_5 - v_t)dt + \theta_3 v_t dw_t.$$

Using the generalized method of moments (Iacus (2008)), the estimates of parameters θ_2 and θ_4 are zero approximately and $(\theta_1,\ \theta_3^2,\ \theta_5,\ \rho) = (0.022,$ 0.029, 3.09, -0.84). Here, B_t and w_t are two standard Brownian motions on $[0,T]$ such that $\mathrm{cor}(B_t,\ w_t) = \rho$. This parameter is called leverage factor. The marginal distribution of $\log(\frac{v_i}{v_{i-1}})$ is normal with mean $-\frac{\theta_3^2 T}{2n}$ and variance $\frac{\theta_3^2 T}{n}$.

Under this information, we find that the conditional distribution of R given $\{v_1, ..., v_n\}$ is normal with mean

$$\theta_1 T + \frac{\rho}{\theta_3} \sum_{i=1}^{n} \sqrt{v_i} \{\log(\frac{v_i}{v_{i-1}}) + \frac{\theta_3^2 T}{2n}\},$$

and variance $(1-\rho^2)T\overline{v}$. The conditional expectation of $\overline{v}$ given $R = r$, $E(\overline{v}|r)$, are 0.012, 0.0139 and 0.0144 for $r = 0.039$, 0.05 and 0.055, respectively. It is seen that this index has little conditional average volatility and therefore it is a good siuation.

Remark 2.2. Chib *et al.* (2006) also considered the log-normal form for variance process given by

$$dv_t = \theta_4 v_t(\theta_5 - \log(v_t))dt + \theta_3 v_t dw_t.$$

Here, $\theta_1 \simeq 0$. The estimates of parameters are $\theta_2 = 0.021$, $\theta_3^2 = 0.033$, $\theta_4 = 0.027$, $\theta_5 = 0.72$ and $\rho = -0.83$. It is easy to see that the logarithm of v_t is Ornstein-Uhlenbeck process, that is

$$dy_t = c(\mu - y_t)dt + \sigma dw_t,$$

where $c = \theta_4$, $\mu = \theta_5 - \frac{\theta_3^2}{2\theta_4}$ and $\sigma = \theta_3$. Therefore, $\log(v_t)$ is normal $N(\kappa_t, \pi_2^2)$, where

$$\kappa_t = e^{-ct}[y_0 + \mu(e^{ct} - 1)] \text{ and } \pi_t^2 = \frac{\sigma^2}{2c}(1 - e^{-2ct}).$$

It is easy to see that

$$\text{cov}(y_s, y_t) = \frac{\sigma^2}{2c} e^{-c(s+t)}(e^{2c\min(s,t)} - 1).$$

Therefore, $\log(\frac{v_i}{v_{i-1}})$ is normal with mean $\iota_i = \kappa_{t_i} - \kappa_{t_{i-1}}$ and variance

$$\vartheta_i = \pi_{t_i}^2 + \pi_{t_{i-1}}^2 - 2$$

$\text{cov}(y_{t_i}, y_{t_{i-1}}).The conditional distribution of \log(\frac{s_i}{s_{i-1}})$ given v_i is normal with mean and variance

$$\{\theta_1 + \theta_2 v_i\}(T/n) + \rho \frac{\sqrt{(T/n)v_i}}{\sqrt{\vartheta_i}} \{\log(\frac{v_i}{v_{i-1}}) - \iota_i\},$$

and

$$(1 - \rho^2)T\overline{v},$$

respectively. We consider this volatility process in the following example.

Example 2.3. This example is taken from Andersen *et al.* (2002). The price equation is given by

$$d\log(s_t) = \theta_1 dt + \sigma_t dB_t,$$

where the log-volatility process $\alpha_t = \log(\sigma_t^2)$ follows the Ornstein-Uhlenbeck process

$$d\alpha_t = -\theta_4(\alpha_t - \theta_5)dt + \theta_3 dw_t.$$

The parameters estimates are

$$\theta_1 = 0.03, \;\; \theta_3 = 0.125, \;\; \theta_4 = 1.389, \;\; \theta_5 = 0, \text{ and } \; \rho = -0.62.$$

The values of $E(\overline{v}|r)$ based on $r = 0.55$, 0.78 and 1.035 are 0.022, 0.0339 and 0.0244, respectively.

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