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# Option pricing within Heston's stochastic and stochastic-jump models

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**Abstract**

The quest to have a model that will be better at approximating market prices and produce fit better than Heston's Stochastic model motivated us to combine jump components to Heston's Stochastic model which we called Heston's Stochastic-Jump model (HSJ). Complete derivation of the Heston's Stochastic-Jump model was presented. Simulation studies were conducted. Pricing performances of Heston's Stochastic and Heston's Stochastic-Jump models were empirically analysed using the NASDAQ index call option price quotations. Results show that Heston's Stochastic-Jump model performed better than Heston's Stochastic model by about 18% reduction in error.

**Keywords:** Heston's Stochastic Model, Heston's Stochastic-Jump Model, Calibration, Fast Fourier Transform, Mean-Squared Error

## 1 Introduction

Financial models are needed in the financial market for the pricing and estimation of fair values of various securities, estimate their risks and show how to control these risks. Since the introduction of the Black Scholes model which was developed for the pricing of financial options, many complex models have been developed such as stochastic volatility models which are used mainly by traders and quantitative analysts for pricing and hedging financial assets. These models were developed with contributions from [1], [2], [3], [4]. Volatility measures the unexpected changes in the value of a financial asset in a certain time period. Volatility is used as a measure of risk of certain financial assets. Therefore, stochastic volatility models treat volatility of the underlying asset as a random process rather than a constant as in the case of Black Scholes model. By calibrating the parameters of a stochastic process, it can be used to estimate prices close to the market values. The assumption of constant volatility in Black-Scholes model has led to numerous attempts of developing new models that would fit the empirical option prices better. In financial markets, the implied volatilities often represent a "smile" or "skew" instead of a straight line. The "smile" reflects higher implied volatilities for in- or out-of-the money options and lower implied volatilities for at-the-money options. It was observed from the work of [6] that Heston's model which is one of the stochastic volatility models is problematic when it comes to fitting short maturities. Reference [7] reported that incorporating both stochastic volatility and jumps in equity returns dynamics makes the short maturity returns less

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Gaussian, so the implied volatilities will fit better. However, the quest to have a model that will be better at producing lowest pricing errors and produce fit better than Heston's Stochastic model motivated us to incorporate jump components to Heston's Stochastic model which we called Heston's Stochastic-Jump model (HSJM). We will present a complete derivation of the Heston's Stochastic-Jump model, empirically analyse its pricing performances and compare it with the original Heston's Volatility model. Fast Fourier Transform (FFT) pricing formula proposed by [8] will be used for calculation of option prices and Euler Monte Carlo simulation for simulating the price paths of the models.

## 2 Option Pricing Models

In this section, a brief introduction to the Black Scholes model will be given which will help in understanding the Heston model.

**The Black-Scholes model**

Many of the techniques and option pricing models used in financial theory and practice are derived from the ideas and methods presented by [9]. The following formulas give Black-Scholes price of a European call at time *t*:

$$C(S,K,T) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{1}$$

where

$$d_1 = \frac{\ln\left(\frac{Se^{-q(T-t}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \;,$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and $N(.)$ is the standard normal cumulative distribution function,

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{\frac{-t^2}{2}}\,dt \;.$$

The stock price follows the Geometric Brownian Motion which has the following dynamics in the risk neutral world:

$$\partial S_t = (r-q)S_t dt + \sigma S_t \partial B_t \tag{2}$$

where

$S_t$: the stock price at time *t*, *t*: current time, $r$: the risk-free interest rate, $q$ : the dividend yield, assumed to be constant, σ: the volatility of the asset's price, which is constant in this case, $B_t$: Brownian Motion, $K$ : the strike price, $(T-t)$: the time to maturity.

**Heston's Model**

This section presents the Heston's Stochastic Volatility option pricing model, which is a type of stochastic volatility model developed by [4] for analysing bond and currency options. The Heston's model is a closed-form solution for pricing options that seeks to overcome the shortcomings in the Black-Scholes option pricing model related to return skewness and strike-price bias. The Heston's model is a tool for advanced investors. It assumes that the underlying stock price $S_t$ follows a Black-Scholes type of stochastic process, but with a stochastic variance $v_t$

$$dS_t = \mu S_t dt + S_t\,\sqrt{v_t}\,dB_1 \tag{3}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB_2 \tag{4}$$

where
$\kappa, \theta, \sigma > 0$ are constant parameters. The two Brownian motions, $B_1$ and $B_2$ are correlated, i.e $\text{corr}[dB_1, dB_2] = \rho dt$.The dynamics of the stock price $S_t$ in (3) is a geometric Brownian motion with time varying volatility. The variance $v_t$ in (4) follows a square root process. The parameter $\theta$ corresponds to the long-run average of $v_t$ and $\kappa$ controls the speed by which the process returns to its long-run mean. The parameter $\rho$ is the correlation between the underlying and the volatility while $\sigma$ is the volatility of volatility (i.e. the volatility of the variance of returns).

**Jump Process**

A jump process is a type of stochastic process that has discrete movements, called jumps, with random arrival times, rather than continuous movement, typically modelled as a simple or compound Poisson process. Reference [10] introduced a jump process to model dynamics as follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t + \sum_{i=1}^{N_t} Y_t \tag{5}$$

where $\mu$ and $\sigma$ are the drift and volatility respectively, $B_t$ is a standard Wiener process, $N_t$ is a Poisson process with jump intensity $\alpha$ and $Y_i \sim N(\theta, \delta^2)$ is the jump size. Reference [10] shows how to price European options with the proposed model. The two basic building blocks of every jump-diffusion model are the Brownian motion (the diffusion part) and the Poisson process (the jump part). The Brownian motion is a familiar object to every option trader since the appearance of the Black-Scholes model.

## 3 Methodology

**Combining Heston's Stochastic Model (HSM) with Jump**

The quest for a model that will be better in the approximation of market prices and produce a better fit than the Heston's Stochastic model motivated us to combine jump components to Heston's model which we called Heston's Stochastic-Jump model (HSJM). Adding jump components to the previous specification of Heston Stochastic model gives

$$dS_t = (r - \alpha R)S_t dt + S_t\sqrt{v_t}dB_1 + R_t S_t dN_t \tag{6}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB_2 \tag{7}$$

where the process $N_t$ represents a Poisson process under the risk neutral measure, with jump intensity $\alpha > 0$. $N_t$ is independent of the two Brownian motions in the stock price and variance processes. The percentage jump size of the stock price is dictated by the random variable $R_t$,

$r$ is the riskless rate,

$\alpha R$ is the expected jump size,

$r - \alpha R = \mu$ ie drift term,

$\kappa$ is the rate of reversion of the variance $v_t$, $B_1$ and $B_2$ are Brownian motions with correlation $\rho$

$$c\text{ov}|dB_1, dB_2| = \rho dt,$$

$$\Pr\left[dN_t = 1\right] = \alpha dt$$
$$\mathrm{E}\left[R_t\right] = R_t$$
$$\mathrm{E}\left[R_t, dN_t\right] = \alpha E\left[R_t\right]dt = \alpha R_t$$

$\log\left(1+R_t\right) \sim N\left(\log\left(1+R\right) - \frac{1}{2}\delta^2, \delta^2\right)$ which defines $1+R_t$ as a log-normal jump with mean $\eta S$ and variance $\sigma_S^2$.

$\theta$ is the mean level of $v_t$ and $\sigma$ is the volatility of $v_t$.

Derivation of Heston's Stochastic-Jump Valuation Equation

Assuming that the stock price and the variance satisfy equations (6) and (7), deriving the Heston's Stochastic-Jump partial differential equation requires forming a riskless portfolio. Setting up a portfolio $\Pi$ which contains the option being priced with its value denoted by $M = M\left(S,v,t\right)$, $\Delta$ units of the stock S, $\varphi$ units of another options $N = N\left(S,v,t\right)$ which hedges the volatility.

$$\Pi = M - \Delta S - \varphi N \tag{8}$$

The change in the portfolio in time $dt$ is given by:

$$d\Pi = dM - \Delta dS - \varphi dN \tag{9}$$

Applying Itô's Lemma to $dM$ and $dN$ differentiating with respect to the variables $S, v,$ and $t$. Following [11], Itô formula for the jump process is given as:

$$df\left(X_t,t\right) = \frac{\partial f\left(X_t,t\right)}{\partial t} + b_t\frac{\partial f\left(X_t,t\right)}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial f\left(X_t,t\right)}{\partial X^2} + \left[f\left(X_{t-} + cX_t\right) + f\left(cX_{t-}\right)\right] \tag{10}$$

Applying (10) to our case of option price function $M\left(S,v,t\right)$, we have

$$\begin{aligned} dM &= \frac{\partial M}{\partial t}dt + \frac{\partial M}{\partial S}dS + \frac{\partial M}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}dt \\ &+ \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}dt + \left[M\left(RS,t\right) - M\left(S,t\right)\right]dN \end{aligned} \tag{11}$$

The term $\left[M\left(RS,t\right) - M\left(S,t\right)\right]dN_t$ describes the difference in the option value when a jump occurs. Applying Itô's Lemma again to $dN$ and differentiating with respect to the variables $S, v,$ and $t$, to obtain:

$$\begin{aligned} dN &= \frac{\partial N}{\partial t}dt + \frac{\partial N}{\partial S}dS + \frac{\partial N}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}dt \\ &+ \sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}dt + \left[N\left(RS,t\right) - N\left(S,t\right)\right]dN_t \end{aligned} \tag{12}$$

Inserting equations (11) and (12) into (9), the change in the value of portfolio $d\Pi$ will now be written as:

$$d\Pi = \left.\begin{array}{l} \frac{\partial M}{\partial t}dt + \frac{\partial M}{\partial S}dS + \frac{\partial M}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}dt \\ +\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}dt + \left[M\left(R_tS_t,t\right) - M\left(S_t,t\right)\right]dN_t - \Delta dS \\ -\varphi\left(\begin{array}{l} \frac{\partial N}{\partial t}dt + \frac{\partial N}{\partial S}dS + \frac{\partial N}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}dt \\ +\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}dt + \left[N\left(RS,t\right) - N\left(S,t\right)\right]dN_t \end{array}\right) \end{array}\right\} \quad (13)$$

Rearranging equation (13),so that $dt$ terms for $M$, $dt$ for $N$, $dS$, $dv$ and $dN_t$ terms are grouped together to have

$$d\Pi = \left.\begin{array}{l} \left(\frac{\partial M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)dt + \\ -\varphi\left(\frac{\partial N}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2} + \sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right)dt \\ +\left(\frac{\partial M}{\partial S} - \varphi\frac{\partial N}{\partial S} - \Delta\right)dS + \left(\frac{\partial M}{\partial v} - \varphi\frac{\partial N}{\partial v}\right)dv \\ +\left\{\left[M\left(R_tS_t,t\right) - M\left(S_t,t\right)\right] - \varphi\left[N\left(RS,t\right) - N\left(S,t\right)\right]\right\}dN_t \end{array}\right\} \quad (14)$$

The two terms $ds$ and $dv$ in (14) contribute to risk in the portfolio according to [4]. However, for the portfolio to be risk free $dS$ and $dv$ must be eliminated by equating their coefficients to zero. The hedge parameters now become

$$\left.\varphi = \frac{\frac{\partial M}{\partial v}}{\frac{\partial N}{\partial v}}, \qquad \Delta = \frac{\partial M}{\partial S} - \varphi\frac{\partial N}{\partial S}\right\} \quad (15)$$

which lives (14) as

$$d\Pi = \left.\begin{array}{l} \left(\frac{\partial M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)dt + \\ -\varphi\left(\frac{\partial N}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2} + \sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right)dt \\ +\left\{\left[M\left(R_tS_t,t\right) - M\left(S_t,t\right)\right] - \varphi\left[N\left(RS,t\right) - N\left(S,t\right)\right]\right\}dN_t \end{array}\right\} \quad (16)$$

The portfolio should also earn a free risk rate, thus:

$$d\Pi = r\left(M - \Delta S - \varphi N\right)dt \quad (17)$$

Equating the right hand side of (16) to right hand side of (17), dividing both side by $dt$,

$$\left(\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)+$$
$$-\varphi\left(\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right) \tag{18}$$
$$+\left\{\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]-\varphi\left[N\left(RS,t\right)-N\left(S,t\right)\right]\right\}dN_t=r\left(M-\Lambda S-\varphi N\right)$$

Plugging the values of $\varphi$ and $\Delta$ from (15), we have

$$\begin{aligned}&\left(\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)\\&-\frac{\partial M/\partial v}{\partial N/\partial v}\left(\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right)\\&+\left\{\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]-\frac{\partial M/\partial v}{\partial N/\partial v}\left[N\left(RS,t\right)-N\left(S,t\right)\right]\right\}dN_t\\&=r\left(M-\frac{\partial M}{\partial S}S+\frac{\partial M/\partial v}{\partial N/\partial v}\left(\frac{\partial N}{\partial S}\right)S-\frac{\partial M/\partial v}{\partial N/\partial v}N\right)\end{aligned} \tag{19}$$

When equation (19) is rearranged, such that $M$ terms will be on one side and $N$ terms will be on other side, then divide both sides by $\frac{\partial M}{\partial v}$ and $\frac{\partial N}{\partial v}$ respectively, we take their expectations over the probability distribution of jumps to obtain:

$$\begin{aligned}&\left\{\begin{aligned}&\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}-rM+rS\frac{\partial M}{\partial S}\\&+\alpha E\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]dN_t\end{aligned}\right\}\Bigg/\frac{\partial M}{\partial v}\\&=\left\{\begin{aligned}&\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}-rN+rS\frac{\partial N}{\partial S}\\&+\alpha E\left[N\left(R_tS_t,t\right)-N\left(S_t,t\right)\right]dN_t\end{aligned}\right\}\Bigg/\frac{\partial N}{\partial v}\end{aligned} \tag{20}$$

Note that:

$$E\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]=\int_0^\infty\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]M\left(R\right)dR \tag{21}$$

Equation (21) is the expected value of the change in the option price with respect to the jump probability distribution function.

Equation (20) now becomes

$$\begin{aligned}&\left\{\begin{aligned}&\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}-rM+rS\frac{\partial M}{\partial S}\\&+\alpha\int_0^\infty\left[M\left(R_tS_t,t\right)-M\left(S_t,t\right)\right]M\left(R\right)dR\end{aligned}\right\}\Bigg/\frac{\partial M}{\partial v}\\&=\left\{\begin{aligned}&\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}-rN+rS\frac{\partial N}{\partial S}\\&+\alpha\int_0^\infty\left[N\left(R_tS_t,t\right)-N\left(S_t,t\right)\right]N\left(R\right)dR\end{aligned}\right\}\Bigg/\frac{\partial N}{\partial v}\end{aligned} \tag{22}$$

The expression in terms of $M$ and that in terms of $N$ in (22) are the same but represent different options. This means that each of the two expressions can be written as a function $M(S,v,t)$ of $S, v,$ and $t$. Following [4], this function can be specified as $M(S,v,t) = -\kappa(\theta - v) + \alpha(S,v,t)$, that is

$$\left\{\begin{array}{l}\frac{\partial M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S} - rM + rS\frac{\partial M}{\partial S} \\ +\alpha\int_0^\infty\left[M(R_t S_t,t) - M(S_t,t)\right]M(R)\,dR\end{array}\right\} \Bigg/ \frac{\partial M}{\partial v} = -\kappa(\theta - v) + \alpha(S,v,t) \quad (23)$$

Multiplying both sides of (23) by $\frac{dM}{dv}$ and rearranging to obtain

$$\begin{array}{l}\frac{M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S} - rM + rS\frac{\partial M}{\partial S} \\ +\kappa(\theta - v)\frac{\partial M}{\partial v} - \alpha(S,v,t)\frac{\partial M}{\partial v} + \alpha\int_0^\infty\left[M(R_t S_t,t) - M(S_t,t)\right]M(R)\,dR = 0\end{array} \quad (24)$$

As written in [4], the market price of risk is a linear function of the volatility, such that:

$$\alpha(S,v,t) = \alpha v.$$

Therefore, equation (24) can be written as

$$\begin{array}{l}\frac{M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S} - rM + rS\frac{\partial M}{\partial S} \\ +\kappa(\theta - v)\frac{\partial M}{\partial v} - \alpha v\frac{\partial M}{\partial v} + \alpha\int_0^\infty\left[M(R_t S_t,t) - M(S_t,t)\right]M(R)\,dR = 0\end{array} \quad (25)$$

Equation (25) is the Heston's Stochastic-Jump Partial Differential Equation with the inclusion of jump component $R_t$ which must be satisfied by the value of an option.

Following [12], the boundary and initial conditions that are imposed are the following:

$M(S,v,t) = \max(S - K, 0)$

$M(S,v,t) = 0$, this means that when the stock price is 0, the call price will also be 0.

$\frac{\partial M}{\partial S}(\infty, v, t) = 1$, this means that as the stock price increases, delta gets closer to 1.

$M(S,\infty,t) = S$, this means that as the volatility increases, the call value gets equal to the stock price.

## 4 Pricing Methods

This section describes some of the pricing techniques that can be used to produce option prices from our models. However, the Fast Fourier Transform (FFT) and Monte Carlo methods will be used in this paper.

**Fast Fourier Transform (FFT)**

This section, therefore introduces a popular method called Fast Fourier Transform (FFT) proposed by [8]. This method will be used for calculation of option prices in this paper. The price of a European call option using Fourier Transform is given by:

$$C_T(k) = \frac{e^{-\alpha k}}{\Pi}\int_0^{\infty}\left[\frac{e^{-rT}\phi_T(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\right]du \tag{26}$$

**Monte Carlo Simulation**

Monte Carlo simulation is a generic algorithm that generates a large number of sample paths according to the model under consideration, then computes the options' payoff for each path in the sample. The average is then taken to find an approximation to the expected present value of the option. The Monte Carlo result converges to the option value in the limit as the number of paths in the sample goes infinity. Monte Carlo simulation has the advantage of being easy to implement and can be used to evaluate large range of European options.

## 5 Calibration of Model Parameters

Calibration means determining the model parameters to match the market prices of a set of options. A model can only be useful in practice if it returns, at least approximately, the current market prices of European options. The purpose of the calibration is to make the model prices fit as closely as possible with the market prices, by reducing the error margin between the estimated model prices and the observed market prices, i.e. is to find the parameter set that minimizes the distance between model predictions and observed market prices. In particular, using the risk-neutral measure, the Heston Stochastic model has five unknown parameters $\Omega=\{v_0,\theta,\sigma,\rho,\kappa\}$ (defined in section 2) which need to be calibrated or estimated while Heston's Stochastic-Jump model has eight unknown parameters $\Theta=\{v_0,\theta,\sigma,\rho,\kappa,\mu_j,\sigma_j,\alpha\}$ which also need to be calibrated. Therefore, by calibrating these parameters values, we seek to obtain an evolution for the underlying asset that is consistent with the current prices of European options. This is called an inverse problem. The most popular approach to solving this inverse problem is to minimise the error or discrepancy between model prices and market prices. This usually turns out to be a non-linear least-squares optimisation problem.

In order to find the optimal parameter $\Omega$, we need to

(i) define a measure to quantify the distance between model and market prices,

(ii) run an optimization scheme to determine the parameter values that minimize such distance. A simple and straightforward approach is to minimize the sum of squared differences.

There are many calibration methods but we will use the Adaptive Simulated Annealing (ASA) method in this paper.

**Adaptive Simulated Annealing (ASA)**

Adaptive Simulated Annealing (ASA) is a calibration method that statistically find the best global fit of a non-linear constrained non-convex cost function over a D-dimensional space. According to [13], Adaptive Simulated Annealing (ASA) can be implemented in MATLAB by downloading the function asamin, written by Shinichi Sakata. asamin is a MATLAB gateway function to ASA. Detailed instructions of installing and the use ASA on one's computer and asamin into MATLAB can be found in [14]. According to [15] in [6] the way the algorithm works is by conducting a guided search, where new iterations are generated by not only considering the previous information, but also by making use of randomization. The main advantage of this optimization method is that it does not exhaust its search on the first minimum attained. It includes stochastic movements in their search pattern, which makes it possible to overcome local minimums and continue searching even if a potential solution has already been found.

# 6 Results

**Calibration of Heston Stochastic Model to Real Market Prices**

Here, we will calibrate Heston Stochastic model to data obtained from real market. The data used was extracted from Bloomberg. The data consists of option market data observed on NASDAQ (National Association of Securities Dealers Automated Quotations) index Call Option Price Quotations on 20th of October 2017. To reduce the absolute errors in the model parameters and consequently to determine whether the model is stable, we run a series of optimization runs and use the values obtained at the end of each run as the initial values for the run which immediately follows thus resetting the optimization until the model parameters converge to the true values used for generating the model prices.

Table 1: Results Obtained with ASA in Heston's Stochastic Model

| Parameters | $v_0$ | $\theta$ | $\sigma$ | $\rho$ | $\kappa$ | Elapsed time |
|---|---|---|---|---|---|---|
| Initial Estimate | 0.050 | 0.050 | 0.20 | -0.40 | 1.40 | |
| $Run_1$ | 0.048 | 0.48 | 0.20 | -0.36 | 1.37 | 153.57 seconds |
| $Run_2$ | 0.067 | 0.067 | 0.20 | -0.33 | 1.34 | 153.57 seconds |
| $Run_3$ | 0.054 | 0.054 | 0.20 | -0.31 | 1.33 | 153.57 seconds |
| $Run_4$ | 0.041 | 0.041 | 0.20 | -0.30 | 1.31 | 153.57 seconds |
| $Run_5$ | 0.04 | 0.04 | 0.20 | -0.30 | 1.30 | 153.57 seconds |

Table 1 shows the values of the Heston Stochastic model parameters after having run the calibration five times on the given set of data.

**Testing the Fit of Heston Stochastic Model**

In testing how well, the Heston's Stochastic model fits the observed market data, we use the 5th parameter set obtained in Table 1. The prices from the Heston's Stochastic model are obtained by using the FFT pricing method. We will require that the difference between model and market prices fall within the observed bid-ask spreads. We will also consider the following set of acceptable solutions:

$$\frac{1}{N}\sum_{i=1}^{N}|(Mid_i - Model_i)| \leq \frac{1}{2N}\sum_{i=1}^{N}|\left(bid_i - Ask_i\right)| \qquad (27)$$

where $Mid_i$ are the mid-market option prices (the average of the current bid and ask prices being quoted) , $Model_i$ are the model prices, $Bid_i$ are the market bid prices and $Ask_i$ are the market ask option prices.

Using the 5th parameter set in Table 1, the Heston's Stochastic model predicted values and its comparison with the market prices are shown in Table 2.

Table 2: Comparison of the Heston's Model Predicted Values and the Market Prices

| Option id | Strike | Bid | Ask | Mid | HSM |
|---|---|---|---|---|---|
| 1 | 500 | 486.05 | 487.00 | 486.525 | 486.6032 |
| 2 | 510 | 476.05 | 477.00 | 476.525 | 476.6013 |
| 3 | 520 | 464.80 | 468.10 | 466.450 | 467.2368 |
| 4 | 530 | 454.80 | 458.18 | 456.490 | 455.9568 |
| 5 | 560 | 424.90 | 428.35 | 426.625 | 426.4991 |
| 6 | 570 | 414.85 | 418.20 | 416.525 | 415.5201 |
| 7 | 590 | 394.90 | 398.00 | 396.450 | 395.547 |
| 8 | 600 | 384.75 | 388.10 | 386.425 | 385.5653 |
| 9 | 605 | 379.75 | 383.20 | 381.475 | 380.5697 |
| 10 | 610 | 374.75 | 378.20 | 376.475 | 375.577 |
| 11 | 620 | 364.75 | 368.20 | 366.475 | 366.6009 |
| 12 | 630 | 354.70 | 358.20 | 356.450 | 355.6086 |
| 13 | 650 | 334.75 | 338.15 | 336.450 | 335.6402 |
| 14 | 660 | 324.75 | 328.20 | 326.475 | 325.6638 |
| 15 | 665 | 319.90 | 323.20 | 321.550 | 320.6653 |
| 16 | 670 | 314.90 | 318.00 | 316.450 | 315.6723 |
| 17 | 675 | 309.75 | 313.20 | 311.475 | 310.6855 |
| 18 | 680 | 304.90 | 308.15 | 306.525 | 305.6915 |
| 19 | 690 | 294.85 | 297.90 | 296.375 | 295.7068 |
| 20 | 700 | 284.75 | 288.15 | 286.450 | 285.7198 |

As can be observed from the Table 2, all Heston's Stochastic model's values have the predicted values that fall within the observed bid-ask spread. Also, when evaluated considering the stated acceptance criterion in equation (27), the Heston's Stochastic model's average distance from the mid-market price is 1.1004, which is less than the average deviation in the bid-ask spreads that yields 3.104.

**Calibration of Heston's Stochastic-Jump Model to Real Market Prices**

Here, we will calibrate Heston's Stochastic-Jump model to the same data using the same procedure used for Heston's Stochastic model. The result obtained is shown in Table 3.

Table 3: Results Obtained with ASA in Heston's Stochastic -Jump Model

| | $v_0$ | $\theta$ | $\sigma$ | $\rho$ | $\kappa$ | $\mu_j$ | $\sigma_j$ | $\alpha$ | Elapsed Time |
|---|---|---|---|---|---|---|---|---|---|
| Initial Estimate | 0.05 | 0.05 | 0.20 | -0.40 | 1.40 | -0.04 | -0.3 | 0.05 | 179.32 seconds |
| $Run_1$ | 0.05 | 0.48 | 0.20 | -0.36 | 1.37 | -0.03 | -0.3 | 0.05 | 179.32 seconds |
| $Run_2$ | 0.07 | 0.07 | 0.20 | -0.33 | 1.34 | 0.2 | -0.12 | 0.02 | 179.32 seconds |
| $Run_3$ | 0.05 | 0.05 | 0.20 | -0.31 | 1.33 | 0.2 | -0.10 | 0.02 | 179.32 seconds |
| $Run_4$ | 0.04 | 0.04 | 0.20 | -0.30 | 1.31 | 0.2 | -0.10 | 0.02 | 179.32 seconds |
| $Run_5$ | 0.04 | 0.04 | 0.20 | -0.30 | 1.03 | 0.2 | -0.10 | 0.02 | 179.32 seconds |

Table 3 shows the values of the Heston's Stochastic-Jump model parameters after having run the calibration on the given set of data five times.

**Testing the Fit of Heston's Stochastic-Jump Model**

In testing the fit of the Heston's Stochastic-Jump model to the observed market data, we follow the same method, procedure and same data as was used in that of Heston's Stochastic model. Using the 5th parameter set on Table 3, the Heston's Stochastic-Jump model predicted values and its comparison with the market prices are shown in Table 4:

Table 4: Comparison of the Heston's Stochastic-Jump Model Predicted Values and the Market Prices

| Option id | Strike | Bid | Ask | Mid | HSJM |
|---|---|---|---|---|---|
| 1 | 500 | 486.05 | 487.00 | 486.525 | 486.6032 |
| 2 | 510 | 476.05 | 477.00 | 476.525 | 476.6037 |
| 3 | 520 | 464.80 | 468.10 | 466.450 | 467.4368 |
| 4 | 530 | 454.80 | 458.18 | 456.490 | 456.5568 |
| 5 | 560 | 424.90 | 428.35 | 426.625 | 427.4991 |
| 6 | 570 | 414.85 | 418.20 | 416.525 | 416.5201 |
| 7 | 590 | 394.90 | 398.00 | 396.450 | 396.547 |
| 8 | 600 | 384.75 | 388.10 | 386.425 | 386.5653 |
| 9 | 605 | 379.75 | 383.20 | 381.475 | 381.5697 |
| 10 | 610 | 374.75 | 378.20 | 376.475 | 376.577 |
| 11 | 620 | 364.75 | 368.20 | 366.475 | 366.6009 |
| 12 | 630 | 354.70 | 358.20 | 356.450 | 356.6086 |
| 13 | 650 | 334.75 | 338.15 | 336.450 | 336.6002 |
| 14 | 660 | 324.75 | 328.20 | 326.475 | 326.5471 |
| 15 | 665 | 319.90 | 323.20 | 321.550 | 322.7023 |
| 16 | 670 | 314.90 | 318.00 | 316.450 | 317.5431 |
| 17 | 675 | 309.75 | 313.20 | 311.475 | 312.0443 |
| 18 | 680 | 304.90 | 308.15 | 306.525 | 307.4100 |
| 19 | 690 | 294.85 | 297.90 | 296.375 | 297.0050 |
| 20 | 700 | 284.75 | 288.15 | 286.450 | 287.6250 |

It can also be observed from Table 4 that all the prices produced by Heston's Stochastic-Jump model fall within the observed bid-ask spread. Also, when evaluated considering the stated acceptance criterion in equation (27), the Heston's Stochastic-Jump model's average distance from the mid-market price is 0.99699, which is less than the average deviation in the bid-ask spreads that yields 3.104. It could be observed from Table 4 that adding jumps to the underlying price process improved the overall fit to market prices. Graphical representation of the fits of Heston's Stochastic and Heston's Stochastic-Jump models is shown in Figure 1. It could be seen that the prices produced by HSJ model gives better fit when compared with the mid prices (market prices) than that of HS model. Also, when the performances of the two models are compared in terms of time to maturity, we observed that their performances are similar at short maturity but HSJ begins to give higher prices as time to maturity increases. Price comparison in terms of maturity time is shown in Figure 2. Therefore, we can say that the new proposed HSJ model is better at pricing options in a long-time maturity than the HS model.

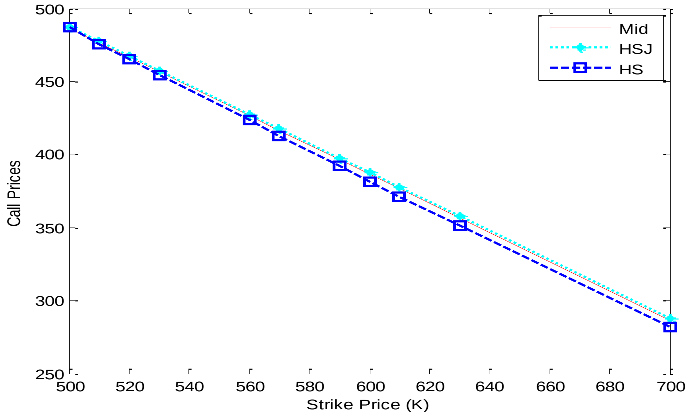

Figure 1: Price Comparison Among Mid, Heston and Heston Jump Model

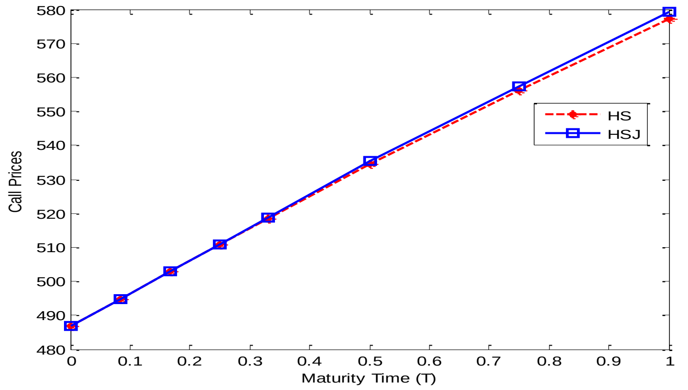

Figure 2: Price Comparison in Terms of Maturity Time

**Error Measurement Comparisons for the Heston's Stochastic and Heston's Stochastic-Jump Models**

In order to measure the accuracy of the model compared to the market prices, we employ error measure statistic as a criterion for comparison. A value closer to 0 indicates that the model has a smaller random error component, and that the fit will be more useful for prediction. The error measure that will be used in this paper is Root-Mean-Squared Error (RMSE) given as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i}^{N}\left(Model\ Price - Market\ Price\right)^2}$$

Root Mean Squared Error (RMSE) measures the average magnitude of the error. It is the square root of the average of squared differences between prediction and actual observation.

Table 5: Measured Errors of Heston's Stochastic Model and Heston Stochastic-Jump Model

| Error Measure | HSM | HSJM |
|---|---|---|
| RMSE | 0.734884 | 0.603298 |

Table 5 shows the pricing errors produced by Heston Stochastic and Heston Stochastic-Jump models. Heston Stochastic model has the RMSE as 0.7349 while Heston Stochastic-Jump model has its RMSE as 0.6033. This shows that HSJM has approximately 18% reduction in error. It could be observed that the Heston's Stochastic-Jump model performs significantly better than the Heston Stochastic model as indicated by low values of Root-Mean-Squared Error produced by Heston's Stochastic-Jump model.

## 7 Conclusion

We proposed a modified Heston's Stochastic model incorporating Poisson Jump process. Monte Carlo simulations and goodness-of-fit tests were used in the comparison of performances of the two models. Results show that Heston's Stochastic-Jump model performed better than Heston's Stochastic model by about 18% reduction in error.

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