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Low Default Portfolios – From the Usefulness of Pluriannual Data to the Inconsistency of Multi-period Estimation

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Abstract

Estimating conservative default probabilities is crucial when banks opt to utilize an internal ratings-based strategy to calculate their capital requirements. At least five years of historical data should be embraced when adopting internal models, according to the Basel Committee on Banking Supervision.
This paper calls for a conceptual shift between pluriannual data and multi-period estimation. It is shown from a variety of theoretical and practical perspectives that default probabilities computed using the multi-year process do not reflect the real-world banking business and are unrealistic or imprudent when the classical or Bayesian approaches are implemented. Different time periods and data aggregation methods are applied in such approaches to illustrate the inconsistency of multi-period estimation.
As a result, any financial risk management tool should refrain from employing the multi-period methodology recommended by various authors for determining low default probabilities because the outcomes are not prudentially sound. Estimating annual default probabilities via time series (instead of estimating multi-period default probabilities) is the most accepted practice for both the classical and Bayesian approaches, as detailed here. The conclusions remain the same whether the occurrence of default events follows a binomial distribution or a Poisson distribution.

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1. Introduction

In credit risk management, the default probability estimations are the key input for an adequate use of internal models, according to the Basel Accords released by the Basel Committee on Banking Supervision. In low default portfolios, each obligor grading requires a long-run average of one-year default rates (Basel Committee, 2022)[2].

A variety of information sources, specifically external, internal, or pooled data, is available to banks. Regardless of the type of information, they need at least five years of empirical data to adopt internal models (Basel Committee, 2006; and Basel Committee, 2022)[3]. The highest quantity of historical observations naturally corresponds to a time period that includes a whole economic cycle. In accordance with the Basel Committee on Banking Supervision, estimations of default probability are more trustworthy the longer the historical data is available.

European banks are subject to the same principles and rules. For example, in compliance with the prudential requirements for credit institutions and investment firms, the less data an institution has the more conservative it shall be in its estimation (Regulation 2013)[4], and irrespective of whether an institution is using external, internal, or pooled data sources, or a combination of the three, for its probability of default estimation, the length of the underlying historical observation period used shall be a minimum of five years for at least one source (Regulation 2013 too)[5].

Therefore, the Basel Committee on Banking Supervision focuses solely on pluriannual historical data for counting defaults, rather than for estimating a multi-year default probability (compressed into a one-year estimation). The historical data over several years is utilized to evaluate the annual default probability, not to create a multi-period probability.

The foundation of this work is some people's misunderstanding. They evoke asking for multi-year default probability estimations rather than time series data for one-year default probability estimations. It will be shown that choosing multi-period estimations is an incorrect procedure because it results in default probabilities that are too imprudent, violating the basic rule of credit risk management that the corresponding probabilities in a low default portfolio context – involving, specifically, corporate, bank, and sovereign exposures – must be conservative.

The specifics of the data adopted here as well as the cross-sectional correlation and the year-to-year correlation are identified in Section 2. Section 3 presents two strategies for estimating one-year default probabilities for both the classical and Bayesian approaches applying the binomial distribution, namely the annual

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[2] Paragraph 36.63 of the Basel Committee on Banking Supervision (2022).

[3] Paragraph 463 of the Basel Committee on Banking Supervision (2006). Also paragraph 36.79 of the Basel Committee on Banking Supervision (2022).

[4] Article 179(1)(a) of the Regulation 575/2013 of the European Parliament and the Council.

[5] Article 180(1)(h) of the previously mentioned Regulation 575/2013.

estimation with annual data and the annual estimation with pluriannual data. The outcomes of those strategies are also contrasted.

The three next sections provide a full explanation of the multi-period problem. In Section 4, the strategy for estimating default probabilities over a period of time is developed. That section provides details on multi-period estimations along with information on analytical methods for computing joint default probabilities. The outputs of one-year and multi-year default probability projections are compared in Section 5.

In Section 6, further technical issues embracing multi-year calculation of default probabilities are examined to round out the analysis. More specifically the effects of employing the Poisson distribution (instead of the binomial distribution) are studied.

A full summary is provided at the end of the document. Section 7 compiles and arranges the main findings from the extensive paper about the one-year estimations and particularly the multi-year estimations.

When estimating default probabilities over multiple periods, it is crucial to take into account two key references in the literature. Katja Pluto and Dirk Tasche (Pluto and Tasche, 2005) are considered pioneers, as they were the first to look at multi-period default probabilities using the classical approach in 2005. In 2012, Dirk Tasche (Tasche, 2012) described his Bayesian methodology for determining multi-period default probabilities. As a result, the theoretical axis of this paper is built on those 2005 and 2012 investigations.

In order to fully develop the analysis that was previously explained, which entails the identification of both cross-sectional and intertemporal influences and the comparison between the one-year probabilities and multi-year probabilities, it is necessary for this paper to be lengthy and dense. The conclusions of a thorough and in-depth investigation of the multi-year issue would not have been extracted if a brief and general study had been performed. The presentation would become softer and simpler by splitting it into the classical and Bayesian approaches. This would somewhat reduce that length and density, but it would also break the crucial connection between the two approaches and the related explanation.

2. Data and Correlations

Most importantly, it is essential to highlight that this study is mainly centered on estimating default probabilities across multiple time periods (i.e., multi-period estimation of default probabilities). Simply put, the multi-period approach discussed here does not rely solely on historical default rates over a "T" period that covers several years. This historical data can be used in two different ways: estimating default probabilities for either a single year (one-year estimation) or multiple years (multi-year estimation).

The weights listed in Table 1 are applied to the data covering a period of "T" years and the corresponding default probabilities. The weight for each year "t" in the T-time period is denoted by "$w_{t,T}$". Five time periods longer than one year, ranging from two to six years, are evaluated in this paper. The procedure of giving the most recent annual data the utmost importance validates the implementation of those weights. Should this procedure not be accepted right away – meaning if it is assigned the same level of importance throughout the T-time period – the study's findings will remain unchanged.

Table 1: Weights used for pluriannual data and one-year default probabilities

$w_{t,T}$	$w_{t,6}$	$w_{t,5}$	$w_{t,4}$	$w_{t,3}$	$w_{t,2}$	$w_{t,1}$
t						
1	30%	35%	40%	50%	60%	100%
2	25%	25%	30%	30%	40%	
3	20%	20%	20%	20%		
4	15%	15%	10%			
5	7,5%	5%				
6	2,5%					
$w_{t,T}$	Weight adopted for the year "t" within the period of "T" years, t = 1, 2, …, T					

A moderate and an extreme low default portfolio scenario are employed, resulting in a dozen and only one default occurrence in the last six years, respectively. Table 2 displays, for each t-year and for both scenarios, the number of obligors at the beginning of the year (n) and the number of defaults at the end of the year (k), as well as the empirical default rate (k/n). The annual figures corresponding to the number of borrowers and the number of defaults remain the same value for each t-year, regardless of the time period "T" that it is assumed.

Table 2: Number of obligors and defaults, and empirical default rate for each year

t	Moderate scenario			Extreme scenario		
	n	*k*	*k/n*	*n*	*k*	*k/n*
1	240	2	0.83%	250	1	0.40%
2	240	0	0.00%	250	0	0.00%
3	241	1	0.41%	250	0	0.00%
4	245	4	1.63%	250	0	0.00%
5	248	3	1.21%	250	0	0.00%
6	250	2	0.80%	250	0	0.00%
		12			1	

It is assumed that a closed group of 250 borrowers will be part of the credit portfolio at the beginning of the sixth year in every scenario, therefore no new obligor admission is tested[6]. Hence, $\sum_{j=t'+1}^{T} k_j = (n_6 - n_{t'})$ and $n_{t'} = (n_{t'+1} - k_{t'+1})$, $t' = 1, 2, \ldots, 5$. Since assuming closed or open groups only has an impact on "n" and "k" figures in each year, the conclusions of this paper still hold true for groups that are open to new borrowers for the credit portfolio.
Two types of averages (or arithmetic means) are displayed to ensure a deeper understanding of the impact of the time series data on the default probability computation: ordinary arithmetic means and weighted arithmetic means. The former is obtained with Table 2, whereas the latter is obtained with Tables 1 and 2. Table 3 shows those means for data utilizing an upward rounding. Certainly, due to the discrete pattern of the numbers for the obligors and defaults, it is unquestionable more prudent to round up instead of down, since upwards adjustments lead to higher default probabilities.

Table 3: Means rounded upwards of the number of obligors and defaults for each pluriannual period

T	Types of average (for data)	Moderate scenario		Extreme scenario	
		n	k	n	k
2	Ordinary mean	240	1	250	1
	Weighted mean	240	2	250	1
3	Ordinary mean	241	1	250	1
	Weighted mean	241	2	250	1
4	Ordinary mean	242	2	250	1
	Weighted mean	241	2	250	1
5	Ordinary mean	243	2	250	1
	Weighted mean	242	2	250	1
6	Ordinary mean	244	2	250	1
	Weighted mean	242	2	250	1

This paper makes solely use of the numbers listed in Table 3 concerning the weighted means – which will show the annualized data obtained from the pluriannual data to compute the annual default rates (see 3.2.2) as well as the multi-annual default rates (see 4.2.2) – because, for weighted means, the "k/n" ratios exceed (or are equal to) the corresponding ratios for ordinary means.
Furthermore, since the more recent information is often given more weight than the older information, weighted data is commonly chosen instead of straight data usage. Only weighted means will be considered when it is necessary to adopt means, exclusively for the sake of simplification. This is true for both the classical and Bayesian approaches.

[6] Pluto and Tasche (2005) assumed closed groups too.

In the banking sector, it is unrealistic to anticipate the independence hypothesis among borrowers during the same year, as default occurrences are influenced by various systematic economic risk factors. As a result, hereinafter, the 12% asset correlation coefficient – which is one of the coefficients set up in the Basel Accords' asset correlation formula for employing internal models (Basel Committee, 2020) [7] – will always be provided as the cross-sectional dependence among borrowers (i.e., asset correlation, "ρ"). However, the exact same calculations used for the asset coefficient of 12% were also applied to the asset correlation of 0%. For simplicity's sake and due to adherence to reality, only outputs (and formulae too) based on the positive correlation hypothesis among obligors will be given.

Moreover, for the multi-period estimation technique, four over-time dependence among several years – or year-to-year structure (i.e., intertemporal correlation, "τ")[8] –, ranging from 0% to 99.9%[9] as opposing values, will be tested. In addition to these two opposing percentages, the middle intertemporal correlation value of 50% will be assessed. For this 50% parameter, both a constant function (fixed theoretical correlation regardless of what "T" assumes) and an exponential function (decreased correlation depending on "T") will be adopted.

3. One-year Estimation of Default Probabilities

The estimating rule for default probabilities is based on a one-period perspective, whether the data scope is pluriannual or merely annual. The two next subsections examine how these two types of data affect estimations of default probabilities. When data reported over a T-time period is summed up, it is presumed that there is no intertemporal correlation among defaults that occurred in various years. The third subsection compares the outputs.

3.1 Annual Estimation with Annual Data

3.1.1 Formulae

Default probabilities are assessed annually for each t-year, even if data for other years is available. One uses the information for that specific t-year, as shown in Table 2. The assessment is conducted by means of both classical and Bayesian methodologies and takes into consideration that the default occurrence follows a binomial distribution.

[7] Basel Committee on Banking Supervision (2020): the other coefficient is 24%. Pluto and Tasche (2005) utilized correlation coefficients of 12% and 0% in the scenarios of correlated and independent default events, respectively.

[8] Term adopted by Pluto and Tasche (2005).

[9] The joint probability density function for any multi-variate normal distribution does not exist at 100% as a correlation parameter because the determinant of the variance-covariance matrix is equal to zero. Thus, the maximum percentage "τ" will be set to 99.9% in order to avoid the affine mapping of spatial variables in the case of a perfect direct correlation and to apply the same method regardless of intertemporal correlation values.

Classical approach

Consequently, the classical probability that there will be no more than "k_t" defaults within a group of "n_t" borrowers over the course of a t-year, "$\lambda_{t,C}$", is given by:

$$\mathrm{P}(K_t \leq k_t) = \int_{-\infty}^{+\infty} \left\{ \sum_{i=0}^{k_t} \binom{n_t}{i} . \left[G(\lambda_{t,C}, y, \rho)\right]^{i} . \left[1 - G(\lambda_{t,C}, y, \rho)\right]^{n_t - i} \right\} . \phi(y) dy \geq 1 - \omega \tag{1}$$

with

$$G(\lambda_{t,C}, y, \rho) = \Phi\{[\Phi^{-1}(\lambda_{t,C}) - y.\sqrt{\rho}]/\sqrt{1-\rho}\} \tag{2}$$

For each t-year, "K_t" indicates the random variable that represents the number of defaults "k_t" in a group with "n_t" obligors. In such a case, it is assumed that "K_t" has a correlated binomial distribution. "ρ" expresses the positive asset correlation among default events. The selected confidence level is depicted by "ω", indicating the suitable degree of prudence when estimating the default probability for the internal model. The aforementioned "G" function derives from the Vasicek's model (Vasicek, 2002).

A correlated default distribution is based on a structural model that takes the default non-independence hypothesis into account, i.e., it assumes that asset correlation is positive (Chatterjee, 2015)[10]. Since conditional probabilities are linked to the realization of the systematic risk, here the default probability "$\lambda_{t,C}$" used in reduced form models (ρ = 0%) is replaced by the function "$G(\lambda_{t,C}, y, \rho)$" for the adoption of structural models (that do not provide analytical solutions).

"Y" refers to a standard and normally distributed random variable, $Y \sim \mathrm{N}(0, 1)$, for all possible values of "y" which reflect the realization range of the systematic risk. "$\phi(y)$", "$\Phi(.)$", and "$\Phi^{-1}(\lambda_{t,C})$" indicates the standard normal probability density function of "Y", the standard normal cumulative distribution function, and the inversed standard normal cumulative distribution function for "$\lambda_{t,C}$", respectively.

Denoting "${\lambda_{T,C}}^{o}$" the ordinary mean of the one-year default probabilities for the period of "T" years that were estimated utilizing the classical approach, it comes:

$${\lambda_{T,C}}^{o} = \left(\sum_{t=1}^{T} \lambda_{t,C}\right)/T \tag{3}$$

when "$\lambda_{t,C}$" represents the classical default probability for the t-year as previously indicated, which results from Equation (1); consequently, $\lambda_{t,C}$ is the probability that makes $\mathrm{P}(K_t \leq k_t) \geq 1 - \omega$ possible.

The weighted mean of the annual default probabilities for the period of "T" years, "${\lambda_{T,C}}^{w}$" (which are also computed by employing the classical approach), derives from:

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[10] In contrast, from a terminological standpoint, a standard default distribution (as opposed to a correlated default distribution) is built on a reduced form model (as opposed to a structural model) that considers the default independence premise and so assumes a null asset correlation.

$$\lambda_{T,C}{}^{w} = \sum_{t=1}^{T} w_{t,T}.\lambda_{t,C} \quad (4)$$

having "$w_{t,T}$" the previously mentioned meaning (Table 1).

Bayesian approach

Within a single-period related to the t-year, the expected value of the Bayesian probability (or the posterior probability), "λ", of exactly "k_t" defaults – and not no more than "k_t" defaults as in the classical approach – occurring inside a group with "n_t" borrowers, "$\mu_{t,B}$", is expressed by:

$$\mu_{t,B} = \int_0^{u} \lambda. f(\lambda). \int_{-\infty}^{+\infty} \binom{n_t}{k_t}. [G(\lambda, y, \rho)]^{k_t}. [1 - G(\lambda, y, \rho)]^{n_t - k_t}. \phi(y) dy d\lambda.$$
$$.1 / \int_0^{u} f(\lambda). \int_{-\infty}^{+\infty} \binom{n_t}{k_t}. [G(\lambda, y, \rho)]^{k_t}. [1 - G(\lambda, y, \rho)]^{n_t - k_t}. \phi(y) dy d\lambda =$$
$$= \int_0^{u} \lambda. f(\lambda). \int_{-\infty}^{+\infty} [G(\lambda, y, \rho)]^{k_t}. [1 - G(\lambda, y, \rho)]^{n_t - k_t}. \phi(y) dy d\lambda.$$
$$.1 / \int_0^{u} f(\lambda). \int_{-\infty}^{+\infty} [G(\lambda, y, \rho)]^{k_t}. [1 - G(\lambda, y, \rho)]^{n_t - k_t}. \phi(y) dy d\lambda\ , \quad (5)$$

being once again

$$G(\lambda, y, \rho) = \Phi\{[\Phi^{-1}(\lambda) - y.\sqrt{\rho}]/\sqrt{1-\rho}\} \quad (6)$$

The second integral (in the numerator and denominator) refers to the likelihood function, which has a correlated binomial distribution with a positive asset correlation "ρ". The definitions of "y", "$\phi(y)$", "$\Phi(.)$", and "$\Phi^{-1}(\lambda)$" are identical to those given above.

The prior function is depicted by "$f(\lambda)$". In this study, it is assumed that "Λ", the random variable of the default probability "λ", follows one of two possible prior distributions, (1) one of which is a quasi-uninformative function and (2) the other is an informative prior. To be more precise:

(1) Uniform distribution
$\Lambda \sim$Uniform(0, u), with $\lambda \in\ ]0,\ u]$

$$f(\lambda) = 1/u \quad (7)$$

One observes that in the setting of low default portfolios (or non-zero default portfolios), it should be expected that the upper limit of "λ" (denoted by "u") is not particularly high[11] and therefore not equal to 1. Consequently, it will be fixed at $u = 0.1$ since the greatest empirical default rate is 1.63% in the moderate scenario and 0.40% in the extreme scenario.

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[11] Annual empirical default rates range from 0% to 1.63%, according to Table 2.

(2) Beta distribution

$\Lambda \sim \text{Beta}(\alpha, \beta)$, with $\alpha, \beta > 0$ and $\lambda \in]0, 1[$

$$f(\lambda) = \Gamma(\alpha + \beta)/[\Gamma(\alpha).\Gamma(\beta)].\lambda^{\alpha-1}.(1-\lambda)^{\beta-1} \quad , \tag{8}$$

with "$\Gamma(.)$" standing for the gamma function.

Taking into consideration that the mean and the variance of "Λ" are given by $\text{E}(\Lambda) = \alpha/(\alpha + \beta)$ and $\text{V}(\Lambda) = \alpha.\beta/[(\alpha + \beta)^2.(\alpha + \beta + 1)]$, the parameters "$\alpha$" and "$\beta$" come from the following:

$$\alpha = [-\text{E}(\Lambda).\text{V}(\Lambda) + \text{E}(\Lambda)^2 - \text{E}(\Lambda)^3]/\text{V}(\Lambda) \tag{9}$$

$$\beta = \alpha.[1 - \text{E}(\Lambda)]/\text{E}(\Lambda) \tag{10}$$

The mean and variance of "Λ" must be known for Equations (9) and (10) to work correctly. One will regard the empirical default rate when calculating "$\text{E}(\Lambda)$", being "k" and "n" applied to each t-year. For the purpose of determining "$V(\Lambda)$", it will be used the ratio of that mean "$\text{E}(\Lambda)$" to 1.645 (corresponding this denominator to a 90% confidence level of a standard normal distribution function, which is a suitable and rational percentage for the purpose of setting confidence levels).

The Bayesian approach is also in line with earlier findings on "${\lambda_{T,C}}^o$" and "${\lambda_{T,C}}^w$" as shown in Equations (3) and (4). Thus, the ordinary expected value of the annual default probabilities throughout the period of "T" years estimated under the Bayesian approach is generated by:

$${\mu_{T,B}}^o = \left(\textstyle\sum_{t=1}^{T} \mu_{t,B}\right)/T \quad , \tag{11}$$

representing "$\mu_{t,B}$" the Bayesian mean of the default probability for the t-year extracted by Equation (5).

The weighted expected value of the annual default probabilities regarding the period of "T" years for the Bayesian approach is get by:

$${\mu_{T,B}}^w = \textstyle\sum_{t=1}^{T} w_{t,T}.\mu_{t,B} \quad , \tag{12}$$

where "$w_{t,T}$" stands for the weights previously mentioned (Table 1, too).

3.1.2 Results

Classical approach

Applying Equation (1), the default probabilities are now given for each one-period of time at three confidence levels, "ω": 65%, 75%, and 95%[12]. The first two percentages might be seen as the lower and upper conservative bounds. It will be clearly imprudent to estimate default probabilities with a confidence level of less than 60% or 65%, while it will almost surely be exaggerated to estimate default probabilities with a confidence level of more than 75% or 80%.

This statement concerning the adequacy of the lower and upper confidence bounds is vulnerable to criticism because it appears to be made from a subjective viewpoint. When estimating default probabilities using limited default data, a larger margin of conservatism is required (Basel Committee, 2022)[13]. It will be discussed in more detail in this sub-subsection (on the Bayesian approach) why a confidence level of roughly 75% is conservative enough to provide reliable objectivity. Additionally, this sub-subsection indicates that the Bayesian default probabilities using the unform distribution as the prior function are associated with confidence bounds of no less than approximately 75%. Although the 95% confidence level is often suggested as a potential or suitable benchmark when default probabilities are evaluated, the figures show that it is grossly exaggerated, making it economically unrealistic.

Table 4 displays the default probabilities for each t-year in the moderate and extreme scenarios. Equations (1) and (2) and the information from Table 2 are utilized to compute those probabilities. The empirical default rate is equal to 46% or 36% of the default probability regarding the 65% or 75% confidence levels for the highest "k" in the first scenario – k = 4 (for t = 4) – and 27% or 20% in the second scenario – k = 1 (for t = 1).

Table 4: One-year default probabilities under the classical approach for each year (ρ = 12%)

Types of scenario	t	n	k	Confidence levels		
				65%	75%	95%
Moderate scenario	1	240	2	2.291%	3.005%	6.790%
	2	240	0	0.772%	1.117%	3.289%
	3	241	1	1.558%	2.105%	5.178%
	4	245	4	3.575%	4.548%	9.390%
	5	248	3	2.896%	3.733%	8.024%
	6	250	2	2.212%	2.904%	6.590%
Extreme scenario	1	250	1	1.509%	2.042%	5.039%
	2 till 6	250	0	0.745%	1.079%	3.188%

[12] The study carried out by Pluto and Tasche (2005) examined confidence bounds of 50%, 75%, 90%, 95%, 99%, and 99.9%.

[13] Paragraph 36.67 of the Basel Committee on Banking Supervision (2022).

The default probabilities presented in Table 5 are obtained by applying Equations (1) and (2), along with Equations (3) and (4), to the data provided in Table 2[14]. Weighted means of annual default probabilities do not significantly differ from ordinary means, as anticipated[15]. For the moderate scenario and the 75% confidence level, the ordinary mean ranges from 2.06% to 2.90%, and the weighted mean ranges from 2.25% to 2.64%. For the extreme scenario and the same confidence level, they range from 1.24% to 1.56%, and from 1.37% to 1.66%, respectively.

Table 5: Means of default probabilities under the classical approach for each pluriannual period (ρ = 12%)

Types of scenario	T	Types of average (for default rates)	Confidence levels		
			65%	75%	95%
Moderate scenario	2	Ordinary mean	1.53%	2.06%	5.04%
		Weighted mean	1.68%	2.25%	5.39%
	3	Ordinary mean	1.54%	2.08%	5.09%
		Weighted mean	1.69%	2.26%	5.42%
	4	Ordinary mean	2.05%	2.69%	6.16%
		Weighted mean	1.82%	2.41%	5.68%
	5	Ordinary mean	2.22%	2.90%	6.53%
		Weighted mean	1.99%	2.62%	6.04%
	6	Ordinary mean	2.22%	2.90%	6.54%
		Weighted mean	2.00%	2.64%	6.07%
Extreme scenario	2	Ordinary mean	1.13%	1.56%	4.11%
		Weighted mean	1.20%	1.66%	4.30%
	3	Ordinary mean	1.00%	1.40%	3.81%
		Weighted mean	1.13%	1.56%	4.11%
	4	Ordinary mean	0.94%	1.32%	3.65%
		Weighted mean	1.05%	1.46%	3.93%
	5	Ordinary mean	0.90%	1.27%	3.56%
		Weighted mean	1.01%	1.42%	3.84%
	6	Ordinary mean	0.87%	1.24%	3.50%
		Weighted mean	0.97%	1.37%	3.74%

[14] It should be emphasized that Table 5 refers to ordinary and weighted means of default probabilities, whilst Table 3 is associated with ordinary and weighted means (rounded upwards) of "n" and "k".
[15] For instance, when taking into account the extreme scenario and the 75% confidence level, ordinary means of default probabilities are 10-16 basis points (bp) lower than weighted means. For the moderate scenario and the same confidence level, ordinary means are nearly 20 bp lower for T = 2 and T = 3, and nearly 30 bp higher for T = 4, T = 5, and T = 6 than weighted means.

In the moderate scenario, for example, with $T = 2$ and $\omega = 65\%$, the ordinary mean and weighted mean yield outcomes of 1.532% and 1.683%, respectively. It is noted that the default probability for $\omega = 65\%$ at $t = 1$ ($n = 240$ and $k = 2$) is 2.291%, whereas the corresponding default probability at $t = 2$ ($n = 240$ and $k = 0$) is notably lower, at 0.772%. When annualizing the aggregated data for the two years at the same confidence level ($\omega = 65\%$), it evidently results a default probability between 2.291% and 0.772%: 1.532% – (2.291% + 0.772%) / 2 – for the ordinary mean and 1.683% – 60% x 2.291% + 40% x 0.772% – for the weighted mean.

Bayesian approach

Table 6 shows the annual default probabilities estimated using the prior functions established in Sub-subsection 3.1.1 for the mean computing, i.e., uniform distribution (spanning from 0 to 0.1) and beta distribution (assuming that the default probability ranges between 0 and 1). Probabilities are derived from Equations (5), (6), and (7) for the uniform distribution and Equations (5), (6), (8), (9), and (10) for the beta distribution, applying to data in Table 2. The outputs of the likelihood function are also identified[16].

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[16] It should be noted that under the Bayesian theory, the posterior probability is created by combining the likelihood function and the prior function. The mean of the posterior probability is the same as the likelihood function when the uniform distribution between 0 and 1 is adopted as the prior function.

Table 6: One-year default probabilities using the Bayesian approach and corresponding matching confidence levels for each year (ρ = 12%)

Types of scenario	*t*	*n*	*k*	Likelihood function	Types of prior function	
					Uniform distribution regarding]0%, 10%]	Beta distribution regarding EDR as the mean (#)
Moderate scenario	1	240	2	3.396%	3.026%	1.515%
	(mcl)			(79.1%)	(75.2%)	(48.8%)
	2	240	0	1.422%	1.310%	0.424%
	(mcl)			(80.9%)	(79.0%)	(48.2%)
	3	241	1	2.472%	2.296%	0.921%
	(mcl)			(79.8%)	(77.7%)	(46.6%)
	4	245	4	4.894%	4.017%	2.629%
	(mcl)			(77.8%)	(77.0%)	(51.4%)
	5	248	3	4.121%	3.538%	2.050%
	(mcl)			(78.5%)	(73.0%)	(50.5%)
	6	250	2	3.304%	2.963%	1.467%
	(mcl)			(79.3%)	(75.7%)	(49.0%)
Extreme scenario	1	250	1	2.413%	2.249%	0.897%
	(mcl)			(80.0%)	(77.9%)	(46.8%)
	2 till 6	250	0	1.386%	1.278%	0.416%
	(mcl)			(81.1%)	(79.2%)	(48.7%)
EDR - Empirical default rate (*k/n*)						
(#) - Variance of beta distribution corresponds to EDR/1.645. For beta distribution, default probabilities range between 0% and 100%.						
mcl - matching confidence level for the classical approach						

The uniform distribution]0, 0.1] yields default probabilities much more conservative than the ones for the beta distribution: 0.9-1.5 percentage points (pp) higher for both the moderate and extreme scenarios. For low default portfolios[17], it is obvious that posterior probabilities adopting the uniform distribution]0, 0.1] as the prior function are (a) lower than those that apply the purest non-informative prior (the uniform distribution]0, 1[) and (b) higher than the posterior probabilities relying on the beta distribution as prior function based on the empirical default rate – this is observable when comparing (a) the penultimate and antepenultimate columns of the preceding table, as well as (b) the penultimate and final column of the same table.

[17] See footnote 11.

The figures in brackets express to the matching confidence levels, i.e., the confidence levels under the classical approach corresponding to the annual default probabilities utilizing the Bayesian approach[18]. As expected, the matching confidence levels for the likelihood function are a little higher than those for the posterior probabilities employing the uniform function between 0 and 0.1 as prior. The highest matching confidence levels for this last case are 77.0% in the moderate scenario and 79.2% in the extreme scenario, which are adequate from the conservative point of view (against what happens with the posterior probabilities using the beta function as prior since the maximum matching confidence levels range from 51.4% to 48.7%, respectively in the moderate and extreme scenarios).
The second paragraph of this Sub-subsection 3.1.2 is now reviewed. There is unmistakable subjectivity when an assessment of the classical approach is conducted separately from the findings of the Bayesian approach, and vice versa. In the classical approach subjectivity is managed by choosing the confidence level, whereas in the Bayesian approach it results from the prior function selection (from the most non-informative function – the uniform distribution spanning from 0 to 1 as the prior function – to any informative function). Making a combined analysis of two approaches, so matching the classical and Bayesian procedures is a great strategy to significantly mitigate such subjectivity.
In order to avoid this subjectivity, the beta distribution here presented has to be removed as a prior function because it generates default probabilities that are obviously unwise (linked to very low matching confidence levels, around 50%). The non-informative prior function produces probabilities that are related to higher confidence levels. For this reason, a quasi-uninformative prior function like the uniform distribution between 0 and 0.1 is objectively a good option that ensures conservative default probabilities in low default portfolios.
Table 7 shows both ordinary and weighted means of default probabilities for the Bayesian approach, obtained by applying Equations (5) through (10) as well as Equations (11) and (12) to data included in Table 2. This procedure is identical to the one presented previously (in reference about Table 5 linked to the classical approach).

[18] For instance, when n = 240 and k = 2, the mean of the likelihood function is 3.396%. Utilizing these three figures (for t = 1) in Equation (1) for the classical approach yields a confidence level of 79.1%, which is deemed appropriately conservative, by contrasting with empirical default rate of 0.833%.

Table 7: Means of default probabilities using the Bayesian approach for each pluriannual period (ρ = 12%)

Types of scenario	*T*	Types of average (for default rates)	Likelihood function	Types of prior function	
				Uniform distribution regarding]0%, 10%]	Beta distribution regarding EDR as the mean (#)
Moderate scenario	2	Ordinary mean	2.41%	2.17%	0.97%
		Weighted mean	2.61%	2.34%	1.08%
	3	Ordinary mean	2.43%	2.21%	0.95%
		Weighted mean	2.62%	2.36%	1.07%
	4	Ordinary mean	3.05%	2.66%	1.37%
		Weighted mean	2.77%	2.46%	1.18%
	5	Ordinary mean	3.26%	2.84%	1.51%
		Weighted mean	2.98%	2.62%	1.32%
	6	Ordinary mean	3.27%	2.86%	1.50%
		Weighted mean	2.99%	2.64%	1.33%
Extreme scenario	2	Ordinary mean	1.90%	1.76%	0.66%
		Weighted mean	2.00%	1.86%	0.70%
	3	Ordinary mean	1.73%	1.60%	0.58%
		Weighted mean	1.90%	1.76%	0.66%
	4	Ordinary mean	1.64%	1.52%	0.54%
		Weighted mean	1.80%	1.67%	0.61%
	5	Ordinary mean	1.59%	1.47%	0.51%
		Weighted mean	1.75%	1.62%	0.58%
	6	Ordinary mean	1.56%	1.44%	0.50%
		Weighted mean	1.69%	1.57%	0.56%
EDR - Empirical default rate					
(#) - Variance of beta distribution corresponds to EDR/1.645. For beta distribution, default probabilities range between 0% and 100%.					

Ordinary means and weighted means are somewhat equivalent[19]. For the moderate scenario and the uniform distribution between 0 and 0.1 as the prior function, the ordinary mean and the weighted mean range from 2.17% to 2.86%, and from 2.34% to 2.64%, respectively; for the extreme scenario, they range from 1.44% to 1.76%, and from 1.57% to 1.86%.

3.2 Annual Estimation with Pluriannual Data

3.2.1 Formulae

In Subsection 3.1, the annual data collected over a period of "T" years were utilized to generate several annual default probabilities, which were then merged to create a specific default probability for that T-period. This probability was calculated via an ordinary mean or a weighted mean of the annual default probabilities. As a consequence, it is necessary to first gather annual default probabilities for each year in order to compute the means of these probabilities.

On the contrary, in this Subsection 3.2 one gets a single annual default probability (instead of "T" annual default probabilities) employing the pooled empirical data across "T" years (instead of annual data). Such an annual probability is associated with a particular "T" and the respective time series data. In other words, in this alternative (i.e., the second one-period estimation method), $n = \sum_{t=1}^{T} n_t$ and $k = \sum_{t=1}^{T} k_t$ (so, indicating that "n_t" and "k_t" are equally weighted) for the case with aggregated data, whereas $n = \sum_{t=1}^{T} w_{t,T}.n_t$ and $k = \sum_{t=1}^{T} w_{t,T}.k_t$ for the case with annualized data.

The same procedure as described in Subsection 3.1 (that was used to determine one-year default probabilities) is applied here too. Equations (1) and (2) produce classical probabilities, whereas Equations (5) through (10) yield Bayesian probabilities. It is also assumed that the default occurrence follows a binomial distribution.

3.2.2 Results

Classical approach

Table 8 compares the one-year default probabilities for aggregated and annualized data. The annualized data in this table (i.e., "n" and "k") indicates the weighted mean of the annual data presented in Table 3.

[19] Of course, both classical and Bayesian probabilities follow the same pattern. Under the Bayesian approach, ordinary means of default probabilities are 10-17 bp lower than weighted means for the extreme scenario and uniform distributions – either]0, 0.1] or]0, 1[. For the moderate scenario and the same uniform distributions, ordinary means are 15-20 bp lower for $T = 2$ and $T = 3$, and 20-28 bp higher for $T = 4$, $T = 5$, and $T = 6$ than weighted means. Figures for the beta distribution are likewise closer between the ordinary and weighted means.

Table 8: One-year default probabilities with aggregated and annualized data under the classical approach for each pluriannual period (ρ = 12%)

Types of scenario	T	Types of data	n	k	Confidence levels		
					65%	75%	95%
Moderate scenario	2	Aggregated	480	2	1.26%	1.69%	4.08%
		Annualized (#)	240	2	2.29%	3.00%	6.79%
	3	Aggregated	721	3	1.16%	1.54%	3.68%
		Annualized (#)	241	2	2.28%	2.99%	6.77%
	4	Aggregated	966	7	1.68%	2.17%	4.81%
		Annualized (#)	241	2	2.28%	2.99%	6.77%
	5	Aggregated	1214	10	1.83%	2.35%	5.09%
		Annualized (#)	242	2	2.27%	2.98%	6.75%
	6	Aggregated	1464	12	1.80%	2.31%	5.00%
		Annualized (#)	242	2	2.27%	2.98%	6.75%
Extreme scenario	2	Aggregated	500	1	0.83%	1.14%	3.00%
		Annualized (#)	250	1	1.51%	2.04%	5.04%
	3	Aggregated	750	1	0.59%	0.81%	2.22%
		Annualized (#)	250	1	1.51%	2.04%	5.04%
	4	Aggregated	1000	1	0.46%	0.64%	1.78%
		Annualized (#)	250	1	1.51%	2.04%	5.04%
	5	Aggregated	1250	1	0.38%	0.53%	1.51%
		Annualized (#)	250	1	1.51%	2.04%	5.04%
	6	Aggregated	1500	1	0.32%	0.46%	1.31%
		Annualized (#)	250	1	1.51%	2.04%	5.04%

(#) - Weighted mean of the annual data

Due to the law of large numbers, the one-year default probabilities with aggregated data are substantially lower than the corresponding probabilities with annualized data for any scenario and confidence level. The probabilities with aggregated data for the moderate scenario range from 52% (for $T = 3$) to 79% (for $T = 5$) of the probabilities with annualized data at the 75% confidence level[20]. The trend of the ratios of probabilities with aggregated data to probabilities with annualized data is even more obvious for the extreme scenario because probabilities with annualized data are always computed utilizing the same "n" and "k" (i.e., 250 and 1) – at the 75% confidence level, the default probabilities using aggregated data range from 56% (for $T = 2$) to 22% (for $T = 6$) of the default probabilities calculated with annualized data.

Probabilities with aggregated data are nonsensical since the default probability computation must bear in mind information from one-year basis instead of multiple years. One observes that such information (over a period of one-year) can be derived with a mean, which must always have an annual basis.

Therefore, adopting aggregated data to estimate default probabilities is not an option, leading to the related probabilities at different confidence levels lacking economic significance. Table 8 include default probabilities linked to aggregated data only to assess the imprudence of using aggregated data for default probability calculations as previously mentioned.

Bayesian approach

Similar conclusions are still true utilizing the Bayesian approach, as seen in Table 9. The pooled effect of the aggregated data (about the number of obligors "n" and defaults "k" during the time period "T") also applies to Bayesian probabilities. The annualized data was also extracted from Table 3, i.e., the annualized default probabilities were generated by taking the weighted mean of the annual data and then rounding it upward.

[20] The probabilities with aggregated data range from 51% to 80% and from 54% to 75% of the probabilities with annualized data, respectively, at the 65% and 95% confidence levels.

Table 9: One-year default probabilities with aggregated and annualized data using the Bayesian approach for each pluriannual period (ρ = 12%)

Types of scenario	*T*	Types of data	*n*	*k*	Likelihood function	Types of prior function: Uniform distribution regarding]0%, 10%]	Types of prior function: Beta distribution regarding EDR as the mean (a)
Moderate scenario	2	Aggregated	480	2	2.13%	2.06%	0.89%
		Annualised (b)	240	2	3.40%	3.03%	1.52%
	3	Aggregated	721	3	2.01%	1.96%	0.88%
		Annualised (b)	241	2	3.39%	3.02%	1.51%
	4	Aggregated	966	7	2.72%	2.59%	1.37%
		Annualised (b)	241	2	3.39%	3.02%	1.51%
	5	Aggregated	1214	10	2.91%	2.76%	1.52%
		Annualised (b)	242	2	3.38%	3.01%	1.51%
	6	Aggregated	1464	12	2.88%	2.74%	1.51%
		Annualised (b)	242	2	3.38%	3.01%	1.51%
Extreme scenario	2	Aggregated	500	1	1.52%	1.49%	0.56%
		Annualised (b)	250	1	2.41%	2.25%	0.90%
	3	Aggregated	750	1	1.17%	1.15%	0.44%
		Annualised (b)	250	1	2.41%	2.25%	0.90%
	4	Aggregated	1000	1	0.97%	0.95%	0.38%
		Annualised (b)	250	1	2.41%	2.25%	0.90%
	5	Aggregated	1250	1	0.85%	0.82%	0.33%
		Annualised (b)	250	1	2.41%	2.25%	0.90%
	6	Aggregated	1500	1	0.75%	0.73%	0.31%
		Annualised (b)	250	1	2.41%	2.25%	0.90%

EDR - Empirical default rate (*k/n*)

(a) - Variance of beta distribution corresponds to EDR/1.645. For beta distribution, default probabilities range between 0% and 100%.

(b) - Weighted mean of the annual data

Due to the discrete feature of the default distribution, integer values for "n" and "k" are necessary; nevertheless, decimal numbers may also be applied in an analytical context[21]. In such a case, because the precise k-values are lower than the values rounded upwards, the default probabilities are also lower than the equivalent probabilities indicated in Table 9[22].
One emphasizes that utilizing decimal numbers for "n" and "k" is an analytically feasible exercise for the Bayesian approach because the binomial coefficient, "$\binom{n}{k}$", does not influence the default probability, as can be observed from Equation (5). It can be seen from the analysis of Equation (1) that the same exercise is not analytically possible using the classical approach for two reasons: on the one hand, the binomial coefficient should not be eliminated; on the other hand, the sum (from $i = 0$ to $i = k_t$) can only be solved for integer values of the default numbers.

3.3 Annual Data *versus* Pluriannual Data

By comparing Tables 5 and 8 for the classical approach and Tables 7 and 9 for the Bayesian approach, one may demonstrate that using probabilities linked to the aggregated data scenario is not a reasonable alternative for estimating annual default probabilities. Figures 1 and 2 show the outcomes of such comparisons. In order to simplify the presentation, the confidence level for the classical approach is set at 75%, and the prior function for the Bayesian approach is set at a uniform distribution with a range of 0 to 0.1.

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[21] For instance, in the moderate scenario, for $T = 3$, the outcomes are $n = 240.2$ (240x0.5 + 240x0.3 + 241x0.2) and $k = 1.2$ (2x0.5 + 0x0.3 + 1x0.2), not $n = 241$ and $k = 2$ (means rounded upwards).
[22] If the uniform distribution between 0 and 0.1 were used as the prior function, the probabilities listed in the antepenultimate column of Table 9 (for the annualized data) would be changed to the following: 2.46%, 2.46%, 2.61%, 2.79%, and 2.80% (from $T = 2$ to $T = 6$), for the moderate scenario; and 1.90%, 1.81%, 1.71%, 1.66%, and 1.61% (equally from $T = 2$ to $T = 6$), for the extreme scenario. Therefore, these probabilities are quite comparable to the weighted means of the annual default probabilities (shown in the antepenultimate column of Table 7). Specifically, those ten probabilities with "n" and "k" without upward rounding are 0.096%-0.164% higher for the moderate scenario and 0.037%-0.043% higher for the extreme scenario than the probabilities in Table 7.

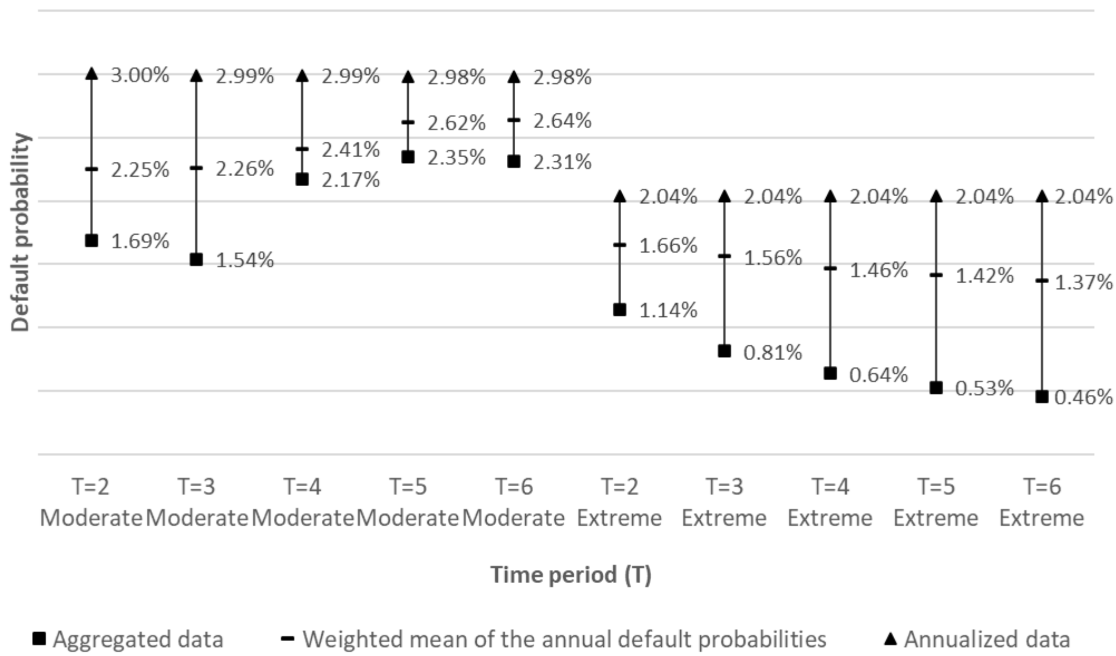

Figure 1: One-year default probabilities under the classical approach at a 75% confidence level for both moderate and extreme scenarios (ρ = 12%)

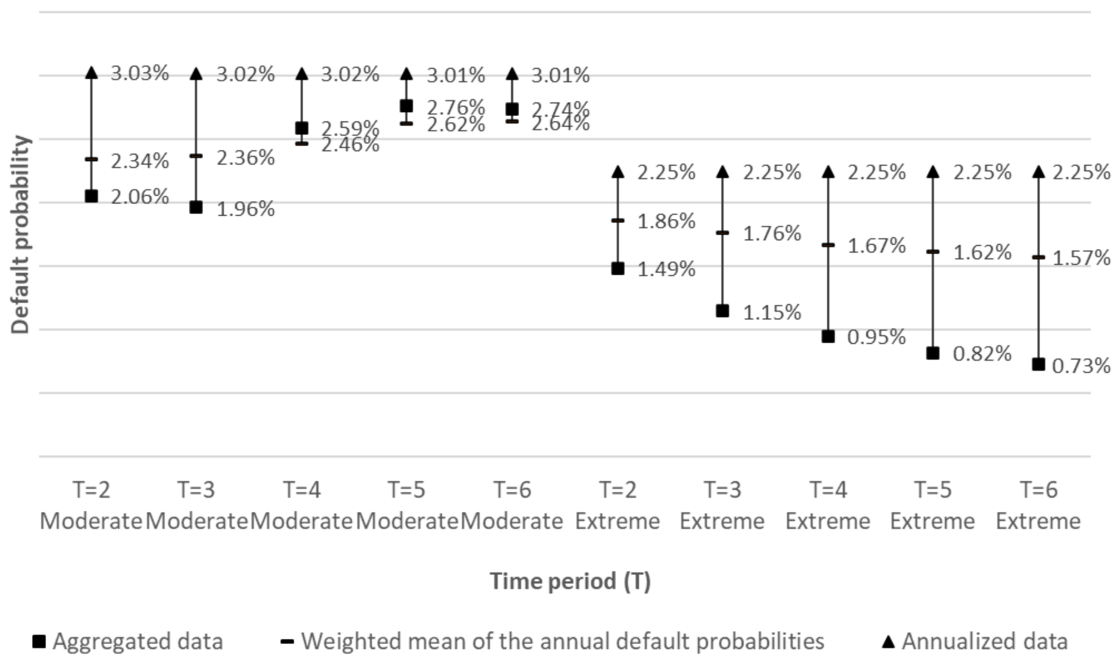

Figure 2: One year default probabilities using the Bayesian approach regarding the uniform distribution]0, 0.1] as the prior function for both moderate and extreme scenarios (ρ = 12%)

Two key points can be drawn from those figures. First, it is clear the rule that, regardless of the time period, the weighted mean of the annual default probabilities (Subsection 3.1) is greater than the annual default probability regarding aggregated data (Subsection 3.2) and lower than the annual default probability taking annualized data with upward rounding (Subsection 3.2 too)[23]. As mentioned earlier, the law of large numbers demonstrates that the annual default probabilities obtained from aggregated data (with $n = \sum_{t=1}^{T} n_t$ and $k = \sum_{t=1}^{T} k_t$) are consistently much lower than those obtained from annualized data (with $n = \sum_{t=1}^{T} w_{t,T}.n_t$ and $k = \sum_{t=1}^{T} w_{t,T}.k_t$).

Second, figures and trends related to the three series of default probabilities are equivalent for both the classical and Bayesian approaches, which allow to see that adopting annual default probabilities under the classical approach with a confidence level of 75% and annual default probabilities employing the Bayesian approach with a uniform distribution]0, 0.1] as the prior function are two very good options[24].

However, neither (i) the probabilities derived from the aggregated data nor (ii) those derived from the annualized data with upward rounding could be considered appropriate methods. (i) Probabilities obtained from aggregated data rely on an inaccurate empirical default rate, $\sum_{t=1}^{T} k_t / \sum_{t=1}^{T} n_t$, that covers several years (instead of a rate that is applicable for just a one-year period). (ii) Furthermore, despite the fact that the empirical default rate get by adding "k_t" and "n_t" (t = 1, 2, …, T) is exactly or nearly equal to "k/n" for a specific t-year[25], the default probability computed from annualized data could be significantly higher than the weighted mean of the annual default probabilities.

In short, when data from default time series is available, for portfolios with few default events it is advisable the mean of several annual default probabilities be utilized as a benchmark for estimating long-run probabilities.

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[23] When $T = 4$, $T = 5$, and $T = 6$, there are minor deviations from the norm in the moderate scenario of the Bayesian approach (Figure 2), explained solely by the weights employed to calculate the weighted mean of the annual default probabilities – the greatest deviation is 14 bp (for $T = 5$). If the ordinary mean had been adopted instead of the weighted mean, there would have been no exceptions to the rule; for $T = 4$, $T = 5$, and $T = 6$, the ordinary means are 2.66%, 2.84%, and 2.86%, respectively (see Table 7), which are higher than the corresponding weighted means of the annual default probabilities as well as default probabilities derived from aggregated data.

[24] Classical and Bayesian default probabilities are very close practically all the time. The maximum deviation is 43 (42.83) bp (for the moderate scenario, aggregate data, and $T = 6$), and the minimum deviation is 0 (0.02) bp (for the moderate scenario, weighted mean of the annual default probabilities, and $T = 6$).

[25] In the extreme scenario, $\sum_{t=1}^{T} k_t / \sum_{t=1}^{T} n_t = k/n$, $\forall T$, happens since "k_t" and "n_t" (rounded upward) remain constant over "t" ($k_t = 1$ and $n_t = 250$ – see Table 3 and, for $T = 1$, Table 2). For $T = 6$, $\sum_{t=1}^{6} k_t / \sum_{t=1}^{6} n_t$ = (6x1) / (6x250) = 6 / 1500 = 0.40%; and for $T = 1$, k_1/n_1 = 1 / 250 = 0.40%.

In the moderate scenario (see Table 3 and, for $T = 1$, Table 2), for $T = 6$, $\sum_{t=1}^{6} k_t / \sum_{t=1}^{6} n_t$ = (2+2+2+2+2+2) / (240+240+241+241+242+242) = 0.8299%; for $T = 3$, $\sum_{t=1}^{3} k_t / \sum_{t=1}^{3} n_t$ = (2+2+2) / (240+240+241) = 0.8322%; and for $T = 1$, k_1/n_1 = 2 / 242 = 0.8333%.

4. Multi-year Estimation of Default Probabilities with Binomial Distribution

4.1 Some Specificities

In Subsections 3.1 and 3.2, annual default probabilities were computed via cross-sectional data. A procedure like this assumes both positive asset correlation (annual correlation among obligors in the same year) and – see the last sentence of the first paragraph in Section 3 – null intertemporal correlation (interannual independence among default events over a number of years).

Conversely, this subsection has a different context. Even with existing time series data, it is now assumed that the default probabilities are calculated on a multi-year basis instead of a year basis. The positive asset correlation remains within the same year, but another type of correlation over a time period of various years is required to allow for the methodological extension of one-period probabilities to multi-period probabilities: the intertemporal correlation.

Multi-period estimation techniques are substantially more complex and require incomparably more difficult programming, time consumption, and memory processing than one-period estimations since numerical computations must be performed under a multi-dimensional integration. Simulation is used to determine the multi-period default probabilities by applying the joint probability density function to the default distributions. Those probabilities include some additional technical details. Such details are developed in both classical and Bayesian approaches and are common for any types of default distribution (namely binomial or Poisson distributions).

One of details is the intertemporal correlation structure, "τ", that was previously described. This correlation coefficient is necessary to know the variance-covariance matrix, "Σ_τ", in order to produce the joint probability density function for the standard multivariate normal distribution of "T" random variables reflecting systematic risk, "Y_1", "Y_2", …, and "Y_T". Because this random vector, $[Y_1 \;\; Y_2 \;\; \ldots \;\; Y_T]$, exists and is connected to "T" standard univariate normal distributions, the joint probability must be calculated utilizing "T" dimensional integrals.

4.2 Joint Default Probabilities

4.2.1 Formulae

Classical approach

So far, the multi-year probability that a group of "n" obligors will experience no more than "k" defaults for the period of "T" years is given by[26]:

$$\mathrm{P}(K_T \leq k) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left\{ \sum_{i=0}^{k} \binom{n}{i} . \left[1 - \prod_{t=1}^{T} \left[1 - G\left(\lambda_{T,C}, y_t, \rho\right)\right]\right]^{i} . \right.$$
$$\left. . \left[\prod_{t=1}^{T} \left[1 - G\left(\lambda_{T,C}, y_t, \rho\right)\right]\right]^{n-i} \right\} . \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) \geq 1 - \omega \ , \quad (13)$$

with

$$G\left(\lambda_{T,C}, y_t, \rho\right) = \Phi\left\{\left[\Phi^{-1}\left(\lambda_{T,C}\right) - y_t . \sqrt{\rho}\right] / \sqrt{1-\rho}\right\} \quad (14)$$

and

$$\phi_T(y_1, \dots, y_T; \tau) = \exp\left[-\left(U^{\mathrm{T}} . \Sigma_\tau^{-1} . U\right)/2\right] / \left[(2\pi)^{T/2} . \sqrt{|\Sigma_\tau|}\right] \quad (15)$$

The random variable "K_T" represents the number of "k" defaults that occur annually within a group of "n" obligors. Again, it is assumed that "K_T" follows a correlated binomial distribution and that default events have a ρ-positive asset correlation. The definitions of "$\Phi(.)$" and "$\Phi^{-1}(.)$" are the same as those that have already been identified. "$\lambda_{T,C}$" indicates the multi-year classical default probability. "ω" expresses the chosen confidence level. Equation (14) is similar to Equation (2).

In this context, "Y_t" refers for a multivariate distribution that contains a set of standard and normally distributed random variables ("Y_1", "Y_2", …, and "Y_T") and encompasses all possible values of "y_t", t = 1, 2, …, T, which stand for the full range of the systematic risk. The joint multivariate standard normal probability density function of "Y_t" is "$\phi_T(y_1, \dots, y_T; \tau)$"; it has a mean of 0 and a variance-covariance matrix "Σ_τ", being "τ" the intertemporal correlation parameter. "U" depicts the column vector of random variables and "U^{T}" denotes its transpose, $U^{\mathrm{T}} = [Y_1 \ \ Y_2 \ \ \dots \ \ Y_T]$. "$\Sigma_\tau^{-1}$" and "$|\Sigma_\tau|$" represent the inverse of the variance-covariance matrix and the determinant of this matrix, respectively.

[26] Similar to formulations (6.8a) and (6.8c) from Pluto and Tasche (2005).

Two different types of function are tested for each "$\tau_{r,c}$", the τ-positive value, $\tau \neq 1$, related to the row "r" and the column "c" of the matrix "Σ_τ", $r \times c$:

C – Constant function

$$\tau_{r,c} = \begin{cases} 1 & , \quad r = c \\ \tau & , \quad r \neq c \end{cases}$$

E – Exponential function (Pluto and Tasche, 2005)

$$\tau_{r,c} = \begin{cases} 1 & , \quad r = c \\ \tau^{|r-c|} & , \quad r \neq c \end{cases}$$

Hence, the variance-covariance matrix of the systematic risk, a $T \times T$, positive, and symmetric matrix, is generated for each type of "$\tau_{r,c}$" function by:

C – Constant function

$$\Sigma_\tau = \begin{bmatrix} 1 & \tau & \cdots & \tau \\ \tau & 1 & \cdots & \tau \\ \vdots & \vdots & \ddots & \vdots \\ \tau & \tau & \cdots & 1 \end{bmatrix} \tag{16}$$

E – Exponential function

$$\Sigma_\tau = \begin{bmatrix} 1 & \tau & \cdots & \tau^{T-1} \\ \tau & 1 & \cdots & \tau^{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tau^{T-1} & \tau^{T-2} & \cdots & 1 \end{bmatrix} \tag{17}$$

Bayesian approach

Given a set of "n" borrowers, the expected value of the multi-year Bayesian probability (or the posterior probability, "λ") that "k" defaults (and not at least "k" defaults as required in the classical approach) will occur within a period of "T" years, "μ_T", is given by[27]:

$$\mu_T = \int_0^{\mathrm{u}} \lambda . f(\lambda) . \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} [\prod_{t=1}^{T} [G(\lambda, y_t, \rho)]^k . [1 - G(\lambda, y_t, \rho)]^{n-k}] .$$
$$. \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) d\lambda.$$
$$. 1/\{\int_0^{\mathrm{u}} f(\lambda) . \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} [\prod_{t=1}^{T} [G(\lambda, y_t, \rho)]^k . [1 - G(\lambda, y_t, \rho)]^{n-k}] .$$
$$. \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) d\lambda\} \; ^{28}, \tag{18}$$

with

$$G(\lambda, y_t, \rho) = \Phi\{[\Phi^{-1}(\lambda) - y_t . \sqrt{\rho}] / \sqrt{1 - \rho}\} \tag{19}$$

The joint multivariate standard normal probability density function of "Y_t" – i.e., "$\phi_T(y_1, \dots, y_T; \tau)$" – is defined by Equation (15). The variance-covariance matrix is obtained by Equations (16) and (17). Equation (19) is similar to Equation (6). The definitions given above apply to the variables "n", "k", "ρ", "$\Phi(.)$", "$\Phi^{-1}(.)$", "Y_t", "y_t", "τ", "U", "U^{T}", "Σ_τ^{-1}", and "$|\Sigma_\tau|$". The prior function of the default probability "λ" is denoted by "$f(\lambda)$".

4.2.2 Results

Classical approach

Table 10 displays the default probabilities estimated for each period of "T" years using "n" and "k" derived from the annualized data (so, the weighted mean of "n" and "k" observed in each t-year, t = 1, 2, …, T, identified in Table 3).

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[27] It is an adaptation of the formula (23c) included in Tasche (2012).

[28] Since the binomial coefficient is a constant and has the same effect on the likelihood function's numerator and denominator, one can eliminate the "$\binom{n}{k}$" factor. See Equation (5).

Table 10: Multi-period default probabilities under the classical approach with annualized data (ρ = 12%)

Types of scenario	T	n	k	k/n	τ	Confidence levels		
						65%	75%	95%
Moderate scenario	2	240	2	0,83%	0% C	0.99%	1.26%	2.66%
					50% E	1.11%	1.45%	3.30%
					50% C	1.11%	1.45%	3.30%
					99.9% C	1.27%	1.69%	4.08%
	3	241	2	0,83%	0% C	0.61%	0.77%	1.56%
					50% E	0.71%	0.93%	2.08%
					50% C	0.73%	0.96%	2.19%
					99.9% C	0.89%	1.20%	3.01%
	4	241	2	0,83%	0% C	0.44%	0.55%	1.08%
					50% E	0.52%	0.67%	1.47%
					50% C	0.55%	0.72%	1.66%
					99.9% C	0.70%	0.95%	2.44%
	5	242	2	0,83%	0% C	0.34%	0.42%	0.82%
					50% E	0.40%	0.51%	1.12%
					50% C	0.44%	0.58%	1.34%
					99.9% C	0.57%	0.78%	2.06%
	6	242	2	0,83%	0% C	0.24%	0.30%	0.58%
					50% E	0.28%	0.36%	0.75%
					50% C	0.31%	0.40%	0.89%
					99.9% C	0.40%	0.53%	1.35%
Extreme scenario	2	250	1	0,40%	0% C	0.64%	0.84%	1.94%
					50% E	0.72%	0.97%	2.41%
					50% C	0.72%	0.97%	2.41%
					99.9% C	0.83%	1.14%	3.00%
	3	250	1	0,40%	0% C	0.40%	0.51%	1.13%
					50% E	0.46%	0.62%	1.51%
					50% C	0.48%	0.64%	1.59%
					99.9% C	0.59%	0.81%	2.22%
	4	250	1	0,40%	0% C	0.28%	0.36%	0.78%
					50% E	0.33%	0.44%	1.07%
					50% C	0.36%	0.48%	1.20%
					99.9% C	0.46%	0.64%	1.79%
	5	250	1	0,40%	0% C	0.22%	0.28%	0.59%
					50% E	0.26%	0.34%	0.81%
					50% C	0.29%	0.39%	0.97%
					99.9% C	0.38%	0.53%	1.51%
	6	250	1	0,40%	0% C	0.18%	0.23%	0.47%
					50% E	0.21%	0.28%	0.64%
					50% C	0.24%	0.32%	0.81%
					99.9% C	0.32%	0.46%	1.32%
C - Constant function used for the variance-covariance matrix								
E - Exponential function used for the variance-covariance matrix								

Some strange and striking conclusions can be drawn from the table. The multi-period default probabilities are lower the longer the time period, "T", regardless of the intertemporal correlation parameter, "τ". In the moderate scenario, for instance, the probability for $T = 2$ with $\tau = 99.9\%$ and a 75% confidence level is 1.69%, whereas it is just 0.53% for $T = 6$ (by contrasting with empirical default rates for this time period, $k/n = 0.83\%$).
Additionally, multi-period default probabilities generally are lower than empirical default rates. The 95% confidence level is the major exception[29]. Another clear exception is observed in both scenarios when $T = 2$ at all confidence levels.
Going back to Table 4 helps to strengthen and demonstrate how highly imprudent and so invalid multi-period default probabilities are. At a 95% confidence level in the moderate scenario, the one-year default probability corresponds to 6.790% when $n = 240$ and $k = 2$. Conversely, in that scenario, for $T = 2$ with the same confidence level and the same "n" and "k", default probabilities calculated by Equation (13) range from 2.664% ($\tau = 0\%$) to 4.082% ($\tau = 99.9\%$). Pay attention to the additional two examples – $T = 2$ and $T = 6$ – for the extreme scenario also for the 95% confidence level. For $n = 250$ and $k = 1$, the annual default probability is 5.039% (Table 4), whilst the multi-period probabilities range from 1.935% ($\tau = 0\%$) to 3.004% ($\tau = 99.9\%$) for $T = 2$, and from 0.470% ($\tau = 0\%$) to 1.315% ($\tau = 99.9\%$) for $T = 6$[30].
The probability estimations in the preceding paragraph were all made under the same asset correlation assumption, $\rho = 12\%$. For $T = 2$ in the moderate scenario, the default probability obtained by Equation (13) would be 6.790% (Table 4) instead of 2.664% or 4.082% (Table 10) if the asset correlation increased, respectively, from $\rho = 12\%$ to $\rho = 33\%$ for $\tau = 0\%$, or from $\rho = 12\%$ to $\rho = 20\%$ for $\tau = 99.9\%$. In the extreme scenario, again for $T = 2$, the probabilities would increase from 1.935% or 3.004% (Table 10) to 5.039% (Table 4) if asset correlation were 32% for $\tau = 0\%$ or 19% for $\tau = 99.9\%$. For $T = 6$, and similarly in the extreme scenario, the probabilities would also be 5.039% instead of 0.470% or 1.315% adopting respectively 69% or 29% as asset correlation, rather than 12%. Therefore, the asset correlation, "ρ", should be higher the longer the time period, "T", in order to balance the one-year estimation and the multi-year estimation, which is another feature that demonstrates the invalidity of the multi-period case.
Now, have a look at Tables 5 and 8. For the moderate scenario with a 75% confidence level, $T = 6$, and $\rho = 12\%$, the weighted mean of the annual default

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[29] Indeed, only for the 95% confidence level the vast majority of the default probabilities – 37 out of 40 figures – exceeds the empirical default rate. They do not exceed for the moderate scenario, at $T = 5$ and $T = 6$.

[30] Thus, for the same "ω", "n", and "k", the annual default probability (5.039%) is 2.6 or 1.7 times higher than the multi-period probabilities (respectively for $\tau = 0\%$ or $\tau = 99.9\%$) when $T = 2$, and 10.7 or 3.8 times higher when $T = 6$. This is another example that highlights the inconsistency of multi-period estimation.

probabilities is 2.64% (Table 5), and the one-year default probability with annualized data is 2.98% (Table 8). It must be stressed that regarding $\rho = 0\%$, those probabilities are 1.43% and 1.61%, significantly exceeding the multi-period probabilities of 0.30% or 0.53% which were obtained with $\rho = 12\%$ (not $\rho = 0\%$), and $\tau = 0\%$ or $\tau = 99.9\%$, respectively, for the same scenario, confidence level, and time period (Table 10).

Consequently, it is concluded that adopting annual default probabilities without considering asset correlation would be significantly and illogically more prudent than using multi-period default probabilities with either asset correlation or intertemporal correlation[31]. Section 5 discusses why the multi-period estimations under the classical approach are completely imprudent and, thus, they should not be employed.

When (a) default probabilities are obtained from empirical data gathered over several years, it is expected to discuss conservatism (or, conversely, imprudence). However, if (b) the goal is not to assess the conservatism level but rather to predict a default probability over a period longer than one year, using multi-period default probabilities can only be described as an economically invalid approach, regardless of its theoretical possibility.

Therefore, when assessing the multi-period issue, it is crucial to distinguish between two immiscible perspectives: (a) recorded data and (b) default probability. It is important to underline that the prudence is fully compatible with the time series. As previously noted, (a) using empirical data from a pluriannual period to estimate a one-year default probability – a logical and expected procedure – greatly contrasts with (b) using that empirical data to estimate a multi-year default probability – which, in this context, is an imprudent and incorrect procedure[32].

The key issue is that the annual default probability might rely on historical data. Employing historical data is an accurate method. Conversely, an incorrect method is to employ historical data to calculate a multi-year default probability[33]. Using the law of large numbers and longer historical data periods to reduce the necessary

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[31] Off course, for $T = 2$ the differences are not very noticeable. The corresponding probabilities for $T = 2$ stated in Tables 5 and 8 at 2.25% and 3.00% would be 1.21% and 1.63%, respectively, if $\rho = 0\%$, compared to 1.26% ($\tau = 0\%$) or 1.69% ($\tau = 99.9\%$) present in Table 10 with $\rho = 12\%$.

[32] It makes sense that the longer the time period concerning default events, the less conservatism is required for estimating the one-year default probabilities.

For instance, one takes $n = 250$ for $t = 1, 2, 3, 4, 5$, with $k = 0$ for $t = 1$ and $t = 4$, $k = 2$ for $t = 2$ and $t = 5$, and $k = 1$ for $t = 3$. Consequently, for $t = 5$ with $n = 250$ and $t = 2$, applying $\rho = 12\%$, the annual default probability at a 75% confidence level is 2.90%. However, using the mean for that five-year period, $n = 5\text{x}250/5 = 250$ and $k = (0+2+1+0+2)/5 = 1$, the corresponding annual default probability (at a 75% confidence level and $\rho = 12\%$ too) is 2.04% (not 2.90%). For $n = 250$ and $k = 2$ (data from the fifth year), a default probability of 2.04% corresponds to a confidence level of 61.9%. As a result, in this situation, utilizing the available pluriannual data allows an unjustified and illogical decrease in the level of prudence from 75% to 61.9%.

[33] In the mentioned case of the previous footnote, over the same five years of historical data for "n" and "k", the multi-period default probability for the extreme scenario ranges from (see Table 10) 0.28% (for $\tau = 0\%$) to 0.53% (for $\tau = 99.9\%$), in contrast to 2.04%.

prudence margin in estimating default probabilities is an arbitrary and wrong criterion[34].
Ultimately, it is important to recognize that it is common for two people to achieve different outcomes, as their simulation processes might not be entirely the same, depending on the number of simulations performed. For instance, employing the same methodology applied to obtain the values presented in Table 10 – thus, Equation (13) – with $T = 5$, $n = 300$, $k = 1$, $\rho = 12\%$, and $\tau = 30\%$ for the exponential function, the results for the confidence levels $\omega = 50\%$, $\omega = 75\%$, $\omega = 90\%$, $\omega = 95\%$, $\omega = 99\%$, and $\omega = 99.9\%$[35] are 0.14%, 0.26%, 0.44%, 0.59%, 0.98%, and 1.65%, respectively. These multi-period default probabilities are quite similar – therefore, they are not precisely the same – at 0.15%, 0.27%, 0.46%, 0.61%, 1.01%, and 1.70% as reported by Pluto and Tasche (2005) for the classical multi-period scenario[36]. Hence, for $\omega = 50\%$ and $\omega = 75\%$, the default probabilities are lower than the empirical default rate (1/300 = 0.33%) for that low default portfolio, which is incorrect because confidence levels should consistently be greater than the empirical default rate.

Bayesian approach
Table 11 shows the multi-period default probabilities under the Bayesian approach for the same scenarios, time periods, and intertemporal correlation coefficients as those previously discussed for the classical approach. The three prior functions described above are utilized – two uniform distributions and one beta distribution.

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[34] The default probabilities of 0.28% and 0.53% – linked to the 75% confidence level for $T = 5$ (not $t = 5$) – correspond to highly unrealistic and imprudent confidence levels of 16.0% and 30.4%, respectively, for the one-year default probability.
[35] See footnote 12.
[36] See Table 14 on page 17 of Pluto and Tasche (2005). Although in that paper of 2005 the joint multivariate standard normal probability density function has not been explicitly mentioned, it was used for calculating the default probabilities corresponding to each confidence level.

Table 11: Multi-period default probabilities using the Bayesian approach with annualized data (ρ = 12%)

Types of scenario	*T*	*n*	*k*	*k/n*	τ	Uniform distribution regarding]0%, 100%[(a)	Uniform distribution Regarding]0%, 10%]	Beta distribution regarding EDR as the mean (b)
Moderate scenario	2	240	2	0,83%	0% C	2.15%	2.14%	1.31%
					50% E	2.59%	2.51%	1.40%
					50% C	2.59%	2.51%	1.40%
					99.9% C	3.11%	2.88%	1.53%
	3	241	2	0,83%	0% C	1.78%	1.77%	1.24%
					50% E	2.22%	2.20%	1.34%
					50% C	2.33%	2.30%	1.36%
					99.9% C	3.01%	2.82%	1.52%
	4	241	2	0,83%	0% C	1.60%	1.60%	1.24%
					50% E	2.01%	2.00%	1.30%
					50% C	2.21%	2.19%	1.34%
					99.9% C	2.97%	2.79%	1.52%
	5	242	2	0,83%	0% C	1.47%	1.47%	1.21%
					50% E	1.85%	1.85%	1.28%
					50% C	2.14%	2.12%	1.34%
					99.9% C	2.94%	2.77%	1.51%
	6	242	2	0,83%	0% C	1.42%	1.42%	1.19%
					50% E	1.75%	1.74%	1.24%
					50% C	2.08%	2.07%	1.31%
					99.9% C	2.85%	2.80%	1.47%
Extreme scenario	2	250	1	0,40%	0% C	1.35%	1.34%	0.69%
					50% E	1.67%	1.66%	0.76%
					50% C	1.67%	1.66%	0.76%
					99.9% C	2.08%	2.01%	0.86%
	3	250	1	0,40%	0% C	1.04%	1.04%	0.63%
					50% E	1.37%	1.37%	0.70%
					50% C	1.44%	1.44%	0.73%
					99.9% C	1.96%	1.92%	0.86%
	4	250	1	0,40%	0% C	0.91%	0.91%	0.63%
					50% E	1.19%	1.19%	0.68%
					50% C	1.34%	1.33%	0.71%
					99.9% C	1.91%	1.87%	0.86%
	5	250	1	0,40%	0% C	0.82%	0.82%	0.61%
					50% E	1.08%	1.08%	0.66%
					50% C	1.27%	1.27%	0.70%
					99.9% C	1.88%	1.85%	0.85%
	6	250	1	0,40%	0% C	0.78%	0.77%	0.61%
					50% E	1.01%	1.00%	0.71%
					50% C	1.25%	1.23%	0.81%
					99.9% C	1.94%	1.82%	1.08%

(a) - Likelihood function

(b) - Variance of beta distribution corresponds to EDR/1.645. For beta distribution, default probabilities range between 0% and 100%.

EDR - Empirical default rate (*k/n*)

C - Constant function used for the variance-covariance matrix

E - Exponential function used for the variance-covariance matrix

The Bayesian approach exhibits the same rule as the classical approach in that the multi-period default probabilities decrease with increasing the time period, regardless of the intertemporal correlation structure. However, the Bayesian approach does not emphasize this rule as strongly as the classical approach does. The cause of this will also be discussed in detail in Section 5.

The significant difference (in comparison to the classical approach) is explained by the probabilities themselves. All multi-period default probabilities are now higher than empirical default rates[37]. Naturally, default probabilities generated with the total uninformative uniform distribution]0, 1[– the antepenultimate column of the table – or the quasi-uninformative uniform distribution]0, 0.1] – the penultimate column – are higher than those produced with the beta distribution as an informative prior function adopting the empirical default rates – the last column[38]. Because of this, the uniform distribution]0, 0.1] as the prior function appears to be the best at first glance, but it is necessary to make sure that its level of conservatism is (in)sufficient.

For this purpose, the matching confidence levels for the classical approach corresponding to multi-period Bayesian probabilities – a procedure similar to Table 6 – can be obtained. Table 12 provides that matching as well as the matching for the weighted mean of one-year default probabilities to facilitate comparisons between multi-period probabilities and annual probabilities. The last two columns of Table 12 display the confidence levels linked to the multi-year and the annual estimations of default probabilities for the classical approach employing Equation (13) and Equation (1), respectively. Those confidence levels were obtained from the data presented in the columns labelled «Bayesian default probabilities».

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[37] In comparison, multi-period default probabilities derived from the classical approach generally are lower than empirical default rates.

[38] The differences between probabilities from a uniform distribution and those from a beta distribution decrease as the time period extends. Such differences are greater the longer the intertemporal correlation.

Table 12: Multi-period default probabilities using the Bayesian approach and corresponding matching confidence levels (ρ = 12%)

Types of scenario	T	n	k	τ	Bayesian default probabilities		Matching confidence levels for the classical approach (a)	
					Multi-period with uniform distribution regarding]0%, 10%]	One-year (b)	Multi-period	One-year
Moderate scenario	2	240	2	0% C	2.14%	2.34%	62.4%	65.8%
				50% E	2.51%		68.5%	
				50% C	2.51%		68.5%	
				99.9% C	2.88%		73.5%	
	3	241	2	0% C	1.77%	2.36%	55.1%	66.4%
				50% E	2.20%		63.6%	
				50% C	2.30%		65.3%	
				99.9% C	2.82%		72.8%	
	4	241	2	0% C	1.60%	2.46%	51.0%	67.9%
				50% E	2.00%		59.9%	
				50% C	2.19%		63.4%	
				99.9% C	2.79%		72.5%	
	5	242	2	0% C	1.47%	2.62%	47.9%	70.4%
				50% E	1.85%		56.9%	
				50% C	2.12%		62.3%	
				99.9% C	2.77%		72.3%	
	6	242	2	0% C	1.42%	2.64%	46.6%	70.6%
				50% E	1.74%		54.5%	
				50% C	2.07%		61.4%	
				99.9% C	2.80%		72.7%	
Extreme scenario	2	250	1	0% C	1.34%	1.86%	61.0%	72.0%
				50% E	1.66%		68.2%	
				50% C	1.66%		68.2%	
				99.9% C	2.01%		74.4%	
	3	250	1	0% C	1.04%	1.76%	52.1%	70.3%
				50% E	1.37%		61.5%	
				50% C	1.44%		63.4%	
				99.9% C	1.92%		73.0%	
	4	250	1	0% C	0.91%	1.67%	47.2%	68.4%
				50% E	1.19%		56.8%	
				50% C	1.33%		60.7%	
				99.9% C	1.87%		72.2%	
	5	250	1	0% C	0.82%	1.62%	44.0%	67.4%
				50% E	1.08%		53.2%	
				50% C	1.27%		59.1%	
				99.9% C	1.85%		71.8%	
	6	250	1	0% C	0.77%	1.57%	41.6%	66.3%
				50% E	1.00%		50.5%	
				50% C	1.23%		57.8%	
				99.9% C	1.82%		71.2%	

(a) - Corresponding to annual default probabilities for each "T", "n", and "k"
(b) - Weighted mean of the annual default probabilities with the uniform distribution regarding]0%, 10%] (see Table 7)
C - Constant function used for the variance-covariance matrix
E - Exponential function used for the variance-covariance matrix

With the exception of adopting the maximum intertemporal correlation (99.9%), the results of the table allow to draw the conclusion that multi-period probabilities under the Bayesian approach (using the uniform distribution]0, 0.1] as the prior function benchmark) are imprudent since the corresponding matching confidence levels do not reach 70% for any time period. The matching confidence levels for other coefficients of intertemporal correlation are almost always less than 65% (except for $T = 2$ in both scenarios and $T = 3$ in the moderate scenario with the fixed intertemporal correlation of 50%). Additionally, it is obvious that as the time period increases, the matching confidence level drops. This is directly evident in the extreme scenario because "n" and "k" assume the same values throughout the T-time period.

Comparing the multi-period and one-year Bayesian default probabilities stated in the table reveals that the latter is nearly higher than the former (with the exception of the cases where $\tau = 99.9\%$, regardless of the scenario and the T-time period, and where $\tau = 50\%$ and $T = 2$, in the moderate scenario). The final column of the table shows that the weighted mean of the Bayesian annual default probabilities with the uniform distribution ranging between 0 and 0.1 as the prior function is suitably prudent since the respective matching confidence levels for the classical approach are between 66% and 71% for the moderate scenario or between 66% and 72% for the extreme scenario.

Another angle on the inconsistency of assessing default probabilities through the multi-period estimations is provided by computing the confidence levels – which are extracted using the cumulative distribution functions via Bayesian likelihood functions of each "T", thus not the matching confidence levels for the classical approach – across "T" for the same default probability value. For $n = 250$, $k = 1$, and $\rho = 12\%$, the annual default probability under the Bayesian approach utilizing the uniform distribution between 0 and 0.1 as the prior function is 2.249% (Table 6), which corresponds to the 62.5% confidence level (for $T = 1$) – once more, not the matching confidence level, which is 77.9% (Table 6 too). For each cumulative distribution function (associated with the different Bayesian likelihood functions related to the T-time period), the confidence levels for the same default probability of 2.249% linked to $T = 2$, $T = 3$, $T = 4$, $T = 5$, and $T = 6$ when $\tau = 0\%$ are, *ceteris paribus*, 84.3%, 93.2%, 97.0%, 98.7%, and 99.9%. With $\tau = 99.9\%$, these five confidence levels are significantly lower, ranging from 67.7% to 71.0%. It is logical to assert that as "τ" increases, the confidence level for a given probability decreases; however, it is incorrect to claim that the probability increases with a greater "T" for the same "n", "k", "ρ", and "τ".

Given the foregoing, one observes that it is theoretically conceivable to estimate Bayesian default probabilities by adopting the multi-period methodology, but it will need extra attention depending on the time period "T" and the intertemporal correlation structure "τ" (only if the correlation figure is high). However, as stated in Section 5, the dependability of the outputs is much dubious.

According to Table 6, the default probability in the moderate scenario for $t = 1$ ($n = 240$ and $k = 2$) is 3.026%, and its matching confidence level is 75.2%; the default probability in the extreme scenario ($n = 250$ and $k = 1$) is 2.249%, and its matching confidence level is 77.9%. Therefore, such default probabilities (and the corresponding matching confidence levels) are greater than any multi-period default probabilities (and their matching confidence levels), regardless of the time period and the intertemporal correlation. As mentioned earlier, incorporating pluriannual data to estimate annual default probabilities is logical and expected; however, leveraging such data to calculate multi-year default probabilities would be a subjective and arbitrary approach, as multi-period estimations result in lower default probabilities than the annual default probabilities, which is an imprudent procedure.

5. One-year versus Multi-year Estimation of Default Probabilities

Table 13 combines some information from Tables 5 and 10 for the classical approach and some information from Tables 7 and 11 for the Bayesian approach. It allows two comparisons: one between the multi-year methodology and the one-year methodology (more precisely, the weighted mean of the one-year default probabilities) and the other between the classical approach and the Bayesian approach. As was seen in the last sentence of the first paragraph of Subsection 3.3, the confidence level of 75% and the uniform distribution between 0 and 0.1 as the prior function were chosen to make the comparison between the classical and Bayesian approaches more understandable. The same choice is suitable for multi-period estimations.

Table 13: Multi-period and one-year default probabilities under the classical and Bayesian approaches (ρ = 12%)

Types of scenario	T	n	k	τ	Multi-year default probabilities		Annual default probabilities		(A) - (C)	(B) - (D)
					Classical Approach (1) (A)	Bayesian Approach (2) (B)	Classical Approach (1) (C)	Bayesian Approach (2) (D)		
Moderate scenario	2	240	2	0% C	1.26%	2.14%	2.25%	2.34%	-1.0%	-0.2%
				50% E	1.45%	2.51%			-0.8%	0.2%
				50% C	1.45%	2.51%			-0.8%	0.2%
				99.9% C	1.69%	2.88%			-0.6%	0.5%
	3	241	2	0% C	0.77%	1.77%	2.26%	2.36%	-1.5%	-0.6%
				50% E	0.93%	2.20%			-1.3%	-0.2%
				50% C	0.96%	2.30%			-1.3%	-0.1%
				99.9% C	1.20%	2.82%			-1.1%	0.5%
	4	241	2	0% C	0.55%	1.60%	2.41%	2.46%	-1.9%	-0.9%
				50% E	0.67%	2.00%			-1.7%	-0.5%
				50% C	0.72%	2.19%			-1.7%	-0.3%
				99.9% C	0.95%	2.79%			-1.5%	0.3%
	5	242	2	0% C	0.42%	1.47%	2.62%	2.62%	-2.2%	-1.2%
				50% E	0.51%	1.85%			-2.1%	-0.8%
				50% C	0.58%	2.12%			-2.0%	-0.5%
				99.9% C	0.78%	2.77%			-1.8%	0.1%
	6	242	2	0% C	0.30%	1.42%	2.64%	2.64%	-2.3%	-1.2%
				50% E	0.36%	1.74%			-2.3%	-0.9%
				50% C	0.40%	2.07%			-2.2%	-0.6%
				99.9% C	0.53%	2.80%			-2.1%	0.2%
Extreme scenario	2	250	1	0% C	0.84%	1.34%	1.66%	1.86%	-0.8%	-0.5%
				50% E	0.97%	1.66%			-0.7%	-0.2%
				50% C	0.97%	1.66%			-0.7%	-0.2%
				99.9% C	1.14%	2.01%			-0.5%	0.1%
	3	250	1	0% C	0.51%	1.04%	1.56%	1.76%	-1.0%	-0.7%
				50% E	0.62%	1.37%			-0.9%	-0.4%
				50% C	0.64%	1.44%			-0.9%	-0.3%
				99.9% C	0.81%	1.92%			-0.7%	0.2%
	4	250	1	0% C	0.36%	0.91%	1.46%	1.67%	-1.1%	-0.8%
				50% E	0.44%	1.19%			-1.0%	-0.5%
				50% C	0.48%	1.33%			-1.0%	-0.3%
				99.9% C	0.64%	1.87%			-0.8%	0.2%
	5	250	1	0% C	0.28%	0.82%	1.42%	1.62%	-1.1%	-0.8%
				50% E	0.34%	1.08%			-1.1%	-0.5%
				50% C	0.39%	1.27%			-1.0%	-0.3%
				99.9% C	0.53%	1.85%			-0.9%	0.2%
	6	250	1	0% C	0.23%	0.77%	1.37%	1.57%	-1.1%	-0.8%
				50% E	0.28%	1.00%			-1.1%	-0.6%
				50% C	0.32%	1.23%			-1.0%	-0.3%
				99.9% C	0.46%	1.82%			-0.9%	0.2%

(1) - Confidence level of 75%

(2) - Uniform distribution regarding]0%, 10%] as the prior function

C - Constant function used for the variance-covariance matrix

E - Exponential function used for the variance-covariance matrix

Four notes are taken from the table. First, the multi-period methodology for the classical approach is obviously dubious or incorrect, hence it should not be employed (as mentioned in the seventh paragraph of Sub-subsection 4.2.2). Second, the multi-period methodology for the Bayesian approach might only be feasible if the intertemporal correlation is set to a very high value (as identified in the penultimate paragraph of Sub-subsection 4.2.2 too)[39]. Third, comparing columns (A) and (B) reveals that there is no valid relationship between the multi-year default probabilities derived from the classical and Bayesian approaches[40]. Fourth, in contrast to what is said in the previous sentence, the annual default probabilities presented in columns (C) and (D) are logical and consistent, as the default probability obtained through the classical approach aligns with the one obtained via the Bayesian approach (regardless of the values of "n" and "k" and the T-time period); therefore, their methodologies are acceptable for internal models[41].
It is necessary to develop those first and second findings for the multi-period methodology. The differences between the classical multi-period probabilities and the weighted mean of the annual probabilities – column "(A) - (C)" – are always negative for any "T" and any "τ", regardless of the scenario. The following analytical reason explains the differences.
Equation (13) illustrates the influence of the joint multivariate standard normal probability density function, "$\phi_T(y_1, \dots, y_T; \tau)$", on the multi-period probability (that there will be no more than "k" defaults in a portfolio with "n" obligors). The univariate standard normal distribution function of the random variable "Y" (so, $T = 1$ year), $\phi(y)$, has the maximum density at $y = 0$, which corresponds to 0.39894, the density of the mean, median, and mode of the distribution. For a multivariate standard normal distribution function of the random variables "Y_1", "Y_2", …, and "Y_T", assuming a null correlation coefficient ($\tau = 0\%$), the density at $y_1 = y_2 = \dots = y_T = 0$ is 0.39894^T – it is the normalization constant derived from Equation (15), i.e., $(2\pi)^{-(T/2)}$. One concludes that, for the same "n" and "k" values, the multi-period default estimations are worse the longer the period of "T" years is, which is economically illogical.
Therefore, the computation of multi-period default probabilities utilizes the joint

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[39] The Bayesian multi-year default probabilities might not serve as a suitable benchmark because, firstly, Equation (18) depends on the joint multivariate standard normal probability density function – although the impact is lessened as it affects both the numerator and denominator – and, secondly, the multi-period default probabilities are unwarrantedly more optimistic compared to the annual probabilities unless "τ" takes very high values.
[40] The default probabilities in column (A) are lower than the corresponding probabilities contained in column (B). In addition, it should be noted that the probabilities in column (A) for the maximum intertemporal correlation ($\tau = 99.9\%$) are consistently much lower than the probabilities in column (B) for the lowest intertemporal correlation ($\tau = 0\%$).
[41] The annual probabilities determined using the Bayesian approach with the uniform distribution ranging from 0 to 0.1 as a prior function (Table 6) are comparable to the annual probabilities computed under the classical approach with a 75% confidence level (Table 4). In a strict sense, the annual Bayesian probabilities match the annual probabilities estimated employing the classical approach with confidence levels between 73% and 79% (Table 6 too).

multivariate standard normal probability density function, whereas the computation of the weighted mean of the annual default probabilities adopts the univariate standard normal probability density function. This happens for both the classical and Bayesian approaches.

However, the differences between the Bayesian multi-period default probabilities and the weighted mean of the annual default probabilities – column "(B) - (D)" – are substantially less negative and even positive when τ = 99.9%[42]. Equation (18) identifies the diverging trend between the classical and Bayesian approaches. Since the Bayesian probability relates to the mean, the effect of the joint multivariate probability density function is evident both in the numerator and denominator[43], which mitigates the abrupt decline in the multi-period estimations previously shown – one can compare the last two columns. For any positive asset correlation, the greater the intertemporal correlation structure, the stronger that mitigation[44].

Even though the multi-period procedure may be a workable methodology for the Bayesian approach, it must be underlined that it is few versatile because the credibility of the outputs depends on the time period and the intertemporal structure, as was already indicated. In the moderate scenario, the multi-period posterior default probability for T = 6 and τ = 99.9% is 2.80% (Tables 11 and 13), whereas the weighted mean of the six annual default probabilities in the classical approach is 2.64% (Table 5)[45]. In order to achieve a multi-period posterior default probability of 2.64%, another intertemporal correlation can be adopted: τ = 87% or τ = 95%, utilizing a constant or an exponential function for "τ" – τ = 63% or τ = 81% in the extreme scenario – to reduce the multi-period posterior default probability of 1.82% (Tables 11 and 13) to 1.37% (Table 5).

For instance, likewise in the extreme scenario and using a constant function for the intertemporal correlation, 62%, 63%, 64%, and 50% are, respectively for T = 5,

[42] Also positive differences for the moderate scenario when T = 2 and τ = 50%.

[43] See footnote 39.

[44] For instance, the annual posterior or Bayesian default probability with the uniform distribution]0, 0.1] as the prior function, n = 250, k = 1, and ρ = 12% is 2.249% (Table 6). For T = 2, T = 4, and T = 6 with ρ = 12% and τ = 0%, the equivalent multi-period posterior probabilities are 1.344%, 0.905%, and 0.768%, respectively (Table 11 and also Table 13), which correspond to 60%, 40%, and 34% of 2.249%. The posterior probabilities for the same three T-time periods at τ = 99.9% correspond to 89%, 83%, and 81% of 2.249%.

For ρ = 0%, the annual posterior default probability would be 0.794% (instead of 2.249%), and the multi-period posterior probabilities (also for T = 2, T = 4, and T = 6) would be 0.598%, 0.499%, and 0.466% – 75%, 63%, and 59% of 0.794% –, regardless of the intertemporal parameter, "τ". One notes that when ρ = 0%, $G(\lambda, y_t, \rho) = \lambda$ takes place. Since the values of the systematic risk's realization range, "y_t", do not interact with the likelihood probability when ρ = 0%, the joint multivariate standard normal probability density function, "$\phi(y_1, \ldots, y_T; \tau)$", behaves as a constant value. As a result, "μ_T" is independent of the τ-intertemporal correlation.

[45] In this case (akin to what was applied to get the matching confidence levels displayed in Table 12), two comparisons are performed: different time periods (comparing multi-year probabilities to one-year probabilities) and different approaches (comparing classical probabilities to Bayesian probabilities).

$T = 4$, $T = 3$, and $T = 2$, the τ-coefficients that ensure the balance between the Bayesian multi-period estimations and the classical annual estimations. For the same scenario but employing an exponential function (instead of a constant function), the new intertemporal correlations, "τ", for those four "T" are 79%, 75%, 71%, and 50%[46].

Due to the fact that credit default events are always correlated among borrowers themselves, applying the independence hypothesis simultaneously within the same year (asset correlation) and over a period of various years (intertemporal correlation) is not valid. The one-year default probability computation can only guarantee the first kind of correlation; intertemporal correlation is thus not included. This is not a problem if the mean of several default probabilities is calculated (preferably determined with time series data incorporating an economic cycle, as recommended). On the contrary, using the multi-period methodology to account for the intertemporal correlation is problematic because the corresponding default probability estimations are excessively imprudent or unrealistically low.

In short, the best realistic procedure for low default portfolios would involve computing annual default probabilities utilizing one-period models and then estimating the default probability for the period linked to the time series data via an arithmetic mean of such probabilities. In contrast, using the same pluriannual historical data, changing one-period models to multi-period models for estimating default probabilities, is an economically incorrect strategy.

6. Multi-year Estimation of Default Probabilities with Poisson Distribution

6.1 Formulae

Classical approach

Equation (13) presumes that the default distribution has a correlated binomial distribution. Under the premise of a correlated Poisson distribution, the probability that no more than "k" defaults would occur in a portfolio with "n" obligors is written as follows:

$$\begin{aligned} \mathrm{P}(K_T \leq k) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\{-n.\left[1 - \prod_{t=1}^{T}\left[1 - G\left(\lambda_{T,C}, y_t, \rho\right)\right]\right]\}. \\ .\sum_{i=0}^{k}\{n.\left[1 - \prod_{t=1}^{T}\left[1 - G\left(\lambda_{T,C}, y_t, \rho\right)\right]\right]\}^{i}/i!. \\ .\phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) \geq 1 - \omega \end{aligned} \quad (20)$$

[46] For $T = 2$, there is no difference between the exponential function and the constant function since $\tau^{T-1} = \tau$.

Although D. Tasche's focus (Tasche, 2012) had been on the Bayesian approach, he proposed a different methodology based on the Poisson distribution. The upper confidence bounds are yield by this methodology as seen below[47]:

$$\begin{aligned}&\mathrm{P}(K_T \leq k) = \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\left[-n. \sum_{t=1}^{T} G\left(\lambda_{T,C}, y_t, \rho\right)\right]. \\ &. \sum_{i=0}^{k} \left[n. \sum_{t=1}^{T} G\left(\lambda_{T,C}, y_t, \rho\right)\right]^i / i!. \\ &. \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) \geq 1 - \omega \end{aligned} \quad (21)$$

Bayesian approach
Equation (18) presumes that the default distribution also exhibits a correlated binomial distribution. Similarly, under the assumption that the default probability follows a correlated Poisson distribution, the corresponding expected value of the multi-year posterior probability of occurring "k" defaults in a group with "n" borrowers is represented by:

$$\begin{aligned}&\mu_T = \int_0^{\mathrm{u}} \lambda. f(\lambda). \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} [\prod_{t=1}^{T} \exp[-n. G(\lambda, y_t, \rho)]. [G(\lambda, y_t, \rho)]^k] . \\ &. \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) d\lambda. \\ &. 1/\{\int_0^{\mathrm{u}} f(\lambda). \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} [\prod_{t=1}^{T} \exp[-n. G(\lambda, y_t, \rho)]. [G(\lambda, y_t, \rho)]^k] . \\ &. \phi_T(y_1, \dots, y_T; \tau) d(y_1, \dots, y_T) d\lambda\} \end{aligned} \quad (22)$$

[48]

6.2 Results

Using the Poisson distribution to estimate default probabilities is not particularly adequate since the size of the risk portfolio, "n", is not fixed in this type of distribution. However, the Poisson limit theorem indicates that outputs from the Poisson distribution are extremely comparable to those obtained by employing the binomial distribution, whether the time period is one-year or multiple years, regardless of the classical or Bayesian approaches are being adopted. The variance of the Poisson distribution, "$n. \lambda$", is slightly higher than the variance of the binomial distribution, "$n. \lambda. (1 - \lambda)$", when considering a low default portfolio; the mean is the same for both distributions, "$n. \lambda$".
Table 14 compares the direct outcomes of Equations (13) and (20) for the classical approach, and Equations (18) and (22) for the Bayesian approach. To make the comparison easier, it only includes, for T = 5, the figures 0% and 99.9% as opposing percentages for the intertemporal correlation structure.

[47] Tasche (2012) states on page 318 that "(…) the determination of upper confidence requires another approximation since convolutions of binomial distributions are not binomially but at best approximately Poisson distributed."
[48] The constant factor "$n^k/k!$" – included in $\exp[-n. G(\lambda, y_t, \rho)]. [n. G(\lambda, y_t, \rho)]^k/k!$ – was left out because it has the same impact on both the numerator and denominator of the likelihood function.

Table 14: Multi-period default probabilities under the classical and Bayesian approaches for the binomial and Poisson distributions (T = 5 and ρ = 12%)

Types of approach	Types of scenario	Confidence levels (a) or prior functions (b)	τ	Binomial distribution [Equations (13) (a) and (18) (b)]	Poisson distribution [Equations (20) (a) and (22) (b)]
Classical approach (confidence levels)	Moderate scenario	65%	0% C	0.341%	0.341%
			99.9% C	0.573%	0.573%
		75%	0% C	0.422%	0.423%
			99.9% C	0.783%	0.785%
		95%	0% C	0.815%	0.820%
			99.9% C	2.056%	2.062%
	Extreme scenario	65%	0% C	0.219%	0.220%
			99.9% C	0.377%	0.378%
		75%	0% C	0.281%	0.281%
			99.9% C	0.531%	0.532%
		95%	0% C	0.589%	0.593%
			99.9% C	1.509%	1.513%
Bayesian approach (prior functions)	Moderate scenario	Uniform]0%, 100%[	0% C	1.470%	1.475%
			99.9% C	2.936%	2.939%
		Uniform]0%, 10%]	0% C	1.470%	1.475%
			99.9% C	2.767%	2.768%
		Beta (EDR as the mean)	0% C	1.213%	1.213%
			99.9% C	1.508%	1.509%
	Extreme scenario	Uniform]0%, 100%[	0% C	0.824%	0.826%
			99.9% C	1.881%	1.883%
		Uniform]0%, 10%]	0% C	0.824%	0.826%
			99.9% C	1.846%	1.848%
		Beta (EDR as the mean)	0% C	0.609%	0.609%
			99.9% C	0.854%	0.854%

(a) - Classical approach

(b) - Bayesian approach

C - Constant function used for the variance-covariance matrix
EDR - Empirical default rate

One comes to the conclusion that the results of the binomial distribution and the Poisson distribution are extremely similar, with the outputs from the last distribution being just marginally higher than the corresponding outputs from the former[49]. Therefore, the findings concerning the binomial distribution drawn using the classic and Bayesian approaches are still valid for the Poisson distribution.
In order to show that no theoretical methodology assures correct or reliable results when employing multi-period estimations under the classical approach, the results from Equation (21) must also be presented. Table 15 displays the multi-period default probabilities arising from Equations (13) and (20) – see Table 14 – as well as Equation (21).

Table 15: Multi-period default probabilities under the classical approach for the binomial and Poisson distributions (T = 5 and ρ = 12%)

Types of scenario	Confidence levels	τ	Binomial distribution	Poisson distribution	
			Equation (13)	Equation (20)	Equation (21)
Moderate scenario	65%	0% C	0.341%	0.341%	0.340%
		99.9% C	0.573%	0.573%	0.571%
	75%	0% C	0.422%	0.423%	0.421%
		99.9% C	0.783%	0.785%	0.781%
	95%	0% C	0.815%	0.820%	0.814%
		99.9% C	2.056%	2.062%	2.051%
Extreme scenario	65%	0% C	0.219%	0.220%	0.219%
		99.9% C	0.377%	0.378%	0.377%
	75%	0% C	0.281%	0.281%	0.281%
		99.9% C	0.531%	0.532%	0.530%
	95%	0% C	0.589%	0.593%	0.589%
		99.9% C	1.509%	1.513%	1.508%
C - Constant function used for the variance-covariance matrix					

There are two main notes that might be made. On the one hand, the divergence between the two final columns – which both employ the Poisson distribution – is immaterial. On the other hand, default probabilities generated by the binomial and Poisson distributions are fairly comparable. As a result, any analytical proposal to use multi-period estimations under the classical approach is illogical and must be abandoned.

[49] The greatest difference between the last two columns is only 0.006%.

7. Final Summary

7.1 Key Findings

7.1.1 One-year Estimation

The analysis has started with the one-period estimation of default probabilities regardless of the available time series. Under the assumption that the default distribution follows a binomial distribution, two different types of data were examined for both the classical and Bayesian approaches. On the one hand, the annual estimation with annual data was performed (then obtaining ordinary and weighted means of the annual default probabilities). On the other hand, the annual estimation with pluriannual data was made (utilizing aggregated as well as annualized data).

Comparing those alternatives[50] to getting annual default probabilities (given in Subsections 3.1 and 3.2), one came to the conclusion that, applying the prudence criteria required for portfolios with few default events, the weighted mean of the one-year default probabilities estimated with annual data is the best option. Using this option as a benchmark, the annual default probabilities were contrasted with the default probability yield by the multi-year estimation.

Combining the classical and Bayesian approaches – which are related to the confidence level and the prior function, respectively – is an acceptable and prudent strategy to eliminate subjectivity in each of them. In such a situation, it was observed that a quite conservative default probability derives when one adopts the 75% confidence level for the classical approach and the uniform distribution]0, 0.1] as the prior function for the Bayesian approach. The matching confidence levels for the classical approach corresponding to the Bayesian probabilities are around 75%, which is a prudent precaution.

7.1.2 Multi-year Estimation

For the binomial distribution, two multi-period methodologies were presented (in Sub-subsection 4.2.1): one methodology for the classical approach – "binomial classical normal"[51] – and one for the Bayesian approach – "binomial Bayesian normal"[52]. Additionally, three other methodologies were identified for the Poisson distribution (in Subsection 6.1): two methodologies for the classical approach – "Poisson classical normal" and "Poisson classical Tasche"[53] – and one for the Bayesian approach[54]. Given that there is negligible difference between the binomial distribution and the Poisson distribution, they are equivalent (although the

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[50] The ordinary mean of the one-year default probabilities estimated with annual data, the weighted mean of the one-year default probabilities estimated with annual data, the aggregated data for the one-year default probabilities estimated with pluriannual data, and the annualized data for the one-year default probabilities estimated with pluriannual data.

[51] Equation (13)

[52] Equation (18)

[53] Equations (20) and (21), respectively.

[54] Equation (22)

former is conceptually better than the latter regarding the portfolio of obligors, which is assumed as a fixed group).

The outputs have proved that the multi-period estimations could not be applied to the classical approach; they are unrealistic estimations, which confirms that the classical multi-year methodologies (related to confidence levels) should never be utilized to estimate default probabilities[55]. The alternative theoretical formula derived from "Poisson classical Tasche" yields results that are nearly identical to those of the "Poisson classical normal" (and "binomial classical normal" too).

If one assumes an exaggeratedly conservative confidence level, the one-year estimation has an excessively high default probability, but for that same confidence level, the multi-year default probability is unrealistically imprudent. This is explained by the fact that the joint probability density drastically decreases as the dimension of the multivariate standard normal distribution rises, making it impossible to compare the corresponding default probabilities to those obtained from a univariate standard normal distribution. It also clarifies why multi-period estimation should never be applied in the classical approach.

The multi-year methodology produces worthless default probabilities, making it useless as a risk management tool. By adopting this methodology, the classical default probabilities are utterly distorted, and the Bayesian default probabilities are significantly changed[56]. When utilizing such a methodology, the probabilities derived from the classical approach and the Bayesian approach are inconsistent.

Finishing a paper or any document with tables or figures is unorthodox. However, due to the complexity of the multi-period problem and the range of results presented here, it is preferable to attempt to condense the important findings into two figures: Figures 3 and 4, for the classical and Bayesian approaches, respectively[57]. The cumulative distribution functions were obtained applying the aforementioned "binomial classical normal" and "binomial Bayesian normal", specifically the expression of Equation (13) and the denominator of Equation (18) regarding the uniform distribution between 0 and 0.1 as the prior function.

[55] For the classical approach the outputs closely resemble those presented by Pluto and Tasche (2005). See the last paragraph of the «Classical approach» of Sub-subsection 4.2.2.

[56] When the Bayesian approach is employed, the multi-period technique should be applied with some caution. This approach might be theoretically feasible, in particular, if a very strong intertemporal correlation is anticipated.

[57] Only two multi-period cases – $T = 2$ and $T = 4$ – are provided because they both make the presentation easy to compare the one-single-period and the multi-period contexts and do not add unnecessary complexity to the presentation.

The four solid, thicker, and horizontal lines at the top of each image refer to four cumulative probabilities (or confidence levels), three of these were utilized in some tables in this paper (65%, 75%, and 95%).

It should be noted that the cumulative probabilities shown in those images have no relationship to the matching confidence levels presented in Tables 6 and 12.

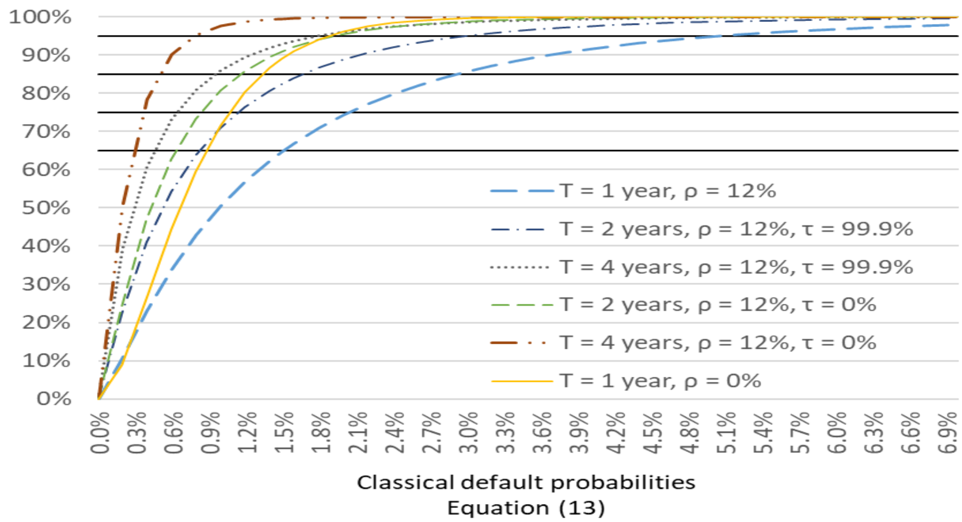

Figure 3: Cumulative distribution functions for the default distribution under the classical approach for various periods of "T" years (n = 250 and k = 1)

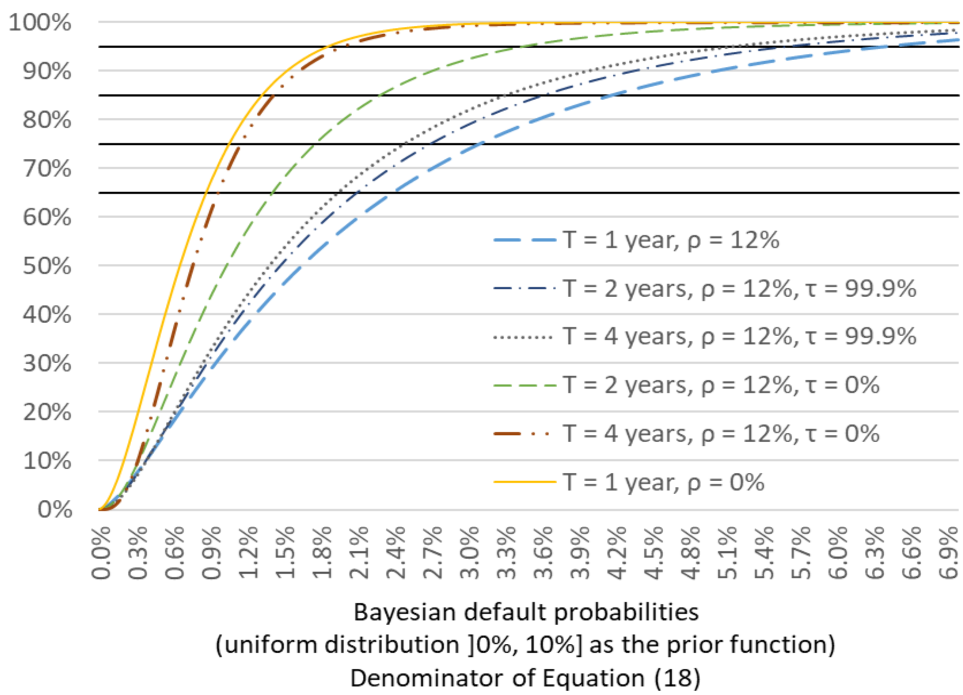

Figure 4: Cumulative distribution functions for the default distribution using the Bayesian approach for various periods of "T" years (n = 250 and k = 1)

The figures serve as an additional example of why it is exceedingly imprudent to employ multi-year estimations, thus, such estimations are an unrealistic option. They share certain similarities. First of all, it is clear how the outcomes for one-period and multi-period differ. Moreover, annual default probabilities (T = 1) with $\rho = 12\%$ are always higher than those for $T \neq 1$, regardless of the intertemporal correlation. An extra point that must be emphasized is that, under the classical approach, annual default probabilities with $\rho = 0\%$ are higher than pluriannual default probabilities with $\rho = 12\%$ and $\tau = 99.9\%$ up to confidence levels of 75%[58]. Using the Bayesian approach, the cumulative distribution function for annual default probability with $\rho = 0\%$ and the analogous function for $T = 4$ with $\rho = 12\%$ and $\tau = 0\%$ are quite similar. Finally, it should be underlined that the one-year cumulative distribution functions for the classical and Bayesian approaches are very close to one another when there is no asset correlation ($\rho = 0\%$).

7.2 Concluding Remarks

Multi-period estimations are crucial for the banking sector due to the potential for significant imprudent effects on the amounts of credit provisioning and the economic capital requirements. This study, which includes the distinction between the pluriannual data and the pluriannual estimating as one of its foundations, deepened analytical issues and practical challenges about multi-period computation. Despite the fact that pluriannual data and pluriannual estimating represent two quite different perspectives, some people almost always view them as being the same.

Pluriannual estimation is not an acceptable substitute for the classical approach and should not be used to calculate default probabilities. In order to produce prudent probabilities, the application of the pluriannual estimation for the Bayesian approach must be accurate because the intertemporal correlation should be stronger the longer the time period is, solely to ensure the theoretical validity of the pluriannual estimation.

So, utilizing the one-year estimations is the only option to assure conservative default probabilities that are consistent between the classical and Bayesian approaches. There is no reasonable relationship between the multi-year probabilities obtained from the classical and Bayesian approaches, on the one hand, and those multi-year probabilities and one-year probabilities, on the other hand.

Finally, it should be emphasized that the more data, the better. The estimations of default probabilities are more accurate the more the historical experience there is. However, one concluded also that the classical or Bayesian estimation processes require a yearly basis regardless of the amount of data that is available and the corresponding length of the time series, therefore aggregated data may not be accepted as a hypothesis for the estimating process. For the purpose of determining long-term default probabilities, all data time series should be included in the one-year estimation, preferably employing a weighted mean of the annual probabilities, thus it is not appropriate to use pluriannual data to compute multi-period estimations.

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[58] With the exception of the 75% confidence level and $T = 2$: the multi-year default probability is 1.15%, which is somewhat higher than the annual default probability (for $\rho = 0\%$) of 1.08%.

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