

Downside Risk and Average Returns: A Condensed Cross-Sectional Analysis Using Semi-Continuous Complex Wavelet Frames

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Abstract

We study the cross-sectional pricing of systematic downside risk within a semi-continuous complex wavelet frame. Unlike conventional downside asset-pricing models built on the real-valued Discrete Wavelet Transform (DWT/MODWT) which are tied to dyadic scales and cannot disentangle amplitude from phase we adopt a redundant, translation-invariant complex representation whose coefficients are complex-valued. This lets us extract, at every investment horizon, the instantaneous amplitude (synchronous co-movement) and the instantaneous phase (lead-lag, delayed transmission) of asset returns. Splitting each complex coefficient into its real and imaginary parts, we build multiscale downside beta and downside co-skewness measures and price them across 30 U.S. industry portfolios (1926–2025). Classical time-domain and MODWT specifications leave downside risk essentially unpriced ($R^2 < 3\%$). The complex representation reverses this: a four-factor model combining synchronous and phase-shifted downside beta and co-skewness explains up to $R^2 \approx 60\%$ of the cross-section at business-cycle horizons. The results show that amplitude and phase information must be considered jointly when assessing systematic downside risk.

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Keywords: Downside beta; Downside co-skewness; Continuous Wavelet Transform; Semi-continuous frame; Complex wavelet coefficients; Cross-sectional asset pricing; Time–frequency analysis.

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1. Introduction

Since Markowitz (1952), Sharpe (1964), Lintner (1965) and Mossin (1966) formalized the Capital Asset Pricing Model (CAPM), systematic risk has usually been quantified by the market beta, under the assumptions of jointly normal returns and a time-invariant covariance structure. Decades of evidence contradict these assumptions: returns exhibit heavy tails, excess kurtosis, asymmetry, volatility clustering and horizon-dependent dependence (Mandelbrot, 1963; Fama, 1965; Cont., 2001; Ane and Geman, 2000). Because investors are more sensitive to losses than to gains (Tversky and Kahneman, 1992), downside fluctuations carry more economic information than symmetric volatility, which following safety-first reasoning (Roy, 1952) and lower-partial-moment theory (Fishburn, 1977; Hogan and Warren, 1974) motivated *downside* risk measures. Downside beta (Bawa and Lindenberg, 1977; Harlow, 1991; Ang et al., 2006; Estrada, 2002; Post and van Vliet, 2006) captures sensitivity to adverse markets only, while downside co-skewness, building on skewness preference (Kraus and Litzenberger, 1976), adds tail asymmetry (Harvey and Siddique, 2000; Galagedera and Brooks, 2007).

These measures remain time-domain constructs and ignore the multi-horizon, time-varying character of financial interaction (Merton, 1973; Fama and French, 1992). Wavelet methods especially the Maximal Overlap Discrete Wavelet Transform (MODWT) decompose returns by investment horizon (Percival and Walden, 2000; Gençay et al., 2002; Galagedera et al., 2008; Rua, 2010). But the MODWT has four structural limits for downside pricing: (i) it yields only real-valued coefficients, so no phase information; (ii) its dyadic scales impose economically inflexible frequency bands; (iii) its frequency resolution is coarse; and (iv) lacking phase, it cannot identify synchronization, lead-lag structure, or delayed shock propagation (Torrence and Compo, 1998; Grinsted et al., 2004).

To overcome these limits, we use a *semi-continuous complex wavelet* representation built on the Morlet analyzing wavelet. It is redundant yet stable, is translation-invariant in magnitude, allows arbitrary (continuous) time localisation, and—being complex-valued—delivers instantaneous amplitude and phase at every horizon. Each downside measure is then decomposed into a real component (synchronous, instantaneous co-movement with the market) and an imaginary component (phase-shifted, delayed or anticipatory transmission). Feeding 30 U.S. industry portfolios into a hierarchy of cross-sectional models, we find that downside systematic risk is highly heterogeneous across scales, is poorly explained by real-valued or time-domain models, and is best captured by a four-factor complex specification that jointly prices synchronous and phase-shifted downside beta and co-skewness.

2. Downside Risk Measures

Under the CAPM, $\mathbb{E}(R_i) - R_f = \beta_i [\mathbb{E}(R_m) - R_f]$ with $\beta_i = \text{Cov}(R_i, R_m) / \text{Var}(R_m)$, where R_f is the risk-free rate and R_m the market return. Because the CAPM treats gains and losses symmetrically and assumes normality, safety-first investors (Roy, 1952) instead care about returns below a reference. Downside beta isolates co-movement in down markets:

$$\beta_i^d = \frac{\mathbb{E}[(R_i - R_f) \min(R_m - R_f, 0)]}{\mathbb{E}[\min(R_m - R_f, 0)^2]}.$$

Downside co-skewness captures the response of an asset's excess return to large market downturns (higher-order, left-tail dependence):

$$\gamma_i^d = \frac{\mathbb{E}[(R_i - R_f) \min(R_m - R_f, 0)^2]}{\mathbb{E}[\min(R_m - R_f, 0)^3]}.$$

A high γ_i^d signals amplified crash risk; a low value indicates relative resilience to stress. Cross-sectional pricing of a factor is assessed with the Fama–MacBeth regression $\bar{R}_i = \lambda_0 + \lambda_1 \hat{\theta}_i + \eta_i$, where a positive, significant λ_1 is the premium paid for exposure to the risk measure $\hat{\theta}_i$.

3. Complex Wavelet Theory and the Morlet Transform

3.1 The Continuous Complex Wavelet Transform

Let $f \in L^2(\mathbb{R})$ and let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be an admissible wavelet satisfying $\hat{\psi}(0) = \int_{\mathbb{R}} \psi(t) dt = 0$ and the admissibility condition $C_\psi = \int_{\mathbb{R}} |\hat{\psi}(w)|^2 / |w| dw < \infty$ (Mallat, 1998; Daubechies, 1988; Antoine et al., 2004). The continuous wavelet transforms (CWT) of f at scale $s \in \mathbb{R}_+^*$ and position $b \in \mathbb{R}$ is

$$W_f(s, b) = \int_{\mathbb{R}} \psi_{s,b}^*(t) f(t) dt = \int_{\mathbb{R}} \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-b}{s}\right) f(t) dt = (f \star \psi_s)(b),$$

where ψ^* is the complex conjugate and f is reconstructed from W_f through the standard resolution of identity. When the analysing wavelet ψ is *complex-valued*, each coefficient is complex and can be written in polar form

$$W_f(s, b) = |W_f(s, b)| e^{i\varphi(s,b)}, \quad \varphi(s, b) = \arg W_f(s, b),$$

so, the transform separates, at every scale, the *instantaneous amplitude* $|W_f(s, b)|$ (the local energy, or envelope) from the *instantaneous phase* $\varphi(s, b)$. Real-valued

wavelets collapse this information and cannot recover the phase, hence cannot identify synchronisation, lead–lag structure or delayed propagation—which is precisely why a complex analysing wavelet is required here (Morlet et al., 1982; Gençay et al., 2001; Rua, 2010).

3.2 The Morlet Wavelet

The canonical complex analyzing wavelet is the Morlet wavelet (Morlet et al., 1982; Torrence and Compo, 1998; Grinsted et al., 2004), a plane wave modulated by a Gaussian envelope,

$$\psi(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-t^2/2}, \quad \hat{\psi}(w) = \pi^{-1/4} \sqrt{2} e^{-(w-\omega_0)^2/2},$$

where ω_0 is the central (dimensionless) frequency; the usual choice $\omega_0 = 6$ makes the wavelet effectively admissible (the negative-frequency and zero-mean corrections are numerically negligible) and gives a near-optimal balance between time and frequency resolution. Because $\hat{\psi}(w) \approx 0$ for $w < 0$, the Morlet wavelet is *analytic* (progressive): it is the archetypal wavelet for extracting amplitude and phase jointly, attaining the Heisenberg time–frequency localisation bound through its Gaussian envelope. Its complex output makes it the natural tool for measuring both the strength and the timing of an asset’s response to market movements.

3.3 Semi-Continuous Representation

For the empirical analysis we keep the translation parameter b continuous and discretize only the scale on the a -adic grid $s_j = a^j$, $j = 1, 2, \dots, J$ (dyadic, $a = 2$), yielding a redundant, translation-invariant *semi-continuous* representation of the return series. This retains the fine time–frequency localisation and the complex (amplitude–phase) structure of the CWT while indexing the coefficients by economically meaningful investment horizons j . Denoting by $W_i(j, t)$ the resulting complex wavelet coefficient of asset i at scale j and time t , the whole multiscale downside-risk analysis below is built on the real and imaginary parts of $W_i(j, t)$.

4. Downside Risk in the Complex Wavelet Domain

The semi-continuous complex wavelet representation produces coefficients that split into real and imaginary parts,

$$W_i(j, t) = W_i^{\Re}(j, t) + i W_i^{\Im}(j, t),$$

where the real part $W_i^{\Re}$ carries the in-phase (synchronous) reaction of asset i and the imaginary part $W_i^{\Im}$ the out-of-phase (lead–lag) reaction. This in-phase/quadrature interpretation of complex coefficients is well established in signal processing and time–frequency finance (Gençay et al., 2001; Rua, 2010; Morlet et

al., 1982). The real component measures instantaneous transfer of market information into prices (microstructure at small scales, macro fundamentals at large scales); the imaginary component encodes delayed propagation from illiquidity, information lags and gradual rebalancing.

Denoting by $w_{it,j}^{\Re}, w_{it,j}^{\Im}$ the real/imaginary wavelet coefficients of asset i (and $w_{mt,j}, w_{Rf,t}$ those of the market and risk-free rate) at scale j , the *real* and *imaginary* wavelet downside betas are

$$\beta_{i,j}^{d,\Re} = \frac{\mathbb{E} \left[\left(w_{it,j}^{\Re} - \mathbb{E}(w_{it,j}^{\Re}) \right) \min \left(w_{mt,j}^{\Re} - w_{Rf,t}^{\Re}, 0 \right) \right]}{\mathbb{E} \left[\min \left(w_{mt,j}^{\Re} - w_{Rf,t}^{\Re}, 0 \right)^2 \right]},$$

$$\beta_{i,j}^{d,\Im} = \frac{\mathbb{E} \left[\left(w_{it,j}^{\Im} - \mathbb{E}(w_{it,j}^{\Im}) \right) \min \left(w_{mt,j}^{\Im} - w_{Rf,t}^{\Im}, 0 \right) \right]}{\mathbb{E} \left[\min \left(w_{mt,j}^{\Im} - w_{Rf,t}^{\Im}, 0 \right)^2 \right]}.$$

The real downside beta captures shared, synchronous vulnerabilities that materialize together with market declines; the imaginary downside beta isolates the delayed, lagged or anticipatory component of downside systematic risk (a positive $\beta_{i,j}^{d,\Im}$ flags assets that fall *after* a correction, a negative one those that decline *before* a crash). The corresponding downside co-skewness measures are

$$\gamma_{i,j}^{d,\Re} = \frac{\mathbb{E} \left[\left(w_{it,j}^{\Re} - \mathbb{E}(w_{it,j}^{\Re}) \right) \min \left(w_{mt,j}^{\Re} - w_{Rf,t}^{\Re}, 0 \right)^2 \right]}{\mathbb{E} \left[\min \left(w_{mt,j}^{\Re} - w_{Rf,t}^{\Re}, 0 \right)^3 \right]},$$

$$\gamma_{i,j}^{d,\Im} = \frac{\mathbb{E} \left[\left(w_{it,j}^{\Im} - \mathbb{E}(w_{it,j}^{\Im}) \right) \min \left(w_{mt,j}^{\Im} - w_{Rf,t}^{\Im}, 0 \right)^2 \right]}{\mathbb{E} \left[\min \left(w_{mt,j}^{\Im} - w_{Rf,t}^{\Im}, 0 \right)^3 \right]}.$$

The real co-skewness measures exposure to synchronous extreme crashes; the imaginary co-skewness captures delayed or anticipatory tail risk—contagion that spreads in waves across horizons, especially relevant for emerging and illiquid sectors.

5. Data

We use daily continuously-compounded returns of 30 U.S. industry portfolios from the Kenneth R. French Data Library (Fama and French, 1993), 1 July 1926 to 29 August 2025—over a century spanning the Great Depression, Black Monday, the dot-com bubble and COVID-19. The market excess return has a positive mean (0.0427%/day) and the risk-free rate is nearly constant. Portfolio volatility is strongly heterogeneous (Servs, Coal, Autos and Steel are the most volatile; Food, Hlth and Util the least). Most portfolios show mild negative skewness, and excess kurtosis is very high throughout (e.g. Servs 445, Whlsl 165), confirming heavy tails and strong non-normality—the stylised facts that motivate tail- and asymmetry-based risk measures. For each scale j we estimate a sequence of cross-sectional regressions,

$$\bar{R}_i = \lambda_{0,j} + \lambda_{k,j} \theta_{i,j}^{(k)} + \varepsilon_i, \quad H_0: \lambda_{k,j} = 0 \text{ vs. } H_1: \lambda_{k,j} > 0,$$

progressively combining the real and imaginary downside measures. Standard errors are White heteroscedasticity-consistent; the Fisher statistic F tests joint significance.

6. Empirical Results

6.1 Time Domain and MODWT: Downside Risk Is Not Priced

Following the Fama–MacBeth procedure (Fama and MacBeth, 1973; Pettengill et al., 1995), in the aggregate time domain (Table 1) standard, downside and skewness betas all have significant intercepts but insignificant slopes ($p > 0.34$) and $R^2 < 3.2\%$: downside risk is *not* priced when horizons are aggregated. The MODWT decomposition confirms this at every scale 1–7, with R^2 falling uniformly from a maximum of only $\approx 5.3\%$ (downside beta, scale 2) to below 1% at long scales. Real-valued dyadic coefficients do not create a systematic downside pricing mechanism.

Table 1: Cross-sectional regressions in the time domain (alternative beta specifications)

Beta	λ_0 Estimate	$t(\lambda_0)$	λ_1 Estimate	$t(\lambda_1)$	F	R^2
Standard	0.039148	4.44	0.007105	0.95	0.3459	0.031779
Downside	0.040056	4.33	0.006086	0.57	0.4277	0.022611
Skewness	0.042607	4.54	0.003402	0.41	0.6856	0.005941

6.2 Real and Imaginary Components Separately

Pricing the SCF *real* component alone lifts explanatory power only at long horizons: synchronous downside risk is significantly priced at scale 6 (monthly–bimonthly), with $F = 3.57$ and $R^2 \approx 11.3\%$, but stays below 3.5% for scales 1–5. The *imaginary* component tells the opposite story: phase-shifted downside exposure is systematically priced across medium frequencies. For scales 1–5, imaginary downside beta and skewness yield an R^2 ranging from 3% to 8.4%; scale 2 (weekly-to-monthly) is the most informative, with imaginary skewness $R^2 = 8.44\%$ and imaginary beta $R^2 = 6.27\%$. Delayed downside transmission thus carries priced risk that the real component misses, but its weak standalone significance shows it should be read *together* with the synchronous effect.

6.3 Two-Factor and Four-Factor Complex Specifications

Combining real and imaginary downside betas (Table 2) sharply raises explanatory power: real beta turns negative ($\lambda_1 < 0$) and imaginary beta positive ($\lambda_2 > 0$) at short scales, implying a microstructure mechanism that *punishes* immediate downside while *compensating* delayed downside exposure. R^2 reaches $\approx 35\%$ at scales 1–2 and peaks at $R^2 = 58.6\%$ ($F > 14$) at scale 6.

Table 2: Two-factor SCF regressions using real and imaginary downside beta (selected scales)

Scale	λ_0	λ_1 (real)	λ_2 (imag.)	$t(\lambda_1)$	F	R^2
1	0.03864	−0.15270	0.16334	−3.564	7.305	0.35113
2	0.03201	−0.15665	0.17270	−3.457	7.272	0.34990
6	0.03359	0.06297	−0.05284	6.185	19.13	0.58630
7	0.04487	−0.04012	0.04153	−1.513	1.206	0.08179

The full four-factor specification prices synchronous and phase-shifted downside beta *and* co-skewness jointly,

$$\bar{R}_i = \lambda_0^j + \lambda_1^j \beta_{i,j}^{d,\Re} + \lambda_2^j \gamma_{i,j}^{d,\Re} + \lambda_3^j \beta_{i,j}^{d,\Im} + \lambda_4^j \gamma_{i,j}^{d,\Im} + \varepsilon_i,$$

and dominates all others (Table 3). At scale 1 the real coefficients are highly significant while the imaginary ones take opposite signs ($F = 8.50$, $R^2 = 57.6\%$): short-horizon pricing is driven by synchronous reactions with priced counter-shocks in the delayed direction. A structural inversion appears at scale 2 ($\lambda_1 < 0$, $\lambda_3 > 0$, $R^2 = 52.3\%$). Explanatory power peaks at scale 6 (about 32–64 trading days) with $F = 9.46$ and $R^2 = 60.2\%$, where real and imaginary components jointly determine returns, mapping onto macroeconomic information cycles. The coefficients remain economically meaningful at scale 7 ($R^2 = 43.9\%$).

Table 3: SCF four-factor cross-sectional regressions using real and imaginary components

Scale	λ_0	λ_1	λ_2	λ_3	λ_4	F	R ²
1	0.03596	0.15371	0.41920	-0.36630	-0.18684	8.496	0.57568
2	0.03590	-0.16374	-0.05347	0.13711	0.09206	6.850	0.52291
3	0.03620	-0.24019	-0.20792	0.28377	0.17528	3.039	0.32672
4	0.03488	-0.05576	-0.00646	0.01208	0.06147	0.948	0.13175
5	0.03642	0.10947	0.24288	-0.19180	-0.15017	5.386	0.46289
6	0.03260	-0.01149	-0.03215	0.07192	-0.01439	9.460	0.60217
7	0.03594	-0.00755	0.10629	-0.00026	-0.08750	4.882	0.43859

7. Discussion

Three regularities emerge. First, adding complex (semi-continuous) wavelet components sharply increases the explanatory power of cross-sectional downside models relative to both the time domain and MODWT: the discrete, real-valued, dyadic structure of MODWT blocks phase extraction and leaves downside risk essentially unpriced. Second, the real and imaginary components are complementary the imaginary (phase-shifted) structure is often weakly significant on its own but becomes decisive once combined with the synchronous effect, so delayed transmission cannot be interpreted in isolation. Third, the strongest and most stable pricing appears at long (business-cycle) horizons, consistent with downside risk being macro-driven at low frequencies and dominated by microstructure noise at short horizons. Sign patterns (negative real, positive imaginary premia at short scales; the reverse at scale 6) indicate that markets differentiate immediate losses from delayed or anticipatory reactions and demand distinct, horizon-dependent premia for each.

8. Conclusion

We introduced a semi-continuous complex wavelet frame for systematic downside risk. Unlike time-domain or discrete multiscale models, it delivers complex coefficients with amplitude and phase. Decomposing downside beta and co-skewness into synchronous (real) and phase-shifted (imaginary) parts, and pricing them across 30 U.S. industry portfolios, we find that downside systematic risk is strongly scale-dependent, is missed by real-valued and time-domain models, and is best explained by a four-factor complex specification (R^2 up to $\approx 60\%$ at business-cycle scales). Amplitude and phase must be assessed jointly. The framework opens further work on volatility synchronisation, financial contagion and dynamic systematic-risk transfer, from developed to emerging markets.

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