

Consolidation of a Layer under a Strip Loading and Resting on an Impermeable Foundation

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Abstract

In this study the distributions of the stresses, the pore pressure and the displacements at various locations of a layer subjected to a strip loading and resting on either a smooth or a rough impervious base are found. For the case of rough base, non-slipping condition is examined in view of Coulomb's law of friction. For both smooth and rough base cases, the pore pressure seems to have a parabolic distribution with depth but when the base becomes rough the pore water pressure increases. As the layer is consolidated, the pore pressure gets smaller. Additionally, the normal stress distribution appears to be independent of time while the shear stress distributions vary with time.

Keywords: Behaviour of a soil layer, Strip loading, Impermeable foundation.

1 Introduction

In studying the three dimensional consolidation problems for a soil medium, two types of mathematical models are proposed. In one of these theories, the soil is characterized by some rheological models [1, 2]. In view of the experimental studies, these models for the volumetric and the shear deformations are chosen as standard and Maxwell models respectively. In the other theory, which is proposed by Biot [3], the soil is considered as a two-phase material that consists of a solid soil skeleton and the pore water. The soil skeleton is assumed to obey Hooke's law while the flow of the water through the pores follows Darcy's law.

In a paper by Gibson et al. [4], displacement functions representation is used to analyze the consolidation problem of a clay layer on a smooth impervious base by employing Biot's theory. However, in [4], the distribution of the vertical displacement with time is given at the centre of the loaded region only. Recently,

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Booker [5] studied the consolidation of a layer resting on either smooth or rough rigid foundations by solving Biot's equations. In this work, again, only the time variation of the vertical displacement at the mid-point of the loaded region is found. It appears that there are some discrepancies between the results of Gibson et al. and Booker due to their different mathematical approaches. Recently, Qin, Pan and Xia et al. [8, 9, 10] have done some works in this area as well.

The present study aims at finding not only the vertical displacement at the centre of the loaded area but also the distributions of the stresses and the pore pressure as well as the vertical displacement at various locations of a layer subjected to a strip loading and resting on a rigid impervious foundation which is either smooth or rough. Governing equations are those of Biot's theory and they are solved by employing Fourier sine, cosine transforms for the space and Laplace transform for the time variable. The inversions of both transforms are carried out numerically. The time variation of the vertical displacement at the centre of the loaded region found in the present study appears to agree with that found by Gibson rather than with that by Booker.

For the case of rough base supporting, it is assumed that no-slipping occurs between the layer and the foundation. This no-slipping condition is examined by considering the weight of the layer and by studying its imposed relation with the coefficient of friction. It is found that for a given value of the ratio of the pressure due to the weight and the applied pressure, there exists a critical coefficient of friction such that slipping occurs if the coefficient of friction is less than this critical value.

2 Formulation of the Problem

We consider a porous elastic layer of thickness H , loaded uniformly by a strip pressure p_0 of width $2a$. It is assumed that the layer rests on an impervious rigid foundation which may be either smooth or rough (see Fig. 1).

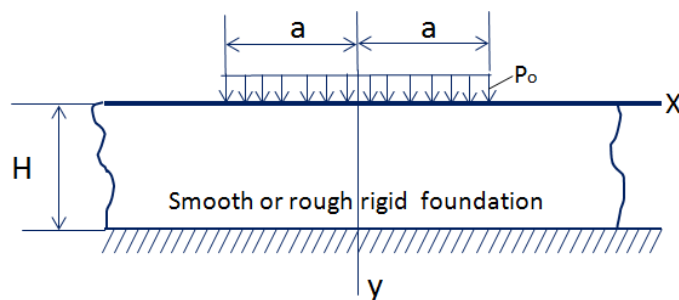


Figure 1: Configuration of the layer

The body is referred to a right handed Cartesian coordinate system in which $x - z$ plane coincides with the upper surface of the layer, the y axis is directed downwards and the loading is uniform in z direction.

The following equations which govern the displacements and the diffusion of the pore pressure are derived by Biot [3]:

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + (\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial \sigma}{\partial x} = 0, \quad (1a)$$

$$\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + (\lambda + \mu) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial \sigma}{\partial y} = 0, \quad (1b)$$

$$k \left(\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} \right) = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (1c)$$

where u, v are the horizontal and the vertical components of the displacement vector, σ the excess pore water pressure, k the coefficient of permeability; λ and μ are Lamé's constants.

The boundary conditions of the problem for smooth and rough bases are:

$$(\lambda + 2\mu) \frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x} = -p(x)U(t), \quad (y = 0, -\infty < x < \infty) \quad (2)$$

$$\sigma = 0, \quad (y = 0, -\infty < x < \infty) \quad (3)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad (y = 0, -\infty < x < \infty) \quad (4)$$

$$\frac{\partial \sigma}{\partial y} = 0, \quad (y = H, -\infty < x < \infty) \quad (5)$$

$$v = 0, \quad (y = H, -\infty < x < \infty) \quad (6)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad (y = H, -\infty < x < \infty) \text{ for smooth base,} \quad (7a)$$

$$u = 0, \quad (y = H, -\infty < x < \infty) \quad \text{for rough base, (without slipping)} \quad (7b)$$

Where:

$$\begin{cases} p(x) = p_0 & \text{for } |x| \leq a \\ = 0 & \text{for } |x| > a \end{cases} \quad (8)$$

and $U(t)$ is the Heaviside step function, and the initial condition is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{at } t = 0, \quad (9)$$

In order to solve eqs. (1.a), (1.b), and (1.c) subjected to boundary conditions, eqs. (2-7), and the initial condition given by eq. (9) we express the unknown functions u, v and σ as:

$$u(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{2}{\pi} \int_0^{\infty} u^{LS}(\alpha, y, s) \sin \alpha x d\alpha \right] e^{st} ds, \quad (10.a)$$

$$v(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{2}{\pi} \int_0^{\infty} v^{LC}(\alpha, y, s) \cos \alpha x d\alpha \right] e^{st} ds, \quad (10.b)$$

$$\sigma(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{2}{\pi} \int_0^{\infty} \sigma^{LC}(\alpha, y, s) \cos \alpha x d\alpha \right] e^{st} ds. \quad (10.c)$$

where u^{LS} , v^{LC} , and σ^{LC} are repeated Fourier (sine or cosine) and Laplace transforms of u , v , and σ . They are defined as:

$$u^{LS}(\alpha, y, s) = \int_0^{\infty} \left[\int_0^{\infty} u(x, y, t) \sin \alpha x d\alpha \right] e^{-st} dt, \quad (11.a)$$

$$v^{LC}(\alpha, y, s) = \int_0^{\infty} \left[\int_0^{\infty} v(x, y, t) \cos \alpha x d\alpha \right] e^{-st} dt, \quad (11.b)$$

$$\sigma^{LC}(\alpha, y, s) = \int_0^{\infty} \left[\int_0^{\infty} \sigma(x, y, t) \cos \alpha x d\alpha \right] e^{-st} dt. \quad (11.c)$$

We note that, in eq. (10), the symmetry or anti-symmetry conditions are used to utilize Fourier sine or cosine transforms. Through the substitution of eqs. (10.a), (10.b), and (10.c) into eqs. (1.a), (1.b), and (1.c) and employing the initial condition, eq. (9), one obtains a system of ordinary differential equations for u^{LS} , v^{LC} , and σ^{LC} . The solution to this system is:

$$u^{LS}(\alpha, y, s) = A_1 e^{\alpha y} + A_2 y e^{\alpha y} + A_3 e^{-\alpha y} + A_4 y e^{-\alpha y} + A_5 e^{py} + A_6 e^{-py}, \quad (12.a)$$

$$v^{LC}(\alpha, y, s) = -A_1 e^{\alpha y} + A_2 \left(-y + \frac{1}{\alpha} \right) e^{\alpha y} + A_3 e^{-\alpha y} + A_4 \left(y + \frac{1}{\alpha} \right) e^{-\alpha y} + A_5 \frac{p}{\alpha} e^{py} + A_6 \frac{p}{\alpha} e^{-py}, \quad (12.b)$$

$$\sigma^{LC}(\alpha, y, s) = -2A_2 \mu e^{\alpha y} + 2A_4 \mu e^{-\alpha y} - A_5 \frac{s}{k\alpha} e^{py} - A_6 \frac{s}{k\alpha} e^{-py}. \quad (12.c)$$

Where:

$$p = \sqrt{\alpha^2 + \frac{s}{k(\lambda + 2\mu)}}.$$

The unknown function $A_i(\alpha, s)$, ($i = 1, \dots, 6$) are to be determined from the boundary conditions. eqs. (2-7). For the smooth base, A_i 's are the solutions of the linear system $\overline{CA} = \overline{F}$,

$$(13)$$

Where:

$$\overline{A} = (A_1, A_2, A_3, A_4, A_5, A_6), \quad (14)$$

$$\overline{F} = \left(-\frac{p_0 \sin \alpha a}{s\alpha}, 0, 0, 0, 0, 0 \right), \quad (15)$$

and \overline{C} is a (6×6) square matrix:

$$\begin{bmatrix} -2\mu\alpha & 0 & -2\mu\alpha & 0 & -\left(2\mu\alpha + \frac{s}{k\alpha}\right) & -\left(2\mu\alpha + \frac{s}{k\alpha}\right) \\ 0 & -2\mu & 0 & 2\mu & -\frac{s}{k\alpha} & -\frac{s}{k\alpha} \\ \alpha & 0 & -\alpha & 0 & p & -p \\ 0 & -2\mu\alpha e^{\alpha H} & 0 & -2\mu\alpha e^{-\alpha H} & -\frac{sp}{k\alpha} e^{pH} & \frac{sp}{k\alpha} e^{-pH} \\ e^{-\alpha H} & \frac{1}{\alpha}(1-\alpha H)e^{\alpha H} & e^{-\alpha H} & \frac{1}{\alpha}(1+\alpha H)e^{-\alpha H} & -\frac{p}{\alpha} e^{pH} & \frac{p}{\alpha} e^{-pH} \\ \alpha e^{\alpha H} & \alpha H e^{\alpha H} & -\alpha e^{-\alpha H} & -\alpha H e^{-\alpha H} & p e^{pH} & -p e^{-pH} \end{bmatrix} \quad (16)$$

For the rough base, the last row of the square matrix, eq. (16), should be replaced by:

$$\left(e^{\alpha H}, H e^{\alpha H}, e^{-\alpha H}, H e^{-\alpha H}, e^{pH}, e^{-pH} \right), \quad (17)$$

Through the use of the constitutive equations for porous media

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \sigma, \quad (18a)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \sigma, \quad (18b)$$

$$\tau_{zz} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \sigma, \quad (18c)$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (18d)$$

the relevant stress components take the forms:

$$\tau_{xx}(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{2}{\pi} \int_0^\infty \left[2\mu \alpha u^{LS} + \lambda \left(\alpha u^{LS} + \frac{dv^{LC}}{dy} \right) - \sigma^{LC} \right] \cos \alpha x d\alpha \right\} e^{st} ds, \quad (19a)$$

$$\tau_{yy}(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{2}{\pi} \int_0^\infty \left[2\mu \frac{dv^{LC}}{dy} + \lambda \left(\alpha u^{LS} + \frac{dv^{LC}}{dy} \right) - \sigma^{LC} \right] \cos \alpha x d\alpha \right\} e^{st} ds, \quad (19b)$$

$$\tau_{zz}(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{2}{\pi} \int_0^\infty \left[\lambda \left(\alpha u^{LS} + \frac{dv^{LC}}{dy} \right) - \sigma^{LC} \right] \cos \alpha x d\alpha \right\} e^{st} ds, \quad (19c)$$

$$\tau_{xy}(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{2}{\pi} \int_0^\infty \mu \left(\frac{du^{LS}}{dy} - \alpha v^{LC} \right) \sin \alpha x d\alpha \right\} e^{st} ds. \quad (19d)$$

At time $t = \infty$, the pore pressure becomes zero, thus, the solution is reduced to the elastostatic solution which reads:

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \left[(B_1 + B_2 y) e^{-\alpha y} + (B_3 + B_4 y) e^{\alpha y} \right] \sin \alpha x d\alpha, \quad (20a)$$

$$v(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ \left[B_1 + \left(\frac{\chi}{\alpha} + y \right) B_2 \right] e^{-\alpha y} + \left[-B_3 + \left(\frac{\chi}{\alpha} - y \right) B_4 \right] e^{\alpha y} \right\} \cos \alpha x d\alpha, \quad (20b)$$

where $B_i(\alpha)$, ($i=1, \dots, 4$) are to be determined from the appropriate boundary conditions and $\chi = 3 - 4\nu$, ν being the Poisson ratio. The corresponding stress components can be obtained from Hooke's law.

3 Solutions and some Numerical Results

The stress, displacement and the pore pressure distributions are obtained through eqs. (10.a), (10.b), and (10.c) and eqs. (19.a), (19.b), (19.c), and (19.d) by employing numerical inversion techniques for both Fourier and Laplace transforms. For time $t = \infty$ since there is no time dependency, only the use of Fourier inversion technique in eqs. (20.a), and (20.b) and the resulting stress expressions are sufficient. To

evaluate the infinite integrals appearing in Fourier inversion expressions, the numerical integration quadrature given by Filon [6] is used. For the numerical inversion of the Laplace transform, a method proposed by Krylov. V. I. and Skoblya, N. S. [7] is utilized. This method is now reviewed very briefly. The Laplace transform $f^L(s)$ of a function $f(t)$ is defined by:

$$f^L(s) = \int_0^{\infty} f(t)e^{-st} dt \tag{21}$$

having the inversion relation:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^L(s)e^{st} ds. \tag{22}$$

We assume the form of $f^L(s)$ to be expressed as:

$$f^L(s) = \frac{1}{s^m} \varphi(s),$$

thus the integral relation, eq. (22), becomes:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-m} \varphi(s)e^{st} ds. \tag{23}$$

By approximating $\varphi(s)$ by an n^{th} order polynomial at $(n+1)$ equally spaced points $s_k = k+1, (k = 0, \dots, n)$ eq. (23) takes the form

$$f(t) = \sum_{k=0}^n \left\{ \sum_{j=0}^n \frac{a_{kj} t^{m+j-1}}{\Gamma(m+j)} \right\} \varphi(k+1), \tag{24}$$

where the values of a_{kj} are tabulated in [7].

In Figures. (2-7) the variations of the vertical displacements, the pore pressure and some stress components at various times are displayed for a layer supported by a smooth base. For a rough base, the same quantities are plotted in Figs. (8-13). The non-dimensional time \bar{t} which appears in the figures is defined by the relation:

$$\bar{t} = \frac{2\mu k}{a^2} t \tag{25}$$

As it is seen in Figs. (2) and (8), for both types of foundations the final settlement is

approximately reached at $\bar{t} = 1$ and the settlement at the point $(x = 0, y = 0)$ is larger in the case of smooth base. It should be observed that the lifting of the soil on the top surface for the rough base appears to be greater than that of the smooth base.

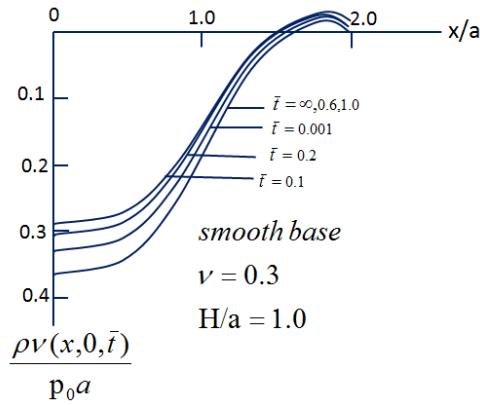


Figure 2: Variation of the vertical displacement with x at the top surface for the smooth foundation

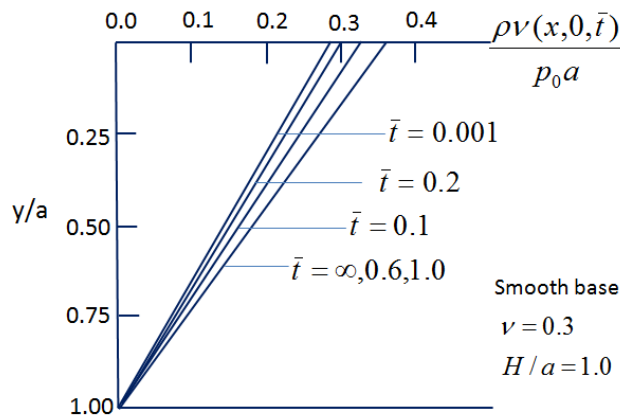


Figure 3: Variation of the vertical displacement with depth at $x = 0$ for the smooth foundation

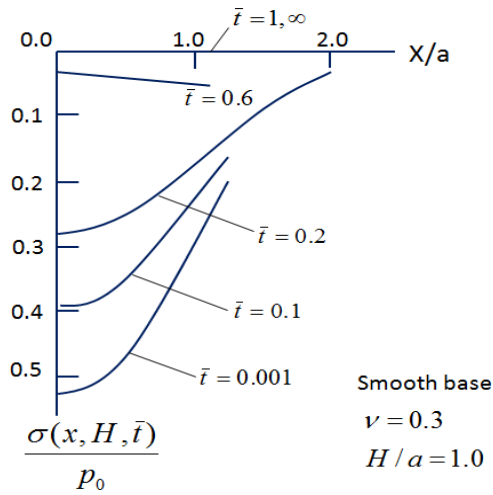


Figure 4: Variation of the pore pressure with x at the base for the smooth foundation

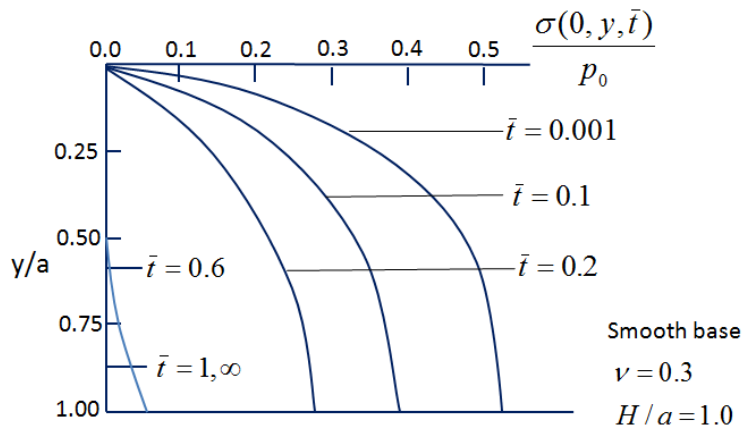


Figure 5: Variation of the pore pressure with depth at $x = 0$ for the smooth foundation

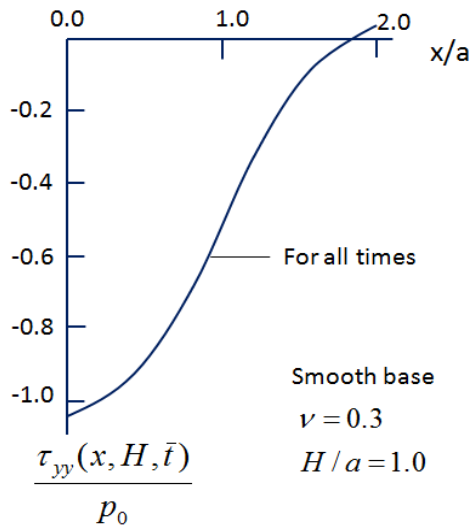


Figure 6: Variation of the normal stress with x at the base for the smooth foundation

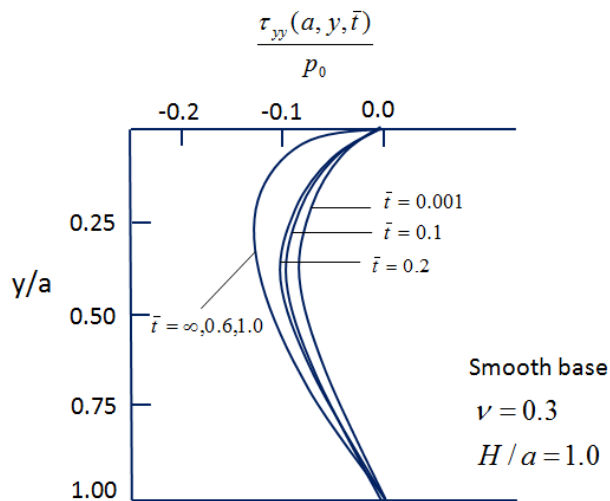


Figure 7: Variation of the shear stress with depth at $x = a$ for the smooth base

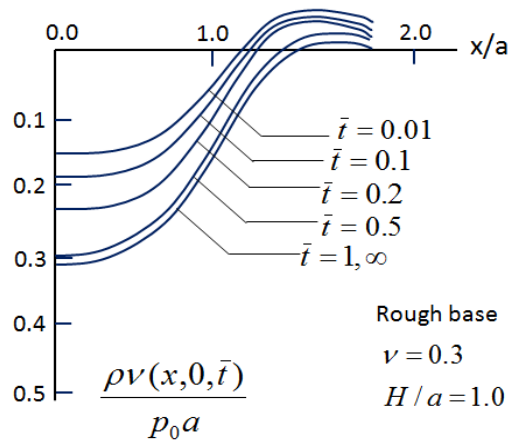


Figure 8: Variation of the vertical displacement with x at the top surface for the rough foundation

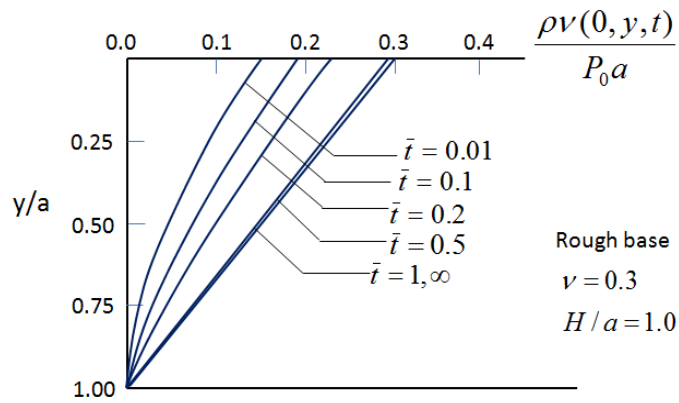


Figure 9: Variation of the vertical displacement with depth at $x = 0$ for the rough foundation

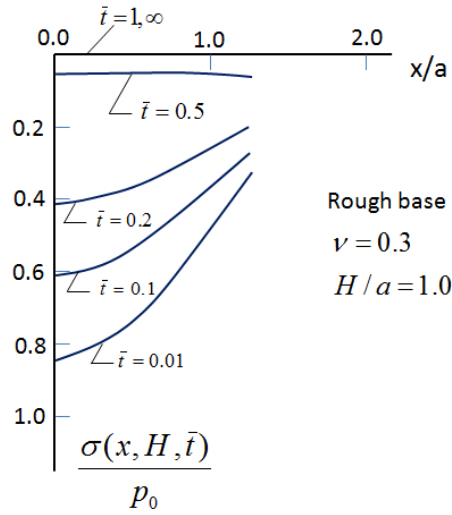


Figure 10: Variation of the pore pressure with x at the base for the rough foundation

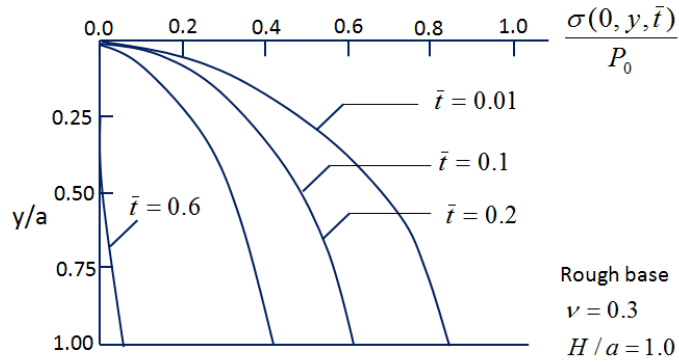


Figure 11: Variation of the pore pressure with depth at $x = 0$ for the rough foundation

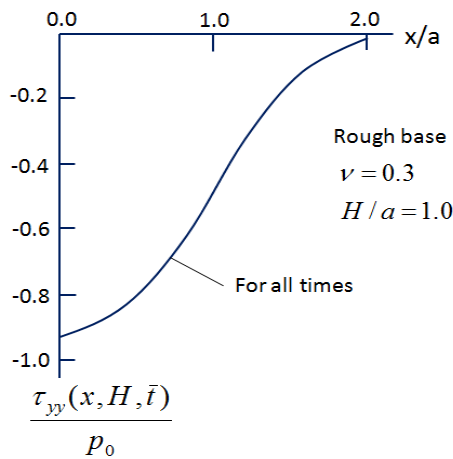


Figure 12: Variation of the normal stress with x at the base for the rough foundation

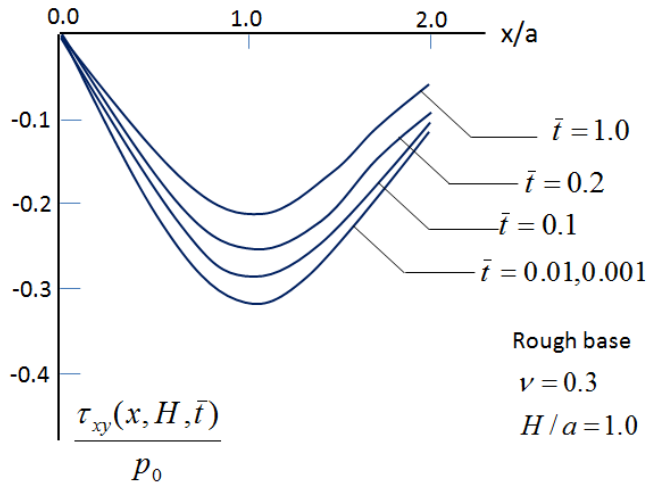


Figure 13: Variation of the shear stress with x at the base for the rough foundation

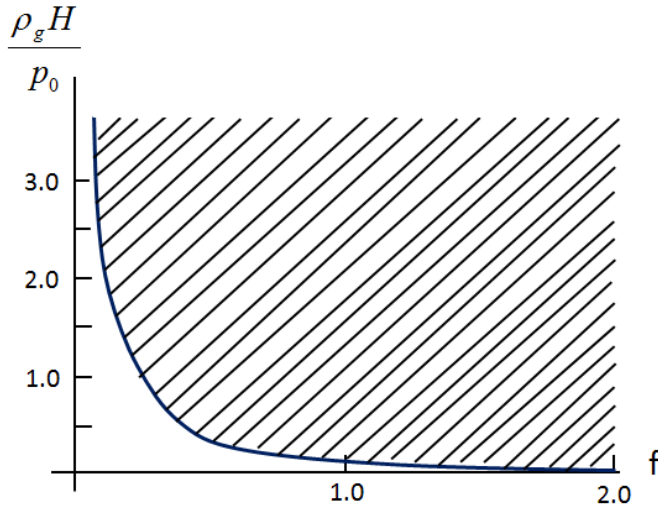


Figure 14: Sketch of the region for no slipping

For both smooth and rough base cases, the pore pressure seems to have a parabolic distribution with depth [see Figs. (5) and (11)] and from Figs. (4) and (10) one can conclude that when the base becomes rough the pore pressure increases. We also note that as the layer is consolidated, the pore pressure gets smaller. The normal stress distribution appears to be independent of time [Figs. (6) and (12)] while the shear stress distributions vary with time [see Figs. (7) and (13)].

For the case of a rough base, it was assumed that no-slipping occurs between the layer and the foundation. This condition is valid provided that the ratio of the shear stress to the total normal pressure, which is the sum of the uniform normal pressure due to the weight of the layer and $\tau_{yy}(x, H)$, is less than the coefficient of friction, that is

$$\frac{|\tau_{xy}|}{\rho g H + |\tau_{yy}|} \leq f, \quad (26)$$

where f is the coefficient of friction, ρ is the mass density and g is the acceleration of gravity. The condition, eq. (26), can be re-written as:

$$\frac{\rho g H}{p_0} \geq \frac{|\tau_{xy}|}{f} - |\bar{\tau}_{yy}|, \quad (27)$$

where

$$\bar{\tau}_{xy} = \frac{\tau_{xy}}{p_0}, \quad \bar{\tau}_{yy} = \frac{\tau_{yy}}{p_0}.$$

From a study of Fig. (12) and (13) it follows that the critical range and for x/a and the critical time at which slipping occurs are-, respectively,

$$\frac{x}{a} \geq 1, \quad \bar{t} = 0.$$

When the condition, eq. (27), is examined in view of the normal and the shear stress distributions [Figs. (12) and (13)], we find that the base can be assumed to be a rough

base without slipping if the point $\left(f, \frac{\rho g H}{p_0}\right)$ on the $\left(f - \frac{\rho g H}{p_0}\right)$ plane is in the

shaded region shown in Fig. (14). For a given value of $\frac{\rho g H}{p_0}$, we can and determine

from Fig. (14) a critical value for the coefficient of friction, f_c , so that if f is larger than f_c , slipping will not occur. It should emphasize that difficult types of boundary conditions for the layer can be handled by just making appropriate changes in eq. (13). This is one of the advantages of the numerical method used in this study.

4 Conclusions

In this paper the stress, displacement and the pore pressure distributions are obtained by using numerical inversion techniques for both Fourier and Laplace transforms at different locations of a layer subjected to a strip loading and resting on either a smooth or a rough impermeable base. For the case of rough base, non-slipping condition is examined in view of Coulomb's law of friction. It is observed that the lifting of the soil on the top surface for the rough base is greater than the smooth base. For both smooth and rough base cases, the pore pressure seems to have a parabolic distribution with depth but when the base becomes rough the pore water pressure increases. Furthermore, as the layer is consolidated, the pore pressure gets smaller as well as the

normal stress distribution appears to be independent of time but the shear stress distributions vary with time.

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