Journal of Applied Finance & Banking, vol.1, no.2, 2011, 173-187
ISSN: 1792-6580 (print version), 1792-6599 (online)
International Scientific Press, 2011

# Portfolio Theory and Cone Optimization

**Marcus Davidsson**[1]

## Abstract

This paper will discuss portfolio optimization, Quadratic Programming (QP) and Second Order Cone Programming (SOCP). We will use simulated and empirical data to compare the two optimization routines. Daily data for SP500 stocks from 2005 to 2010 was used to show that a 20-days rebalanced portfolio strategy with an expected portfolio return of 60 percent of the maximum expected return for all stocks produced an 8.4 percent return premium on an annual basis if we used QP and 11.2 percent return premium on an annual basis if we used SOCP.

**JEL classification:** G11

**Keywords:** QP, SOCP, optimization, portfolio theory, cone

## 1 Introduction and Literature Review

Markowitz [7], Sharpe [9], Ross [8], Black and Litterman [1], Fama and French [5] and Carhart [3] have all made significant contributions to portfolio

[1] Department of Economics, Newcastle University Business School, 5 Barrack Road, Newcastle upon Tyne, NE1 4SE, UK, e-mail: davidsson_marcus@hotmail.com

Article Info: *Revised* : September 8, 2011. *Published online* : September 30, 2011

theory. The main objective for portfolio diversification is to minimize portfolio variance. In the basic model portfolio variance is a function of the return volatility for each security in the portfolio and the cross correlation of returns. Since cross correlation can be negative return variance can be cancelled out. However, the same idea can also be applied to highly positive correlated stock return portfolio by artificially creating negative cross correlation in return by short selling. Portfolio variance can also be thought as the amount of return noise around the portfolios expected return. Diversification can to a large extend eliminate such return noise.

Markowitz [7] mainly looks at diversification from an asset class perspective where an investor that spreads his risk between different asset classes will achieve a greater “diversification” and hence a smoother equity curve. Brinson, Hood and Beebower [2] found that asset class allocation (compared to market timing and stock picking) can explain on average 93.6 per cent of the variation in total return. It is also interesting to note that the bond returns in general tend to be the only return that will not become negative during a market crash [6]. This means that bonds provides a good source of diversification due to return stability especially when markets has become more positive cross correlated during the last thirty years and even though the return on “risk free” government bonds has steadily been declining for the last 40 years. The Capital Asset Pricing Model (CAPM) which was introduced by Sharpe [9] points out that market risk also plays an important role for the smoothness of the equity curve. A portfolio with a large beta (i.e., highly sensitive to changes in market returns) will have more risk than a portfolio with a zero beta. An investor can reduce such market risk by balancing long and short positions. Market risk plays an important role when it comes to investing in financial markets because market returns accounts for a large fraction of stock returns [4]. The impact of a market risk should not be underestimated. Taleb [10] explains that in general people tend to overprice equity and under price options due to the abundance of volatility in financial markets.

Ross [8] introduced the so called Arbitrage Pricing Theory which illustrates that asset returns can be modelled as linear functions of various factor indices. Black and Litterman [1] introduced the so called Black-Litterman model which starts by assuming that the benchmark index is mean-variance efficient and from such assumption derive the expected return of the benchmark portfolio. Fama and French [5] introduced the three factor models which includes beta, book-to-market-ratio and stocks size which they claim will reduced return noise even further. Finally Carhart [3] extend such a three-factor model to a four-factor model which also includes a momentum component which explains even more of the return variance.

## 2 Theoretical Modelling and Simulation

The traditional portfolio optimization model introduced by Markowitz is solved by using Quadratic Programming (QP). The objective is to minimize portfolio risk for a given portfolio expected return $pr$. The objective function with its corresponding constraint for QP can be expressed as follows

$$\min[w^T Q w] \quad \text{subject to} \quad w^T \cdot ER \geq pr, \quad S^T w \leq 1, \quad 1 \geq w[i] \geq -1$$

where $w$ is a column vector containing the portfolio weights, $T$ is the transpose notation, i.e., convert a column vector to a row vector, $Q$ is the covariance matrix, $ER$ is a column vector with expected returns and $S$ is a column vector of 1's. Note that $w^T Q w$ represent the portfolio variance and $w^T \cdot ER$ represent the expected portfolio return. The portfolio variance is simply given by the sum of all elements in the weighted covariance matrix. Such sum includes quadratic terms i.e. $w[2]^2 \cdot \text{cov}[2,2]$ hence QP is appropriate to use. The problem with the above model is that the performance of QP decreases when we have a large portfolio. It requires a long time to calculate the portfolio weights for such portfolio.

One way to get around that is to use Second Order Cone Programming (SOCP). In order to use a quadratic objective function $w^T Qw$ in SOCP we have to introduce a new variable let say $r$. The objective then becomes to minimize $r$ given the constraint that the quadratic objective function must be smaller than the square of $r$ which means that. The constraint also needs to be converted to SOCP form. This is done by noting that the covariance matrix $Q$ can be written as $Q = RR^T$ where $R$ is the Cholesky Decomposition matrix and that

$$norm(x) = \sqrt{x^T x} = \sqrt{[2,\ 3]^T [2,3]} = \sqrt{2^2 + 3^2} = \sqrt{13}\ .$$

The constraint can be rewrite as:

$$w^T Qw < r^2 \Rightarrow w^T RR^T w < r^2 \Rightarrow (Rw)^T Rw < r^2 \Rightarrow \sqrt{(Rw)^T Rw} < \sqrt{r^2} \Rightarrow norm(Rw) < r$$

Again the objective is to minimize portfolio risk for a given portfolio expected return $pr$. The objective function and its constraint for SOCP can therefore be expressed as follows:

$$\min[r] \quad \text{subject to} \quad norm(R,w) < r, \quad w^T \cdot ER \geq pr, \ S^T w \leq 1, \quad 1 \geq w[i] \geq 0$$

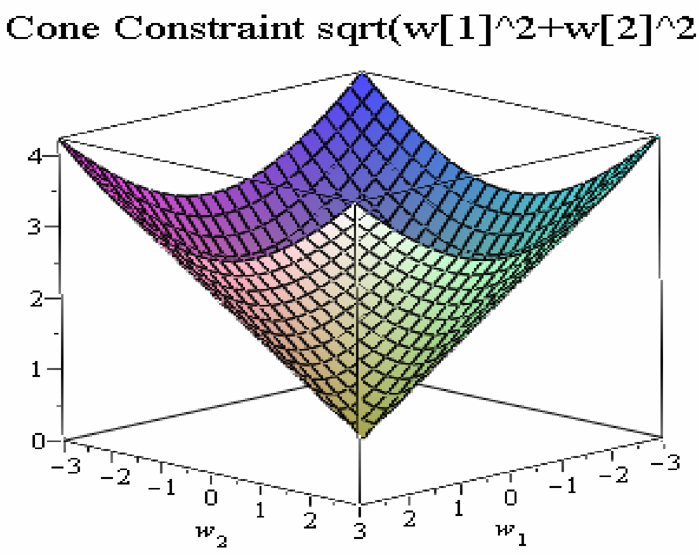



Figure 1: Cone Constraint and Weight Vector

Note that we are not allowing for any short positions since $w[i] \geq 0$ and the values for $norm()$ need to be non-negative. The benefit of using a cone constraint like $\sqrt{\left(w[1]^2 + w[2]^2\right)^2} < r$ compared to a non-cone constraint like $\sqrt{\left(w[1]^2 - w[2]^2\right)^2} < r$ is that the cone constraint is convex. For a convex problem any locally optimal point is a globally optimal hence the optimization becomes fast. SOCP also works for Linear Programming (LP) i.e. a linear objective functions and constraints. Then the weight vector can for example be constrained to be inside a cone as seen in Figure 1, i.e.,

$$w \in C \Rightarrow \sqrt{w[2]^2 + \cdots + w[n]^2} < w[1] \Rightarrow norm(w[2]^2 + \cdots + w[n]^2) < w[1]$$

and $w[1] \geq 0$

or you can assume for example that the norm of two weights is never larger than 0.5, i.e.,

$$\sqrt{w[1]^2 + \cdots + w[2]^2} < 0.5\ , \quad \sqrt{w[2]^2 + \cdots + w[3]^2} < 0.5\ , \quad \sqrt{w[3]^2 + \cdots + w[1]^2} < 0.5$$

We can now start by simulating some cross correlated random walks by again using Cholesky Decomposition as seen in Figure 2.

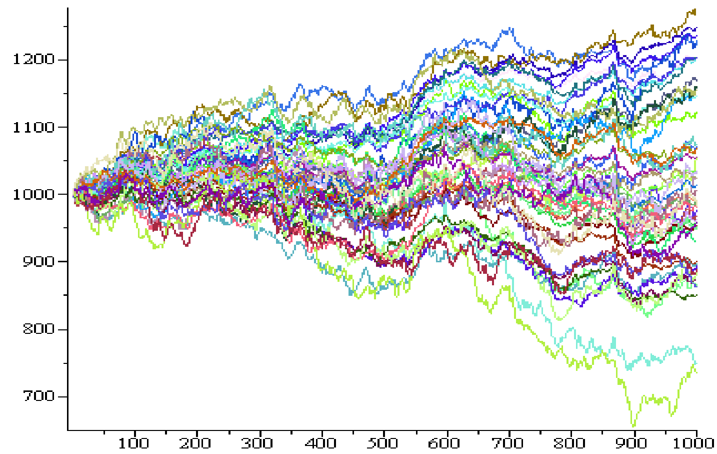



Figure 2: Simulated Random Walks

In order to get some return with negative cross correlation and some with positive cross correlation we multiply half of the returns with -1 and half of the returns with +1. We simulate in total 50 stocks where each stock has 1000 observations. The standard deviation of return and expected return are randomly assigned. We further assume that the standard deviation and the expected return remain constant over time. We then run the QP and SOQP algorithms to get the return distributions and the historical or optimized equity curves as seen in Figure 3. The weights can be found in Appendix 1.

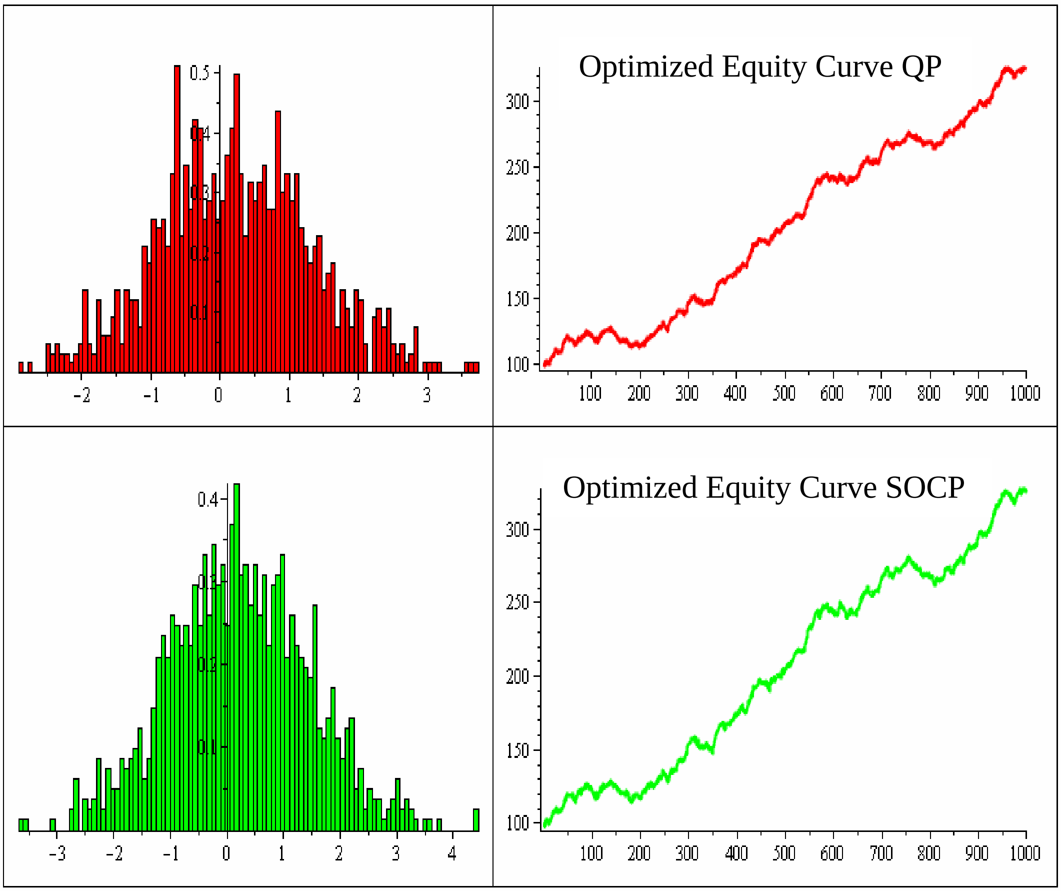



Figure 3: Simulated Data with QP and SOCP-1

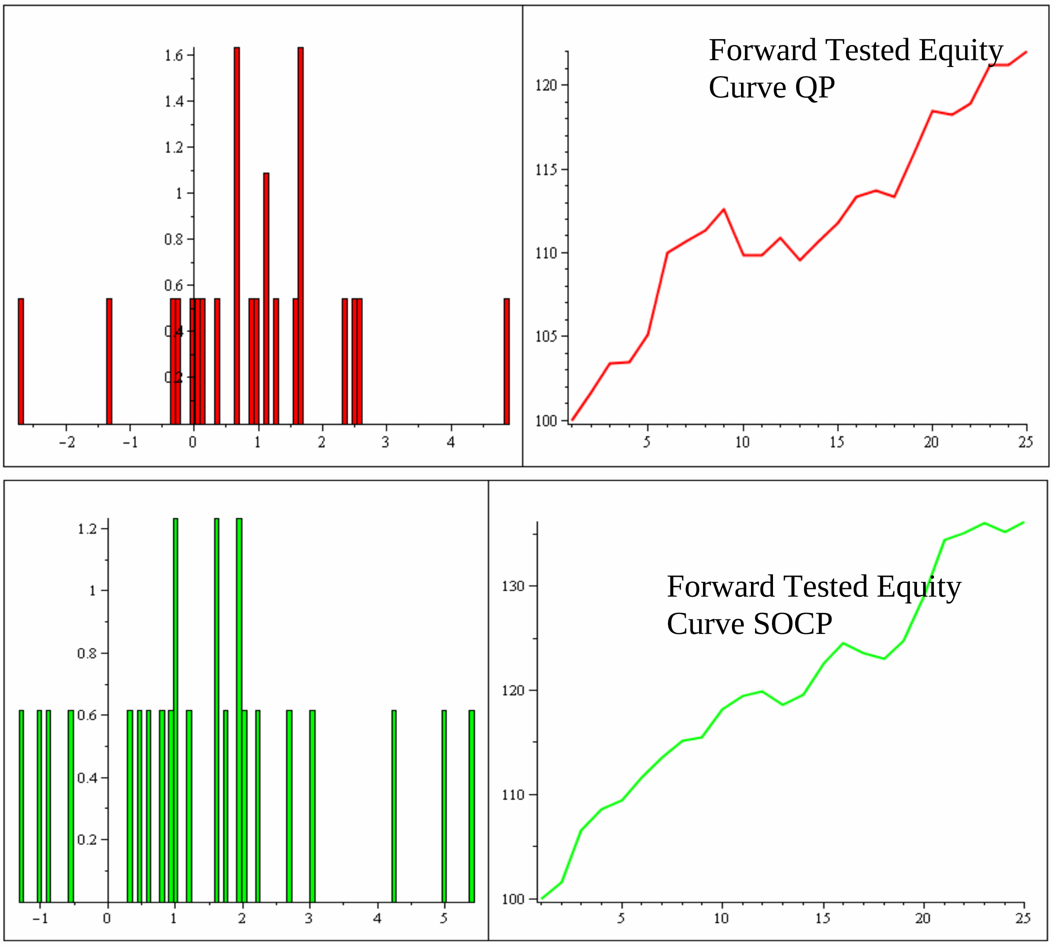



Figure 4: Simulated Data with QP and SOCP-2

We can see that both QP and SOCP produce optimized equity curves that are smooth and upward sloping hence the risk adjusted returns is large. Further, the threshold for the expected return both for the simulated and the empirical portfolio was calculated as the $0.6 \cdot \text{maximum}$ expected return for all stocks. Hence, our portfolio will have an expected return that is sixty percent of the expected return for the stock with the largest expected return. It is also important to note that the optimized equity curves can look very different from a forward testing equity return i.e. forward testing cumulative return. We can now forward test our two portfolio algorithms. We assumed that the portfolio is rebalanced every 20 days and only the data for the last 20 days is used in the portfolio optimization algorithm. We can see in Figure 4 that the forward testing equity curves does contain more volatility.

## 3 Empirical Analysis

We can now test our QP portfolio algorithms on some empirical data extracted from datastream which consists of daily data for 100 SP500 stocks from 2005 to 2010.

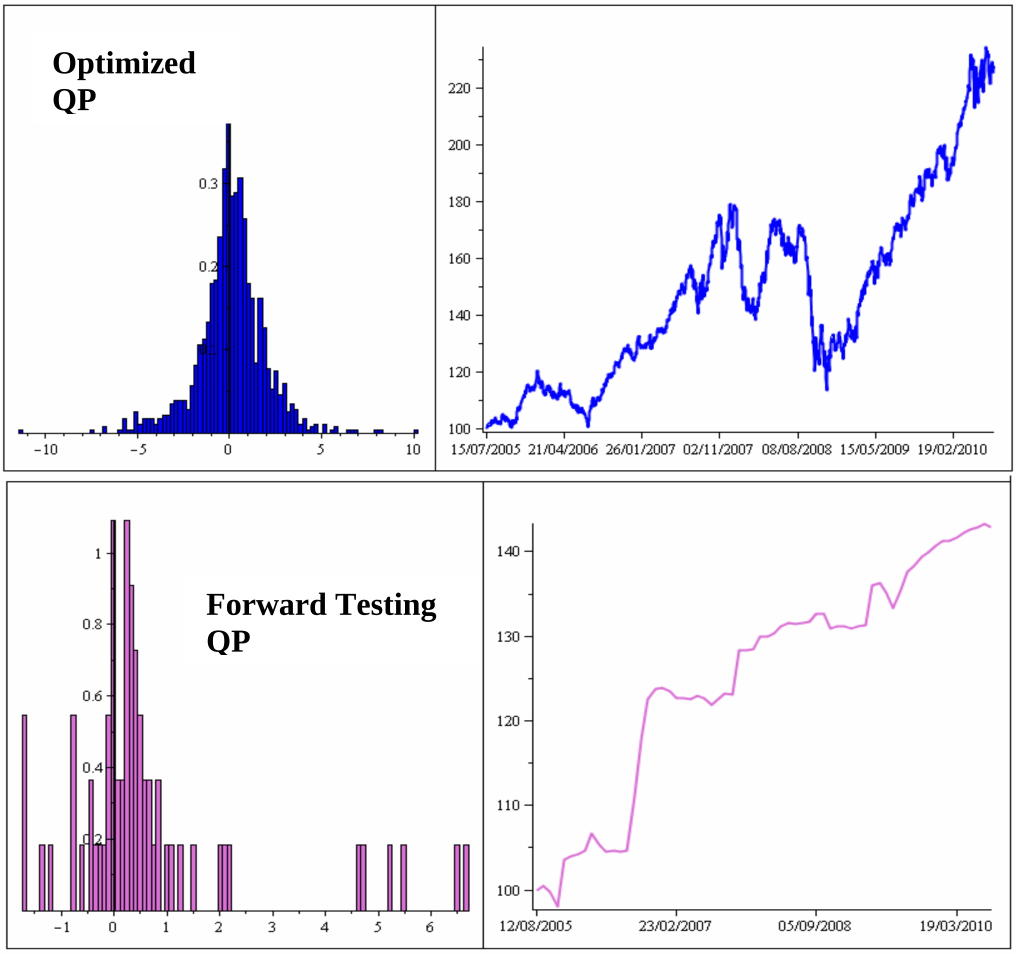



Figure 5: Empirical Data and QP

We can in Figure 5 see the optimized and forward testing return distribution with its corresponding equity curves for the empirical dataset. We can see that the forward testing equity curve is upward sloping and rather smooth. From the period 2005 to 2010 the QP algorithm produced a return of approximately 50 percent or 8.4 percent on an annual basis. We can also see that the forward testing return distribution is asymmetrical i.e. positive returns are both larger and more frequent than negative returns. See Appendix 2 for QP allocation matrix.

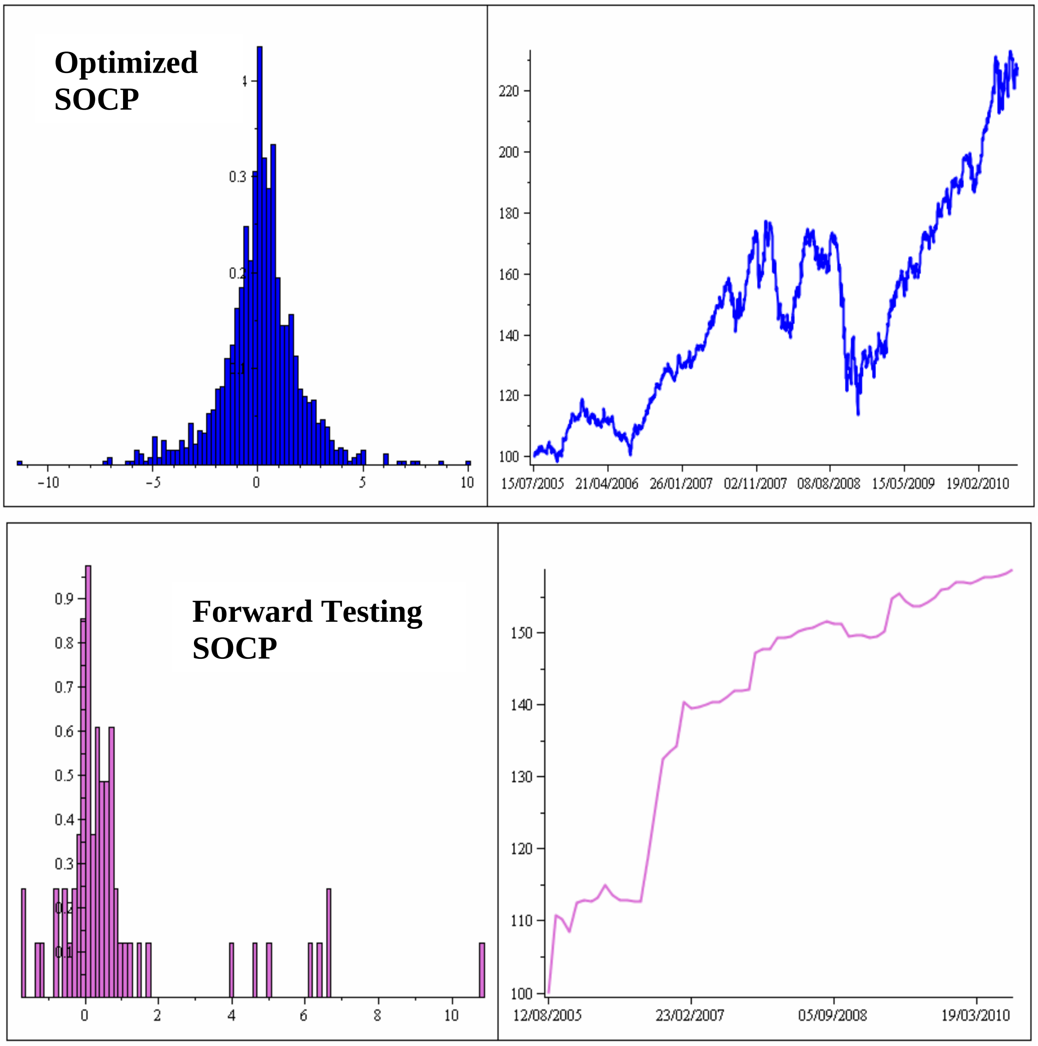



Figure 6: Empirical Data and SOCP

We can now test our SOCP portfolio algorithms on the same empirical data. We can in Figure 6 see the optimized and forward testing return distribution with its corresponding equity curves for the empirical dataset. We see that the optimized equity curve looks similar to the QP example. We can also see that the forward testing equity curve is upward sloping and rather smooth. From the period 2005 to 2010 the SOCP algorithm produced a return of approximately seventy percent or 11.2 percent on an annual basis. We can again see that the forward testing return distribution is asymmetrical i.e. positive returns are both larger and more frequent than negative returns. See Appendix 3 for SOCP allocation matrix.

## 4 Conclusion and Final Discussion

Portfolio optimization problems have traditionally been solved by using Quadratic Programming (QP). However, such optimization method has always had performance constraints attached to it. In the 1980's second order cone programming (SOCP) was introduced to the world. SOCP is a generalisation of Linear Programming (LP). LP has always played an important role in economics for its efficiency and practical relevance. Today SOCP has become an important tool for financial optimization due to its powerful nature. Investment firms need to optimize large portfolios and take into consideration a large universe of assets in order to find the most optimal portfolios. However, the basic theory behind cone optimization is highly mathematical dense so it can be hard to understand some of the basic ideas. This paper aims at being a simple introduction to cone optimization. We have also in this paper shown that a 20-day rebalanced portfolio strategy with an expected portfolio return of 60 percent of the maximum expected return for all stocks produced an 8.4 percent return premium on an annual basis if we used QP and 11.2 percent return premium on an annual basis if we used SOCP. The annual return was calculated as $\mathrm{PV}[t](1+r)^n = \mathrm{PV}[t+1]$, where $r$ is the annual percentage return. For simplicity we assumed a transaction cost equal to zero.

It is also worth noting that theoretically the QP and SOCP algorithm should produce the same asset allocations. However, in this case it clearly did not. Another drawdown with the simple examples introduced in this paper must be the sample size. We have only considered an asset universe of 100 stocks hence many of the benefits SOCP ie large scale optimization was lost. The same examples could be tested with a much more powerful solver to investigate assets universes of a couple of thousands of stocks. If a more powerful solver were introduced then transaction costs would be interesting to include in the modelling framework. It would also be interesting to look at portfolio rebalancing frequency and how the frequency relates to the expected return of the portfolio.

## References

[1] F. Black and R. Litterman, Global Portfolio Optimization, *Financial Analysts Journal*, **48**(5), (1992), 28-43.

[2] G. Brinson, R. Hood and G. Beebower, Determinants of Portfolio Performance, *Financial Analysts Journal*, **42**(4), (1986), 39-44.

[3] M. Carhart, On Persistence in Mutual Fund Performance, *Journal of Finance*, **52**(1), (1997), 57-82.

[4] E. Fama and K. French, The Cross-Section of Expected Stock Returns, *Journal of Finance*, **47**(2), (1992), 427-465.

[5] E. Fama and K. French, K Common Risk Factors in the Returns on Stocks and Bonds, *Journal of Financial Economics,* **33**(1), (1993), 3-56.

[6] F. Longin and B. Solnik, Is the correlation in international equity returns constant : 1960-1990?, *Journal of International Money and Finance*, **14**, (1995), 3-26.

[7] H. Markowitz, *Portfolio Selection: Efficient Diversification of Investment*, New York: John Wiley & Sons, 1959.

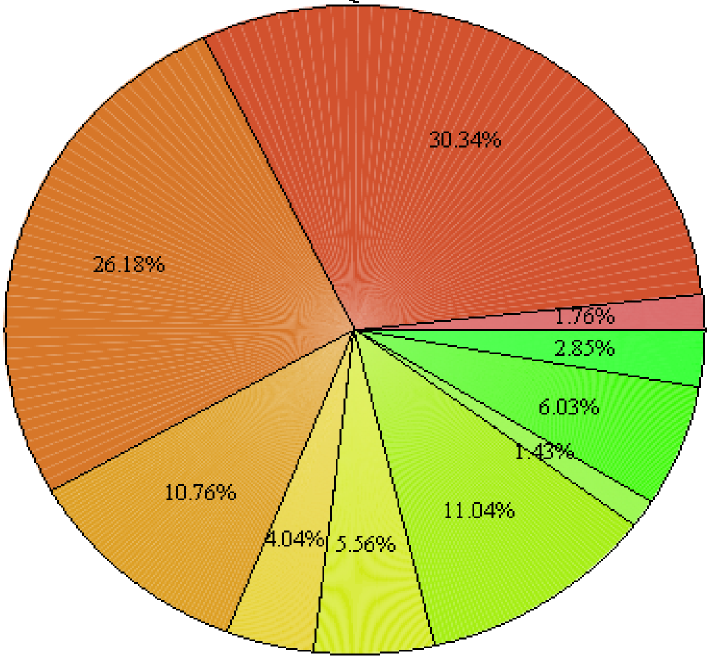
[8] S. Ross, The arbitrage theory of capital asset pricing, *Journal of Economic Theory,* **13**(3), (1976), 341-360.

[9] W. Sharpe, Capital Asset Prices - A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance*, **19**(3), (1964), 425-442.

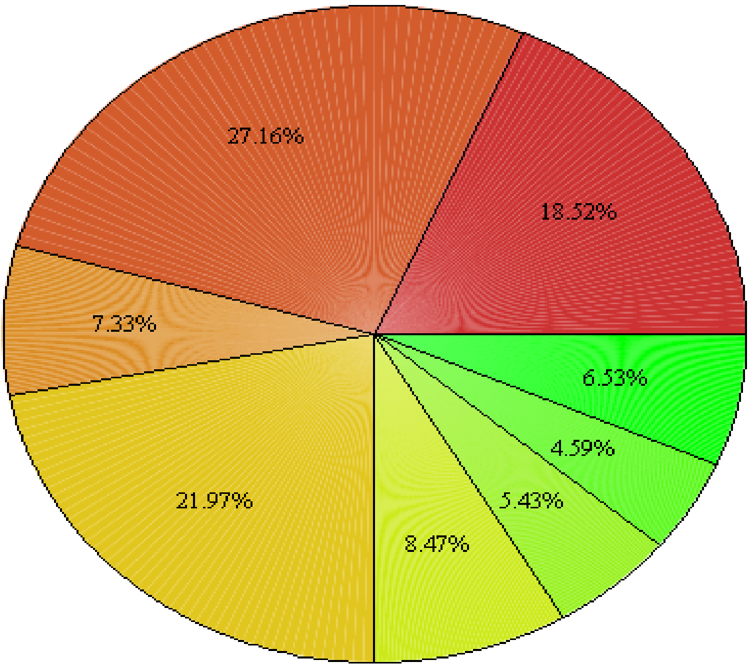
[10] N. Taleb, *The Black Swan: The Impact of the Highly Improbable*, Penguin Books, Limited (UK), 2008.

## Appendix 1 Allocation Simulated Data

Simulation QP Allocation




Simulation SOCP Allocation



# Appendix 2 QP Allocation Empirical Data

| | | |
|---|---|---|
| 0 | 5.66 | [w[22] = .171, w[26] = .199, w[29] = .630] |
| 20 | 0.113 | [w[9] = .109, w[11] = .317, w[43] = .424, w[68] = .116, w[92] = .347e-1] |
| 40 | 9.41 | [w[63] = .443, w[92] = .557] |
| 60 | 26.3 | [w[26] = .378, w[46] = .281, w[92] = .341] |
| 80 | 6.39E-02 | [w[17] = .319, w[18] = .422e-1, w[22] = .187, w[37] = .598e-1, w[64] = .175, w[68] = .216] |
| 100 | 1.61 | [w[8] = .387e-1, w[18] = .153, w[88] = .785, w[100] = .233e-1] |
| 120 | 4.63 | [w[17] = .262, w[50] = .470, w[92] = .268] |
| 140 | 13.5 | [w[26] = .462e-1, w[64] = .467, w[92] = .486] |
| 160 | 8.17 | [w[68] = .578, w[92] = .422] |
| 180 | 10.1 | [w[17] = .558, w[92] = .442] |
| 200 | 3.32E-02 | [w[2] = .185, w[7] = .370e-1, w[9] = .693e-1, w[40] = .691e-1 w[58] = .364, w[80] = .632e-1, w[93] = .694e-1, w[94] = .140] |
| 220 | 4.89E-02 | [w[5] = .215, w[18] = .124, w[36] = .673e-1, w[62] = .302, w[80] = .291] |
| 240 | 0.176 | [w[2] = .246, w[64] = .129, w[73] = .139, w[88] = .558e-1, w[91] = .424] |
| 260 | 42.3 | [w[26] = .536, w[89] = .464] |
| 280 | 11.4 | [w[26] = .531, w[45] = .469] |
| 300 | 15.1 | [w[26] = .322, w[42] = .379, w[92] = .299] |
| 320 | 2.76 | [w[17] = .958, w[85] = .150e-1, w[92] = .266e-1] |
| 340 | 27.6 | [w[17] = .558, w[26] = .442] |
| 360 | 14.7 | [w[14] = .294, w[26] = .528, w[97] = .178] |
| 380 | 18.6 | [w[45] = .524, w[92] = .476] |
| 400 | 7.39E-02 | [w[10] = .719e-1, w[12] = .756e-1, w[36] = .176, w[42] = .140 w[47] = .114, w[78] = .206, w[80] = .317e-1, w[84] = .768e-1, w[89] = .990e-1] |
| 420 | 0.331 | [w[6] = .151, w[24] = .289, w[37] = .355e-1, w[54] = .309, w[60] = .216] |
| 440 | 8.77 | [w[17] = .642, w[26] = .358] |
| 460 | 4.64 | [w[22] = .107, w[26] = .293, w[37] = .600] |
| 480 | 2.56 | [w[22] = .634e-1, w[34] = .203, w[37] = .536, w[92] = .197] |
| 500 | 6.84 | [w[36] = .231, w[37] = .549, w[92] = .221] |
| 520 | 0.118 | [w[33] = .108, w[38] = .347, w[61] = .209, w[67] = .188] |
| 540 | 0.509 | [w[22] = .155, w[58] = .211, w[61] = .142, w[85] = .164, w[88] = .319] |
| 560 | 21.4 | [w[26] = .412, w[37] = .574, w[92] = .142e-1] |
| 580 | 2.15 | [w[34] = .102, w[36] = .173, w[37] = .296, w[40] = .336 w[58] = .212e-1, w[61] = .120e-1, w[75] = .602e-1] |
| 600 | 0.459 | [w[7] = .173, w[58] = .742, w[73] = .658e-1, w[88] = .199e-1] |
| 620 | 11.1 | [w[37] = .548, w[75] = .306, w[92] = .146] |
| 640 | 0.377 | [w[29] = .329e-1, w[35] = .338, w[75] = .940e-1, w[85] = .340] |
| 660 | 0.675 | [w[17] = .251, w[47] = .172, w[78] = .510, w[93] = .658e-1] |

# Appendix 3 SOCP Allocation Empirical Data

| | | |
|---|---|---|
| 0 | 193 | [w[26] = .159, w[29] = .841] |
| 20 | 0.241 | [w[3] = .398e-1, w[11] = .877e-1, w[43] = .589, w[85] = .245 w[90] = .236e-1, w[92] = .143e-1] |
| 40 | 292 | [w[63] = .443, w[92] = .557] |
| 60 | 1.80E+03 | [w[26] = .212, w[37] = .155, w[92] = .633] |
| 80 | 0.205 | [w[22] = .542, w[64] = .520e-1, w[68] = .406] |
| 100 | 6.41 | [w[18] = .160, w[42] = .451e-1, w[85] = .161e-1, w[88] = .777] |
| 120 | 208 | [w[50] = .734, w[92] = .266] |
| 140 | 772 | [w[26] = .203e-1, w[64] = .491, w[92] = .488] |
| 160 | 320 | [w[68] = .578, w[92] = .422] |
| 180 | 480 | [w[17] = .558, w[92] = .442] |
| 200 | 4.48E-02 | [w[7] = .129e-1, w[9] = .259e-1, w[44] = .402e-1, w[58] = .243, w[61] = .297e-1 w[80] = .782e-1, w[85] = .177e-1, w[93] = .138, w[94] = .405] |
| 220 | 0.131 | [w[36] = .669e-1, w[62] = .896e-1, w[80] = .784, w[94] = .592e-1] |
| 240 | 1.71 | [w[21] = .283, w[88] = .259, w[91] = .372, w[99] = .855e-1] |
| 260 | 6.55E+03 | [w[26] = .519, w[37] = .481] |
| 280 | 546 | [w[26] = .531, w[45] = .469] |
| 300 | 906 | [w[26] = .507, w[42] = .297, w[92] = .196] |
| 320 | 51.7 | [w[17] = .983, w[92] = .168e-1] |
| 340 | 3.90E+03 | [w[17] = .381, w[26] = .371, w[92] = .248] |
| 360 | 673 | [w[26] = .520, w[97] = .480] |
| 380 | 1.28E+03 | [w[45] = .524, w[92] = .476] |
| 400 | 0.314 | [w[32] = .928e-1, w[36] = .286, w[42] = .197, w[84] = .222e-1 w[89] = .469e-1, w[99] = .355] |
| 420 | 3.58 | [w[60] = .439, w[99] = .561] |
| 440 | 319 | [w[17] = .642, w[26] = .358] |
| 460 | 168 | [w[22] = .379, w[26] = .270, w[37] = .350] |
| 480 | 25.5 | [w[22] = .945e-1, w[37] = .796, w[92] = .109] |
| 500 | 304 | [w[37] = .840, w[92] = .160] |
| 520 | 0.495 | [w[61] = .342, w[67] = .192, w[96] = .261] |
| 540 | 5.29 | [w[61] = .729, w[88] = .235, w[98] = .354e-1] |
| 560 | 2.84E+03 | [w[26] = .381, w[37] = .548, w[92] = .713e-1] |
| 580 | 47.9 | [w[37] = .372, w[40] = .628] |
| 600 | 5.29 | [w[58] = .909, w[93] = .909e-1] |
| 620 | 850 | [w[37] = .950, w[92] = .503e-1] |
| 640 | 2.46 | [w[29] = .546e-1, w[35] = .592e-1, w[48] = .169e-1 w[75] = .321e-1, w[76] = .986e-1, w[85] = .488] |
| 660 | 3.56 | [w[17] = .344e-1, w[47] = .383e-1, w[93] = .236, w[99] = .691] |