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# Selected Methods of optimized Sampling for Index Tracking – Evidence from German Stocks

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## Abstract

The aim of this study is to verify the tracking quality of four different optimization approaches used for approximate replication (sampling) of a stock index. These approaches include relative optimization, optimization according to Markowitz, the use of regression methods and linear optimization. To test the tracking qualities of these strategies, an empirical analysis of portfolios of 10 stocks included in the German stock index DAX is used to determine the in-sample and out-of-sample results. In addition, a portfolio composition based on market capitalization and an equally weighted portfolio are considered.
The analysis shows that the in-sample results are quite similar for all index tracking methods used in this study. Considering the out-of-sample results, it can be stated that all four index tracking methods lead to a portfolio that initially shows a high degree of similarity to the benchmark. However, it is surprising that the equally weighted portfolio leads to the best overall results. Therefore, the analysis presented here gives the impression that the uncomplicated equal weighting is preferable to the more sophisticated index tracking methods considered in this study.

**JEL classification number:** G11.
**Keywords:** Index tracking, Sampling, Optimization, Tracking error, Residual risk.

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# 1. Introduction

In contrast to active portfolio management, passive management of securities often involves the replication of suitable market indices, which is also referred to as index tracking. In this process, a target portfolio is replicated as closely as possible by a portfolio that is actually to be realized (tracking portfolio). A distinction must be made between full index replication and approximate replication (sampling).

The approximate replication of an index can be done with heuristic methods ("rules of thumb") and with optimization approaches. Optimization approaches include relative optimization, Markowitz optimization, index tracking using regression methods and linear optimization.

With these methods, there is the problem of estimating the parameters entering the model. Therefore, a comparative, theoretical and empirical analysis is useful to verify the tracking qualities of these methods. Accordingly, the aim of this study is to determine the extent to which stock index performance can be successfully tracked with these methods using a small number of stocks. For comparison purposes, a weighting according to the current market capitalization of the stocks in the index as well as an equally weighted portfolio will be included in the empirical analysis.

In the following, firstly the four index tracking methods mentioned above are presented. This is followed by a comparison of the methods based on an empirical analysis. The success of index tracking is to be determined for the period from December 30, 2010 to December 30, 2020 and for a portfolio of 10 stocks from the German stock index DAX. For the portfolios, a semi-annual rebalancing is made according to the respective strategy on the basis of the past 60 monthly stock returns. A minimum weight in the portfolio of 2% and a maximum weight of 50% are assumed. Finally, the in-sample and out-of-sample results are presented.

# 2. Optimized Sampling for Index Tracking

## 2.1 Index tracking by means of relative optimization

A market index or benchmark can be replicated either fully (full replication) or approximately (sampling). In the case of full replication, the proportions of securities in the tracking portfolio are chosen to be the same as the weighting of the respective investments in the benchmark or target portfolio. Due to the associated costs, sampling, also known as partial replication, is often used in practice, where the tracking portfolio deviates from the target portfolio due to the lower number of securities included (Jiang/Perez, 2020). These deviations should be minimized when replicating the target portfolio. In addition to heuristic methods, model-based procedures have also been developed for this purpose, which are based on an optimization approach and can be summarized under the generic term Optimized Sampling. In the context of optimized sampling, different methods of index tracking can be used. A good overview of related research on heuristic methods and optimization approaches can be found at Sant‘Anna, Filomena, Guedes, and

Borenstein (2017) and Mezali/Beasley (2014) In this study, relative optimization is considered first.
Relative optimization is based on a given target portfolio (benchmark, reference index, etc.) whose return and risk are to be replicated as closely as possible by a tracking portfolio. It must be taken into account that the investment universes and restrictions of both portfolios can differ, i.e. that the tracking portfolio can also contain investments that are not in the target portfolio, or that not all investments of the target portfolio can be included in the tracking portfolio, or that weighting restrictions are formulated in the tracking portfolio. Unlike relative optimization in active portfolio management, where the difference between portfolio alpha and the residual risk weighted by the risk aversion parameter is maximized (neglecting the timing component), in index tracking the following characteristics of the tracking portfolio can be intuitively identified in order to replicate the benchmark portfolio as closely as possible (Bruns and Meyer-Bullerdiek, 2020, and Poddig, Brinkmann and Seiler, 2005):

(1) The tracking portfolio must have an alpha of zero, because the alpha of the benchmark is also zero.
(2) The tracking portfolio must have a beta of one, because the beta of the benchmark is also one.
(3) The tracking portfolio must have a minimum residual risk.

Here, the focus is on minimizing the residual risk, which corresponds to minimizing the active risk or the expected tracking error ($TE_{PF}^{exp.}$) while excluding the timing component. This can be shown using the following formula (Bruns/Meyer-Bullerdiek, 2020):

$$TE_{PF}^{exp.} = \sqrt{(ß_{PF} - 1)^2 \cdot \sigma_{BM}^2 + \sigma_{\varepsilon_{PF}}^2} \tag{1}$$

Here, the difference ($ß_{PF}$ – 1) denotes the active beta. For $ß_{PF} = 1$, i.e. in case there is no active shaping of the beta factor by timing activities, the active risk (the expected tracking error) thus corresponds to the residual risk or non-systematic risk. For this risk, the following applies:

$$\sigma_{\varepsilon_{PF}}^2 = \sigma_{PF}^2 - ß_{PF}^2 \cdot \sigma_{BM}^2 \tag{2}$$

From these considerations, the following objective function (OF) and corresponding constraints can be derived (Poddig, Brinkmann and Seiler, 2005, and Ernst and Schurer, 2015):

$$W_{PF}^T \cdot \Sigma \cdot W_{PF} - \left(W_{PF}^T \cdot ß\right)^2 \cdot W_{BM}^T \cdot \Sigma \cdot W_{BM} \rightarrow \min! \tag{3}$$

where

$W_{PF}$ = N×1 vector of security weights in the tracking portfolio
$W^T_{PF}$ = 1×N vector of security weights in the tracking portfolio (transposed)
$\Sigma$ = N×N variance-covariance matrix of (historical) security returns
ß = N×1 vector of the beta factors of the securities against the benchmark
$W_{BM}$ = N×1 vector of security weights in the benchmark portfolio

The central constraints can be formulated as follows:

$$\text{(a)} \quad \sum_{i=1}^{N} w_{PF_i} = 1 \qquad \text{or} \qquad \sum_{i=1}^{N} w_{a_i} = 0 \qquad , \qquad w_{a_i} = w_{PF_i} - w_{BM_i}$$

where

$w_{PFi}$ = weight of security i in the tracking portfolio
$w_{ai}$ = active weight of security i in the tracking portfolio
$w_{BMi}$ = weight of security i in the benchmark portfolio

$$\text{(b)} \quad ß_{PF} = 1 \qquad \Leftrightarrow \qquad \underbrace{ß_{PF} - 1}_{ß_a} = 0 \qquad \text{(no timing)}$$

where

$ß_{PF}$ = beta factor of the tracking portfolio in relation to the benchmark
$ß_a$ = active beta factor of the tracking portfolio relative to the benchmark

(c) $\alpha_{PF} = 0$ (no selection)

where

$\alpha_{PF}$ = alpha of the tracking portfolio relative to the benchmark

Constraint (a) can also be referred to as a budget constraint. In addition, other constraints may be added, such as the following (Scozzari, Tardella, Paterlini and Krink, 2013, Poddig, Brinkmann and Seiler, 2005, and Ernst and Schurer, 2015, further possible constraints can be found e.g. at Derigs and Nickel, 2003):

(d) $w_{PFi} \geq 0$ for all securities i = 1, … , N (no short selling)
(e) $w_{PFi} \leq \max w_i$ for all securities i = 1, … , N (max. permissible proportion)
(f) $w_{PFi} \geq \min w_i$ for all securities i = 1, … , N (required minimum proportion)

It should be noted that, according to constraint (c), there should be no positive or negative alpha and thus no selection effect. However, temporary, unsystematic deviations between the returns of the tracking portfolio and the benchmark cannot be avoided if the tracking portfolio is also composed of securities other than the benchmark itself. These deviations affect the selection risk, which in this case corresponds to the likewise unavoidable active risk, which must be minimized accordingly. Furthermore, it can be pointed out that constraint (d), i.e. the exclusion

of short selling, makes sense because short selling can lead to unstable portfolios. In addition, it should be noted that the consideration of minimum and maximum proportions for the respective securities in the tracking portfolio can lead to the fact that no admissible solution can be found by the optimization process if the number of available securities for the formation of the tracking portfolio is too small (Van Montfort, Visser and van Draat, 2008, and Poddig, Brinkmann and Seiler, 2005).
If the tracking error is expressed as the variance of the active return, then, taking into account the above-mentioned constraint (b), the following can be shown (Poddig, Brinkmann and Seiler, 2005, it should be noted that the term "tracking error" may also refer to the difference between the benchmark return and the portfolio return (Karlow, 2012 or Gavriushina, Sampson, Berthold, Pohlmeier and Borgelt, 2019)):

$$TE = \sigma^2_{\varepsilon_{PF}} \tag{4}$$

Thus, the objective of this approach to determining the tracking portfolio is to minimize the tracking error which is the same as the residual risk because of constraint (b). The problem with this approach, however, is that the expected future alpha and beta factors of the individual investments relative to the benchmark must be estimated. Otherwise, it would not be possible to determine the tracking portfolio.

## 2.2 Index tracking by means of optimization according to Markowitz

Markowitz's approach (Markowitz, 1987) is very similar to the relative optimization approach of index tracking. The target portfolio is to be replicated in the best possible way, ideally resulting in an active return and an active risk of zero for the tracking portfolio. In vector notation, the active return can be represented as follows:

$$r_a = W_{PF}^T \cdot R - W_{BM}^T \cdot R = \left(W_{PF}^T - W_{BM}^T\right) \cdot R = W_a^T \cdot R \tag{5}$$

where

| | | |
|---|---|---|
| $R$ | = | N×1 vector of expected excess returns of the securities included in the tracking portfolio and in the benchmark |
| $W_a$ | = | N×1 vector of active security weights in the tracking portfolio |

The active risk or tracking error can be determined as follows (Poddig, Brinkmann and Seiler, 2005):

$$TE = \sigma_a^2 = W_a^T \cdot \Sigma \cdot W_a \tag{6}$$

where
$\sigma^2_a$ = variance of active returns
$W^T_a$ = 1×N vector of active security weights in the tracking portfolio (transposed)
$\Sigma$ = N×N variance-covariance matrix of (historical) security returns

The objective function (OF) in this approach is accordingly:

$$OF = TE = \sigma_a^2 = W_a^T \cdot \Sigma \cdot W_a \rightarrow \min! \quad (7)$$

As a rule, the central constraint is (Poddig, Brinkmann and Seiler, 2005):

(a) $r_a = W_a^T \cdot R = 0$

In addition, further constraints can be formulated, which have already been mentioned for the index tracking by means of relative optimization presented above:

(b) $\sum_{i=1}^{N} w_{PF_i} = 1$ or $\sum_{i=1}^{N} w_{a_i} = 0$ , $w_{a_i} = w_{PF_i} - w_{BM_i}$
(c) $w_{PFi} \geq 0$ for all securities i = 1, … , N (no short selling)
(d) $w_{PFi} \leq \max w_i$ for all securities i = 1, … , N (max. permissible proportion)
(e) $w_{PFi} \geq \min w_i$ for all securities i = 1, … , N (required minimum proportion)

It should be noted that without constraint (a), a negative active return can theoretically result from the optimization. Nevertheless, dispensing with this constraint would have the advantage that, in addition to simplifying the optimization, no expected returns would have to be estimated. Compared to index tracking with relative optimization, index tracking according to Markowitz would then only require relatively unproblematic variables to be estimated. If the constraint (a) is waived, only the variance-covariance matrix would have to be estimated, whereby the empirical variance-covariance matrix can be used. However, this approach merely ignores the estimation problem that arises in relative optimization. Thus, in addition to a negative active return, a beta factor deviating from one may result, so that an unintended timing component exists (Poddig, Brinkmann and Seiler, 2005, and Ernst and Schurer, 2015).

## 2.3 Index tracking using the regression method

Similar to the above-mentioned approaches, index tracking by means of the regression method aims to achieve the smallest possible difference in return between the target portfolio (benchmark) and the tracking portfolio, whereby this difference ($r_d$) corresponds to the active return, but with the opposite sign. This can be illustrated as follows:

$$r_d = R_{BM} - R_{PF} = -r_a \tag{8}$$

where $R_{BM}$ is the excess return of the benchmark and $R_{PF}$ is the excess return of the tracking portfolio.
According to this approach, the expected squared return differences or the expected mean squared error are to be minimized so that the objective function (OF) is equivalent to the minimization of the tracking error when the expected value of the return difference is zero (Poddig, Brinkmann and Seiler, 2005):

$$OF = E\left(r_d^2\right) = E\left(r_a^2\right) \to \min! \tag{9}$$

Again, the following (or even more) constraints have to be considered:

(a) $\sum_{i=1}^{N} w_{PF_i} = 1$ or $\sum_{i=1}^{N} w_{a_i} = 0$ , $w_{a_i} = w_{PF_i} - w_{BM_i}$
(b) $w_{PFi} \geq 0$ for all securities i = 1, … , N (no short selling)
(c) $w_{PFi} \leq \max w_i$ for all securities i = 1, … , N (max. permissible proportion)
(d) $w_{PFi} \geq \min w_i$ for all securities i = 1, … , N (required minimum proportion)

Since the objective function refers to an ex ante variable, there is also an estimation error problem with this approach. Here, $E(r_d^2)$ is determined directly from the historical returns on the investments of the benchmark portfolio and of the tracking portfolio. The estimator $\hat{E}(r_d^2)$ is then the historical mean of the squared differences between the returns:

$$\hat{E}\left(r_d^2\right) = \frac{1}{T} \cdot \sum_{t=1}^{T} \left(r_{BM_t} - r_{PF_t}\right)^2 \tag{10}$$

where T = number of historical return periods included

While the returns of the benchmark portfolio are available, the returns of the tracking portfolio cannot be observed directly because its final structure is not known until after the optimization, whereby the following applies in principle:

$$r_{PF_t} = \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t} \quad \text{or in matrix notation:} \quad r_{PF_t} = W^T \cdot r_t \tag{11}$$

Thus, those proportions $w_{PFi}$ for which $E(r_d^2)$ is minimized have to be found. If $E(r_d^2)$ is replaced by its estimator $\hat{E}(r_d^2)$ in the objective function, the following objective function (OF) is obtained (Poddig, Brinkmann and Seiler, 2005, and Zhang, Wang and Xiu, 2019):

$$OF = \hat{E}\left(r_d^2\right) = \frac{1}{T} \cdot \sum_{t=1}^{T}\left(r_{BM_t} - r_{PF_t}\right)^2 = \frac{1}{T} \cdot \sum_{t=1}^{T}\left(r_{BM_t} - \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t}\right)^2 \rightarrow \min! \tag{12}$$

This objective function has a structural identity to the objective function in least squares estimation using multivariate linear regression, but in this context constraints have to be considered. Therefore, this approach is also referred to as index tracking using constrained regression.

This method can also be formulated directly as a regression-analytical procedure. In this case, the historical benchmark returns as dependent variable are explained by the returns of N investments as independent variables. Thus, the following applies for any point in time:

$$r_{BM_t} = \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t} + \varepsilon_t \qquad \Leftrightarrow \qquad \varepsilon_t = r_{BM_t} - \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t} \tag{13}$$

Minimizing the regression residual then leads to the "optimal" tracking portfolio, where the residual is squared to avoid negative and positive errors cancelling out:

$$OF = \sum_{t=1}^{T} \varepsilon_t^2 = \sum_{t=1}^{T}\left(r_{BM_t} - \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t}\right)^2 \rightarrow \min! \tag{14}$$

Here, the searched weights $w_{PFi}$ are estimated on the basis of the historical observations. This function basically corresponds to the above-mentioned objective function. While the above-mentioned objective function is based on portfolio theory considerations, the latter objective function is based on data analysis. However, both equations lead to a regression under constraints (Poddig, Brinkmann and Seiler, 2005).

## 2.4 Index tracking by means of linear optimization

Index tracking can also be performed by means of linear optimization. In this case, the considerations on index tracking by means of regression are applied first. For any future point in time, the benchmark return is to be reproduced as best as possible, taking into account an unavoidable residual error or residual $\varepsilon_t$. The corresponding regression equation corresponds to the one above, which can then be transformed into an equation for the active return ($r_a$)

$$r_{BM_t} = \sum_{i=1}^{N} w_{PF_i} \cdot r_{i_t} + \varepsilon_t = r_{PF_t} + \varepsilon_t \qquad \Leftrightarrow \qquad -\varepsilon_t = r_{PF_t} - r_{BM_t} = r_{a_t} \tag{15}$$

Furthermore, it is assumed that an investor does not want to achieve a negative active return, while a positive active return is desirable. Thus, it is necessary to minimize the absolute amounts of all negative active returns, so that the expected negative deviations of the portfolio return from the benchmark return are minimized, but the positive deviation possibilities remain. To determine the tracking portfolio, the following objective function (OF) can be formulated (Poddig, Brinkmann and Seiler, 2005):

$$OF = \sum_{\substack{t=1 \\ r_{a_t}<0}}^{T} \left| r_{a_t} \right| \rightarrow \min! \tag{16}$$

Behind this objective function is a one-sided understanding of risk on the part of the investor, in which only a negative active return is perceived as risk.
To simplify the optimization problem, the two auxiliary variables $r_{a_t}^{+}$ and $r_{a_t}^{-}$ are considered. While the former stands for the positive active rate of return, the latter ($r_{a_t}^{-}$) refers to a negative active rate of return, which is, however, also represented as a positive number. Consequently, the active return for a period t can also be formulated as follows (Poddig, Brinkmann and Seiler, 2005):

$$r_{a_t} = r_{a_t}^{+} - r_{a_t}^{-} \tag{17}$$

where

$$r_{a_t}^{+} = r_{a_t} \quad \text{for} \quad r_{a_t} > 0, \quad \text{otherwise} \quad r_{a_t}^{+} = 0$$
$$r_{a_t}^{-} = -r_{a_t} \quad \text{for} \quad r_{a_t} < 0, \quad \text{otherwise} \quad r_{a_t}^{-} = 0$$

Accordingly, the following relationships apply:

$$r_{a_t} = r_{a_t}^{+} - r_{a_t}^{-} = R_{PF} - R_{BM} \quad \Leftrightarrow \quad R_{PF} - R_{BM} - r_{a_t}^{+} + r_{a_t}^{-} = 0 \tag{18}$$

$$\Leftrightarrow \quad R_{PF} - r_{a_t}^{+} + r_{a_t}^{-} = R_{BM} \quad \Leftrightarrow \quad \sum_{i=1}^{N} w_{PF_i} \cdot R_i - r_{a_t}^{+} + r_{a_t}^{-} = R_{BM} \tag{19}$$

Only negative active returns ($r_{a_t}^{-}$) reduce the investment performance of the investor because the positive active returns ($r_{a_t}^{+}$) mean a return of the tracking portfolio that is above the benchmark return. Thus, the above objective function (OF) can also be represented as follows (Poddig, Brinkmann and Seiler, 2005).

$$OF = \sum_{t=1}^{T} r_{a_t}^{-} \rightarrow \min! \tag{20}$$

The optimization is about determining the following values:

- N weights of the individual securities i in the tracking portfolio ($W_{PFi}$)
- T values for the positive active returns ($r_{a_t}^+$)
- T values for the negative active returns ($r_{a_t}^-$)

In the course of optimization, these values are to be determined in such a way that equation (19) is fulfilled for all T periods and the sum of the negative active returns ($r_{a_t}^-$) is minimized. Thus, the following constraints to the above objective function can be formulated:

(a) $\sum_{i=1}^{N} w_{PF_i} \cdot R_i - r_{a_t}^+ + r_{a_t}^- = R_{BM}$ for all points in time t = 1, … , T

(b) $\sum_{i=1}^{N} w_{PF_i} = 1$

(c) $w_{PFi} \geq 0$ for all securities i = 1, … , N (no short selling)

(d) $r_{a_t}^+ \geq 0$ for all points in time t = 1, … , T

(e) $r_{a_t}^- \geq 0$ for all points in time t = 1, … , T

In addition to these constraints, further restrictions may be added, such as the maximum permissible proportion or the required minimum proportion of securities in the tracking portfolio:

(f) $w_{PFi} \leq \max w_i$ for all securities i = 1, … , N (max. permissible proportion)

(g) $w_{PFi} \geq \min w_i$ for all securities i = 1, … , N (required minimum proportion)

As with the index tracking approaches presented above, the problem here is that the objective function is based on ex ante values, but it is estimated on the basis of historical observation values (Poddig, Brinkmann and Seiler, 2005).
For a discussion of the estimation problem in index tracking also with regard to transaction costs, reference can be made to Poddig, Brinkmann and Seiler (2005), Wu, Kwon and Costa (2017), Choudhary and Sen (2020) and Rowley and Kwon (2015).

# 3 Empirical analysis

## 3.1 Research design

In the empirical analysis, the four index tracking methods presented are compared both with each other and with a portfolio composition based on market capitalization as well as an equally weighted portfolio.
The analysis is performed for the period from December 30, 2010 to December 31, 2020, whereby stock price data from December 30, 2005 to December 31, 2020 are

required. Based on this price data, the tracking portfolios and the portfolio based on the respective market capitalization are rebalanced every six months, starting on December 30, 2010. For this purpose, the number of individual stocks in the portfolio is recalculated in each case and retained for the coming half-year. A minimum weight of 2% and a maximum weight of 50% are assumed in the analysis. Thus, it is taken into account that each stock is also included in each rebalancing. In addition, it was observed for all stocks that their proportion in the portfolio based on market capitalization did not fall below 2% at any rebalancing date.

The portfolio composition is based on the stock prices adjusted for returns (such as dividend income and income from subscription rights) and the calculated monthly discrete returns of the respective stocks of the preceding 5 years. Monthly returns are more likely to be normally distributed than weekly or daily returns. Thus, the portfolio composition is based on 60 monthly returns of the respective stocks at each rebalancing point.

The analysis relates to 10 stocks from the DAX index, which were selected on the basis of their respective market capitalization as of December 30, 2015, i.e. halfway through the period under review. In principle, the largest stock corporations at this date were used. However, companies with a lower market capitalization that belong to the same or similar industry were excluded for the purpose of achieving a broad diversification. This applies to BASF, Volkswagen, BMW, Munich Re, Continental and Fresenius. Linde was not included because no stock prices were available for the entire period. Henkel was not included in order to include E.ON (although its market capitalization was slightly higher than E.ON's on December 30, 2015, it was usually significantly higher for E.ON in the other years). Thus, the analysis is based on the following stocks: Adidas, Allianz, Bayer, Daimler, Deutsche Bank, Deutsche Post, Deutsche Telekom, E.ON, SAP, Siemens.

The half-yearly recomposition of the portfolios is intended to ensure a regular response to changing market situations. In contrast to a constant portfolio composition defined for the entire period under review, temporary changes in the parameters included are thus taken into account. This means that better results can be expected (Inker, 2010). A regular rebalancing of the portfolios would be relatively expensive in practice due to the transaction costs incurred. Therefore, this study does not include a more frequent restructuring. In addition, transaction costs are not included in the analysis, as they are less important for a half-year horizon than for a weekly or even daily adjustment (Meyer-Bullerdiek, 2016).

The respective weightings and (adjusted) prices of the stocks in the portfolio in the period from December 30, 2010 to December 31, 2020 can then be used to determine the corresponding portfolio values at the end of the month, whereby it should again be noted that the analysis does not use actual stock prices, but stock prices adjusted for dividend payments, payments from subscription rights, stock splits, etc. The portfolio values are then used as the basis for the performance analysis. The monthly logarithmic portfolio returns calculated from the portfolio values then form the basis of the performance analysis. In the following sections, the performance of the portfolios is discussed.

## 3.2 Selection of performance measures

With regard to the assessment of the success of the respective index tracking methods, a distinction is to be made between in-sample and out-of-sample assessments. In this empirical analysis, the weightings are determined on the basis of the preceding 60 monthly returns. If it is now assumed for each method that the weightings determined in this way (i.e. optimal in each case) actually existed in this past 5-year period, the resulting values can be regarded as in-sample results. If, however, the portfolio weights determined at the rebalancing dates are used for the subsequent stock market data (which thus did not form the basis for the determination of the weights) in order to determine the corresponding portfolio values and performance results, these results represent out-of-sample values. The distinction between in-sample and out-of-sample tests is explained, for example, by Kunst (2004) and Tashman (2000).
For the in-sample observations, the following measures based on the monthly logarithmic returns are used:

- Mean value of active returns
- Mean value of squared active returns
- Variance of the Tracking Portfolio returns
- Variance of the DAX returns
- Alpha of the Tracking Portfolio
- Beta of the Tracking Portfolio
- Residual variance
- Tracking error (*here*: the sample variance of active returns)

In addition, the out-of-sample analysis also includes the logarithmic returns and the Sharpe ratio of the tracking portfolio and the DAX as well as the correlation of the tracking portfolio with the DAX. For the Sharpe ratio, a risk-free return of zero is assumed.

## 3.3 Results of the empirical analysis

### 3.3.1 In-sample results of the index tracking methods considered

The in-sample results of the different index tracking methods are shown in % in the following tables – first for the relative optimization. The values shown in Table 1 result from the 60 monthly returns before rebalancing on the basis of the optimal weightings determined in each case.

**Table 1: Relative optimization: in-sample results (data in %)**

| Rebalancing date | 30 Dec 2010 | 30 June 2011 | 30 Dec 2011 | 29 June 2012 | 28 Dec 2012 | 28 June 2013 | 30 Dec 2013 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Mean value of squared active returns | 0.01572 | 0.01510 | 0.01993 | 0.02267 | 0.01747 | 0.02064 | 0.01620 |
| Variance of the Tracking Portfolio returns | 0.35524 | 0.35636 | 0.44805 | 0.46490 | 0.46191 | 0.40866 | 0.34849 |
| Variance of the DAX returns | 0.33926 | 0.34101 | 0.42778 | 0.44185 | 0.44414 | 0.38767 | 0.33202 |
| Alpha of Tracking Portfolio | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Beta of Tracking Portfolio | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 |
| Residual Variance | 0.01599 | 0.01535 | 0.02026 | 0.02305 | 0.01777 | 0.02099 | 0.01647 |
| Tracking error as sample variance of active returns | 0.01599 | 0.01535 | 0.02026 | 0.02305 | 0.01777 | 0.02099 | 0.01647 |

| Rebalancing date | 30 June 2014 | 30 Dec 2014 | 30 June 2015 | 30 Dec 2015 | 30 June 2016 | 30 Dec 2016 | 30 June 2017 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Mean value of squared active returns | 0.00898 | 0.00709 | 0.00687 | 0.01086 | 0.01011 | 0.01279 | 0.01317 |
| Variance of the Tracking Portfolio returns | 0.25170 | 0.23523 | 0.23876 | 0.28891 | 0.30428 | 0.22288 | 0.19612 |
| Variance of the DAX returns | 0.24257 | 0.22802 | 0.23178 | 0.27786 | 0.29401 | 0.20987 | 0.18272 |
| Alpha of Tracking Portfolio | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Beta of Tracking Portfolio | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 |
| Residual Variance | 0.00913 | 0.00721 | 0.00698 | 0.01105 | 0.01028 | 0.01301 | 0.01339 |
| Tracking error as sample variance of active returns | 0.00913 | 0.00721 | 0.00698 | 0.01105 | 0.01028 | 0.01301 | 0.01339 |

| Rebalancing date | 29 Dec 2017 | 29 June 2018 | 28 Dec 2018 | 28 June 2019 | 30 Dec 2019 | 30 June 2020 |
|---|---|---|---|---|---|---|
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Mean value of squared active returns | 0.01147 | 0.01012 | 0.01000 | 0.01108 | 0.00932 | 0.01687 |
| Variance of the Tracking Portfolio returns | 0.19777 | 0.20068 | 0.20853 | 0.22680 | 0.21878 | 0.28496 |
| Variance of the DAX returns | 0.18611 | 0.19040 | 0.19836 | 0.21553 | 0.20930 | 0.26781 |
| Alpha of Tracking Portfolio | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Beta of Tracking Portfolio | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 |
| Residual Variance | 0.01167 | 0.01029 | 0.01017 | 0.01127 | 0.00947 | 0.01715 |
| Tracking error as sample variance of active returns | 0.01167 | 0.01029 | 0.01017 | 0.01127 | 0.00947 | 0.01715 |

As the values in the table show, the beta of the tracking portfolio has a value of 1 (or 100%) at all rebalancing dates. Furthermore, the alpha of the tracking portfolio is zero at all rebalancing dates. Thus, these two constraints are satisfied in each period under consideration. Since $ß_{PF} = 1$ and $ß_a = 0$, the residual variance of the tracking portfolio equals the tracking error (as variance) of the portfolio in each period.

Table 2 presents the in-sample results of index tracking with Markowitz optimization. Again, the values shown in the tables are derived from the 60 monthly returns before rebalancing based on the optimal weightings determined in each case. Compared to the values of the relative optimization, the results according to Markowitz optimization are quite similar. With respect to the weightings, there are deviations of more than 10 percentage points only in three cases (with a maximum of 11.23 percentage points). Since the constraints $ß=1$ and $\alpha=0$ do not apply to the Markowitz optimization, there are corresponding deviations. In all periods, the residual variance and tracking error are below the respective values of the relative optimization, although this cannot always be made directly clear in the tables due to the limited number of decimal places shown.

It can be observed that the mean value of the squared active returns is always higher with relative optimization than with Markowitz optimization. The variance of the tracking portfolio returns is higher in most cases with Markowitz optimization.

Table 3 presents the in-sample results of index tracking with the regression method. Here, too, the values shown in the tables result from the 60 monthly returns before rebalancing on the basis of the optimal weights determined in each case

In contrast to the two methods mentioned above, the mean values of the active returns, when using index tracking with the regression method, are different from zero. In contrast, the mean values of the squared active returns are consistently lower compared to both relative optimization and Markowitz optimization. The same applies to the tracking error and the residual variance.

Table 4 presents the in-sample results of index tracking with linear optimization. Again, the values presented in the tables result from the 60 monthly returns before rebalancing based on the optimal weightings determined in each case.

As the values in table 4 show, the results of index tracking with linear optimization are similar to the values of index tracking with the regression method.

However, it is remarkable that index tracking with linear optimization regularly leads to higher mean values of the active return and to higher alpha values compared to all other index tracking methods presented. Given the objective function of linear optimization, a higher active return can be expected. However, in all periods, index tracking with linear optimization results in higher values for the residual variance and the tracking error compared to the regression method.

**Table 2: Markowitz optimization: in-sample results (data in %)**

| Rebalancing date | 30 Dec 2010 | 30 June 2011 | 30 Dec 2011 | 29 June 2012 | 28 Dec 2012 | 28 June 2013 | 30 Dec 2013 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Mean value of squared active returns | 0.01569 | 0.01509 | 0.01949 | 0.02102 | 0.01746 | 0.02064 | 0.01604 |
| Variance of the Tracking Portfolio returns | 0.35933 | 0.35844 | 0.46609 | 0.49837 | 0.46594 | 0.41017 | 0.33724 |
| Variance of the DAX returns | 0.33926 | 0.34101 | 0.42778 | 0.44185 | 0.44414 | 0.38767 | 0.33202 |
| Alpha of Tracking Portfolio | -0.0035 | -0.0018 | -0.0006 | 0.00580 | -0.0006 | -0.0011 | 0.02196 |
| Beta of Tracking Portfolio | 100.606 | 100.305 | 102.160 | 103.978 | 100.455 | 100.194 | 98.331 |
| Residual Variance | 0.01595 | 0.01534 | 0.01962 | 0.02067 | 0.01774 | 0.02098 | 0.01622 |
| Tracking error as sample variance of active returns | 0.01596 | 0.01535 | 0.01982 | 0.02137 | 0.01775 | 0.02099 | 0.01631 |
| | | | | | | | |
| Rebalancing date | 30 June 2014 | 30 Dec 2014 | 30 June 2015 | 30 Dec 2015 | 30 June 2016 | 30 Dec 2016 | 30 June 2017 |
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Mean value of squared active returns | 0.00898 | 0.00706 | 0.00676 | 0.00992 | 0.00886 | 0.01186 | 0.01150 |
| Variance of the Tracking Portfolio returns | 0.25084 | 0.23194 | 0.23304 | 0.30535 | 0.32418 | 0.23883 | 0.21390 |
| Variance of the DAX returns | 0.24257 | 0.22802 | 0.23178 | 0.27786 | 0.29401 | 0.20987 | 0.18272 |
| Alpha of Tracking Portfolio | 0.00234 | 0.00679 | 0.01373 | -0.0274 | -0.0217 | -0.0491 | -0.0631 |
| Beta of Tracking Portfolio | 99.823 | 99.285 | 98.789 | 103.130 | 103.599 | 104.027 | 105.329 |
| Residual Variance | 0.00913 | 0.00717 | 0.00684 | 0.00981 | 0.00863 | 0.01172 | 0.01118 |
| Tracking error as sample variance of active returns | 0.00913 | 0.00718 | 0.00688 | 0.01009 | 0.00901 | 0.01206 | 0.01170 |
| | | | | | | | |
| Rebalancing date | 29 Dec 2017 | 29 June 2018 | 28 Dec 2018 | 28 June 2019 | 30 Dec 2019 | 30 June 2020 | |
| Mean value of active returns | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | |
| Mean value of squared active returns | 0.01031 | 0.00765 | 0.00673 | 0.00833 | 0.00835 | 0.01092 | |
| Variance of the Tracking Portfolio returns | 0.21282 | 0.22253 | 0.23062 | 0.25187 | 0.23561 | 0.32634 | |
| Variance of the DAX returns | 0.18611 | 0.19040 | 0.19836 | 0.21553 | 0.20930 | 0.26781 | |
| Alpha of Tracking Portfolio | -0.0426 | -0.0526 | -0.0169 | -0.0318 | -0.0258 | -0.0292 | |
| Beta of Tracking Portfolio | 104.360 | 106.397 | 106.407 | 106.465 | 104.256 | 108.855 | |
| Residual Variance | 0.01014 | 0.00700 | 0.00603 | 0.00757 | 0.00811 | 0.00900 | |
| Tracking error as sample variance of active returns | 0.01049 | 0.00778 | 0.00684 | 0.00847 | 0.00849 | 0.01110 | |

**Table 3: Index tracking with the regression method: in-sample results (data in %)**

| Rebalancing date | 30 Dec 2010 | 30 June 2011 | 30 Dec 2011 | 29 June 2012 | 28 Dec 2012 | 28 June 2013 | 30 Dec 2013 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | -0.0389 | 0.04539 | 0.15927 | 0.16327 | 0.08703 | 0.21995 | 0.31804 |
| Mean value of squared active returns | 0.01540 | 0.01490 | 0.01527 | 0.01597 | 0.01673 | 0.01695 | 0.00884 |
| Variance of the Tracking Portfolio returns | 0.35565 | 0.35734 | 0.44779 | 0.45899 | 0.45891 | 0.39964 | 0.33367 |
| Variance of the DAX returns | 0.33926 | 0.34101 | 0.42778 | 0.44185 | 0.44414 | 0.38767 | 0.33202 |
| Alpha of Tracking Portfolio | -0.0395 | 0.04432 | 0.15910 | 0.16346 | 0.08734 | 0.22336 | 0.33052 |
| Beta of Tracking Portfolio | 100.111 | 100.176 | 100.553 | 100.133 | 99.756 | 99.384 | 99.051 |
| Residual Variance | 0.01564 | 0.01513 | 0.01526 | 0.01596 | 0.01693 | 0.01673 | 0.00793 |
| Tracking error as sample variance of active returns | 0.01564 | 0.01513 | 0.01527 | 0.01596 | 0.01694 | 0.01674 | 0.00796 |

| Rebalancing date | 30 June 2014 | 30 Dec 2014 | 30 June 2015 | 30 Dec 2015 | 30 June 2016 | 30 Dec 2016 | 30 June 2017 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | 0.13703 | 0.10797 | 0.08474 | 0.23141 | 0.19856 | 0.25854 | 0.25048 |
| Mean value of squared active returns | 0.00788 | 0.00639 | 0.00637 | 0.00722 | 0.00675 | 0.00780 | 0.00719 |
| Variance of the Tracking Portfolio returns | 0.25073 | 0.23282 | 0.23422 | 0.29583 | 0.31222 | 0.22959 | 0.20283 |
| Variance of the DAX returns | 0.24257 | 0.22802 | 0.23178 | 0.27786 | 0.29401 | 0.20987 | 0.18272 |
| Alpha of Tracking Portfolio | 0.13612 | 0.11127 | 0.09444 | 0.21379 | 0.18653 | 0.22234 | 0.20698 |
| Beta of Tracking Portfolio | 100.069 | 99.653 | 99.144 | 102.009 | 101.998 | 102.970 | 103.675 |
| Residual Variance | 0.00783 | 0.00638 | 0.00639 | 0.00669 | 0.00634 | 0.00707 | 0.00643 |
| Tracking error as sample variance of active returns | 0.00783 | 0.00638 | 0.00641 | 0.00680 | 0.00646 | 0.00725 | 0.00668 |

| Rebalancing date | 29 Dec 2017 | 29 June 2018 | 28 Dec 2018 | 28 June 2019 | 30 Dec 2019 | 30 June 2020 |
|---|---|---|---|---|---|---|
| Mean value of active returns | 0.21494 | 0.11882 | 0.09412 | 0.19573 | 0.14161 | 0.22548 |
| Mean value of squared active returns | 0.00694 | 0.00665 | 0.00620 | 0.00664 | 0.00746 | 0.00838 |
| Variance of the Tracking Portfolio returns | 0.20666 | 0.21604 | 0.22401 | 0.23960 | 0.22777 | 0.30640 |
| Variance of the DAX returns | 0.18611 | 0.19040 | 0.19836 | 0.21553 | 0.20930 | 0.26781 |
| Alpha of Tracking Portfolio | 0.17831 | 0.07774 | 0.08119 | 0.17551 | 0.12558 | 0.20663 |
| Beta of Tracking Portfolio | 103.753 | 104.997 | 104.899 | 104.105 | 102.648 | 105.711 |
| Residual Variance | 0.00633 | 0.00614 | 0.00574 | 0.00600 | 0.00723 | 0.00713 |
| Tracking error as sample variance of active returns | 0.00659 | 0.00662 | 0.00621 | 0.00637 | 0.00738 | 0.00801 |

**Table 4: Index Tracking with linear optimization: in-sample results (data in %)**

| Rebalancing date | 30 Dec 2010 | 30 June 2011 | 30 Dec 2011 | 29 June 2012 | 28 Dec 2012 | 28 June 2013 | 30 Dec 2013 |
|---|---|---|---|---|---|---|---|
| Mean value of active returns | 0.05725 | 0.14644 | 0.28094 | 0.21762 | 0.27644 | 0.41444 | 0.44439 |
| Mean value of squared active returns | 0.01996 | 0.01763 | 0.01957 | 0.02383 | 0.02606 | 0.02320 | 0.01158 |
| Variance of the Tracking Portfolio returns | 0.40093 | 0.38850 | 0.46401 | 0.50501 | 0.50245 | 0.42515 | 0.34522 |
| Variance of the DAX returns | 0.33926 | 0.34101 | 0.42778 | 0.44185 | 0.44414 | 0.38767 | 0.33202 |
| Alpha of Tracking Portfolio | 0.02185 | 0.11997 | 0.28034 | 0.22412 | 0.27176 | 0.40327 | 0.43759 |
| Beta of Tracking Portfolio | 106.103 | 104.366 | 102.001 | 104.459 | 103.668 | 102.015 | 100.517 |
| Residual Variance | 0.01900 | 0.01706 | 0.01893 | 0.02287 | 0.02512 | 0.02169 | 0.00976 |
| Tracking error as sample variance of active returns | 0.02026 | 0.01771 | 0.01910 | 0.02375 | 0.02572 | 0.02185 | 0.00977 |
| | | | | | | | |
| Rebalancing date | 30 June 2014 | 30 Dec 2014 | 30 June 2015 | 30 Dec 2015 | 30 June 2016 | 30 Dec 2016 | 30 June 2017 |
| Mean value of active returns | 0.36891 | 0.33389 | 0.28649 | 0.44179 | 0.46405 | 0.43752 | 0.39066 |
| Mean value of squared active returns | 0.01287 | 0.00988 | 0.00946 | 0.00956 | 0.01062 | 0.01052 | 0.00935 |
| Variance of the Tracking Portfolio returns | 0.26633 | 0.23287 | 0.24108 | 0.29124 | 0.30098 | 0.22839 | 0.19615 |
| Variance of the DAX returns | 0.24257 | 0.22802 | 0.23178 | 0.27786 | 0.29401 | 0.20987 | 0.18272 |
| Alpha of Tracking Portfolio | 0.33607 | 0.34235 | 0.28523 | 0.43289 | 0.46573 | 0.40913 | 0.37293 |
| Beta of Tracking Portfolio | 102.484 | 99.109 | 100.111 | 101.016 | 99.721 | 102.329 | 101.498 |
| Residual Variance | 0.01156 | 0.00890 | 0.00879 | 0.00771 | 0.00861 | 0.00863 | 0.00792 |
| Tracking error as sample variance of active returns | 0.01171 | 0.00892 | 0.00879 | 0.00774 | 0.00861 | 0.00875 | 0.00796 |
| | | | | | | | |
| Rebalancing date | 29 Dec 2017 | 29 June 2018 | 28 Dec 2018 | 28 June 2019 | 30 Dec 2019 | 30 June 2020 | |
| Mean value of active returns | 0.38384 | 0.20314 | 0.19282 | 0.36216 | 0.43687 | 0.45919 | |
| Mean value of squared active returns | 0.00968 | 0.00798 | 0.00736 | 0.00825 | 0.01249 | 0.01139 | |
| Variance of the Tracking Portfolio returns | 0.20616 | 0.20976 | 0.21601 | 0.23095 | 0.21036 | 0.29002 | |
| Variance of the DAX returns | 0.18611 | 0.19040 | 0.19836 | 0.21553 | 0.20930 | 0.26781 | |
| Alpha of Tracking Portfolio | 0.35314 | 0.17796 | 0.18580 | 0.35260 | 0.45090 | 0.45132 | |
| Beta of Tracking Portfolio | 103.145 | 103.064 | 102.658 | 101.941 | 97.682 | 102.385 | |
| Residual Variance | 0.00817 | 0.00752 | 0.00696 | 0.00697 | 0.01065 | 0.00929 | |
| Tracking error as sample variance of active returns | 0.00835 | 0.00769 | 0.00710 | 0.00705 | 0.01076 | 0.00944 | |

### 3.3.2 Out-of-sample results of the index tracking methods considered

Out-of-sample results are of essential importance for the practical use of index tracking methods. For this purpose, the monthly logarithmic returns of the tracking portfolios are determined, which would have resulted in each case with the new weightings due to the half-yearly rebalancing. For example, after the rebalancing on 30 December 2010, the stock prices (adjusted for dividend payments, subscription rights proceeds and stock splits) at the subsequent month-ends are used to determine the monthly portfolio values. This then results in the six logarithmic portfolio returns (starting on 31 January 2011) until the next rebalancing on 30 June 2011. The logarithmic returns of all periods are needed to calculate the out-of-sample performance for the entire period from 30 December 2010 to 30 December 2020. The values for the four index tracking strategies are presented in Table 5.

**Table 5: Out-of-sample results for the entire period**

| | Relative optimization | Markowitz | Regression | Linear optimization |
|---|---|---|---|---|
| Mean value of active returns | 0.02919% | 0.01990% | 0.04622% | 0.09635% |
| Mean value of squared active returns | 0.0180% | 0.0173% | 0.0117% | 0.0100% |
| Variance of the Tracking Portfolio returns | 0.30170% | 0.31457% | 0.30869% | 0.29661% |
| Variance of the DAX returns | 0.27796% | 0.27796% | 0.27796% | 0.27796% |
| Alpha of Tracking Portfolio | 0.02348% | 0.00022% | 0.02671% | 0.08743% |
| Beta of Tracking Portfolio | 1.01000 | 1.03446 | 1.03416 | 1.01563 |
| Residual Variance | 0.01815% | 0.01713% | 0.01142% | 0.00990% |
| Tracking error as sample variance of active returns | 0.01818% | 0.01746% | 0.01175% | 0.00996% |
| Mean logarithmic return | 0.60018% | 0.59089% | 0.61721% | 0.66734% |
| Mean logarithmic return of the DAX | 0.57099% | 0.57099% | 0.57099% | 0.57099% |
| Correlation with the DAX | 0.96945 | 0.97239 | 0.98132 | 0.98318 |
| Sharpe ratio | 10.92682% | 10.53525% | 11.10883% | 12.25340% |
| Sharpe ratio of the DAX | 10.83032% | 10.83032% | 10.83032% | 10.83032% |

Not surprisingly, the mean of the active returns and also the alpha are highest for index tracking by means of linear optimization (as is also the case for the in-sample results). The beta values are comparable and are slightly above one in each case. In contrast, residual variance and tracking error are quite low for all methods, with index tracking by means of linear optimization again leading to the best values. The same applies to the Sharpe ratio. The latter is due to the highest average return with the lowest risk when linear optimization is used for index tracking. In general, the risk is slightly higher for all index tracking methods than for the DAX, but the positive active returns indicate higher returns than the DAX. All methods show a very high correlation with the DAX, which underlines the high similarity of the portfolios. In addition, the alpha values (positive in each case) are close to zero.

For comparison purposes, a portfolio is now used which is weighted at the same rebalancing dates according to the then applicable market capitalization, and a portfolio which is equally weighted at all rebalancing dates. In the latter case, it should be noted that the number of stocks to be held changes at each rebalancing date due to changes in the price of the individual stocks in the portfolio. The results presented in table 6 can be obtained for this.

**Table 6: Out-of-sample results for the entire period**

| | Market capitalization weighted | Equally weighted |
|---|---|---|
| Mean value of active returns | 0.02188% | 0.13162% |
| Mean value of squared active returns | 0.0079% | 0.0112% |
| Variance of the Tracking Portfolio returns | 0.29998% | 0.27670% |
| Variance of the DAX returns | 0.27796% | 0.27796% |
| Alpha of Tracking Portfolio | 0.00744% | 0.14438% |
| Beta of Tracking Portfolio | 1.02529 | 0.97765 |
| Residual Variance | 0.00779% | 0.01103% |
| Tracking error as sample variance of active returns | 0.00796% | 0.01117% |
| Mean logarithmic return | 0.59287% | 0.70261% |
| Mean logarithmic return of the DAX | 0.57099% | 0.57099% |
| Correlation with the DAX | 0.98694 | 0.97987 |
| Sharpe ratio | 10.82464% | 13.35695% |
| Sharpe ratio of the DAX | 10.83032% | 10.83032% |

The result is surprising if, in addition to the portfolio based on market capitalization, the equally weighted portfolio is also included in the analysis. This portfolio leads to the highest mean logarithmic return and thus to the highest mean value of the active returns and at the same time to the lowest variance of returns. This in turn results in the highest Sharpe ratio of all portfolios considered.
The most important results for investors are ranked in Table 7.

**Table 7: Out-of-sample results for the entire period – ranking**

| | Relative optimization | Markowitz | Regression | Linear optimization | Market capitalization | Equally weighted |
|---|---|---|---|---|---|---|
| Mean value of active returns | 4 | 6 | 3 | 2 | 5 | 1 |
| Variance of the Tracking Portfolio returns | 4 | 6 | 5 | 2 | 3 | 1 |
| Alpha of Tracking Portfolio | 4 | 6 | 3 | 2 | 5 | 1 |
| Residual Variance | 6 | 5 | 4 | 2 | 1 | 3 |
| Tracking error as sample variance of active returns | 6 | 5 | 4 | 2 | 1 | 3 |
| Mean logarithmic return | 4 | 6 | 3 | 2 | 5 | 1 |
| Correlation with the DAX | 6 | 5 | 3 | 2 | 1 | 4 |
| Sharpe ratio | 4 | 6 | 3 | 2 | 5 | 1 |
| Average rank | 4.75 | 5.63 | 3.50 | 2.00 | 3.25 | 1.88 |

The ranking in Table 7 shows that, surprisingly, the equally weighted portfolio, which is often also used as a naïve portfolio for comparison purposes, leads to the best ranking on average. In contrast, index tracking according to Markowitz is often ranked last. Of the index tracking methods examined, linear optimization performs best with an overall second place.

At least for the study presented here, it can thus be observed that all four index tracking methods examined lead to a portfolio which initially shows a high degree of similarity to the benchmark (in this case the DAX). However, this is also true for a portfolio that is weighted according to the market capitalization at the respective rebalancing dates as well as for a portfolio in which the stocks (also in the benchmark) are equally weighted at each rebalancing date. Due to the best overall results for the equally weighted portfolio, the analysis presented here gives the impression that the uncomplicated equal weighting is preferable to the more complex index tracking methods considered here (relative optimization, optimization according to Markowitz, regression methods and linear optimization).

# 4 Conclusion

In the context of approximate replication of stock indices, known as index tracking, a target portfolio is mimicked as closely as possible by a portfolio that is actually to be realized (tracking portfolio). Heuristic methods and optimization approaches can be used for this purpose. The best-known optimization approaches include relative optimization, optimization according to Markowitz, the use of regression methods and linear optimization.

For index tracking with relative optimization, the objective is to minimize the residual risk of the tracking portfolio under the constraint of a portfolio alpha of zero and a portfolio beta of one. Index tracking based on Markowitz optimization aims at minimizing the tracking error (*here*: the sample variance of active returns). Similar to these two methods, index tracking using the regression method aims to minimize the difference in returns between the target portfolio and the tracking portfolio. For index tracking using linear optimization, the objective is to minimize the absolute amounts of all negative active returns, while preserving the positive deviations.

For all methods, there is the problem of estimating the parameters that enter the model. To test the tracking qualities of the methods, an empirical analysis of the German stock market is used to determine the in-sample and out-of-sample results of the four strategies for the period from 30 December 2010 to 30 December 2020. In addition, a portfolio composition based on market capitalization and an equally weighted portfolio are considered. The basis for this is a portfolio of 10 DAX stocks, for which a half-yearly rebalancing is implemented in accordance with the respective strategy on the basis of the 60 previous monthly stock returns. A minimum weight in the portfolio of 2% and a maximum proportion of 50% are assumed.

The in-sample results are quite similar for all index tracking methods used in this study. Due to the constraints, relative optimization yields a beta of one and an alpha of zero. While the mean values of the active returns are zero for this method and for Markowitz optimization, the mean values of the active returns are almost exclusively positive when using regression and linear optimization. They are highest when linear optimization is used. The same applies to the alpha of the tracking portfolio.

Of essential importance for practical use is the consideration of the out-of-sample results. At least for this study, it can be stated that all four index tracking methods considered lead to a portfolio that initially shows a high degree of similarity to the benchmark (in this case the DAX). The mean value of the active returns and also the alpha are highest for index tracking by means of linear optimization. Looking at all portfolios, it is surprising that the equally weighted portfolio leads to the best overall results. Therefore, the analysis presented here gives the impression that the uncomplicated equal weighting is preferable to the more sophisticated index tracking methods considered in this study.

This conclusion would have to be verified on the basis of further investigations. It would be advisable to apply the analysis to other DAX stocks, to other time periods and also to other investment universes (with corresponding benchmarks).

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