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Option Pricing: Five Notes

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Abstract

In this paper, five notes about the option pricing are presented. The first note is concerned about application of downside-delta hedging to the binomial tree. In the second note, the delta-gamma neutral portfolio involving a derivative is considered. The third note considers the dynamic hedging cases. A differential equation based relation is derived between the dynamic and static deltas. The fourth note search for the best simple derivative for hedging another complex derivative. In the last note, an approximated formulae is given for the price of a derivative which its payoff function is twice differentiable.

Keywords: Option pricing; Risk neutral; Downside delta; Martingale equivalent measure

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1 Introduction

Hedging is the taking opposite position of a derivative in a portfolio to offset the risk of derivative exists in the portfolio. Delta hedging is one of important and widely used method of hedging. It is applied in discrete time (binomial tree) and continuous time processes of price. In this paper, different type of delta hedging in the format of five notes are considered and in each case the no arbitrage condition gives an interesting result. The first note, studies the downside delta hedging in a binomial tree setting. The second note considers the simultaneous delta and gamma hedging in the Black and Scholes (1973) continuous time frame work. The Third one is concerned with dynamic delta hedging. A differential equation based relation is derived between the dynamic and static deltas. In the fourth note, the functional form for the best derivative with simple structure is derived for offsetting the risk of a complex derivative existing in portfolio. In the last note, an approximated formulae is given for the price of a derivative which its payoff function is twice differentiable.

2 Downside Delta of Binomial Tree

Binomial trees are useful, mathematically simple technique for option pricing problems. They use the no arbitrage arguments by constructing a risk neutral portfolio containing the derivative and a portion of underlying financial asset such as stock. Indeed, they apply the delta-hedging arguments. Delta belongs to the collection of Greeks, which is the sensitivity of the price of derivatives to a change of value of stock.

Here, the concept of downside delta is introduced and it is tried to remove it in a binomial tree setting. The idea behind this part of paper comes from the concept of semi-variance of Markowitz *et al*. (1993) in portfolio management. In fact, downside risk is the financial risk associated with losses. That is, it is the risk of the actual return being below the expected return, or the uncertainty about the magnitude of that difference. A comprehensive review of downside risk can be found in Nawrocki (1999).

To illustrate more the problem, suppose that the current price of stock is S_0 and after T-period it will go up to S_0u or down to S_0d . Assume that the up and down values of a derivative f with maturity T written on mentioned stock are f_u , and f_d , respectively. Following Hull (2008) construct a portfolio containing a long position in derivative and short position in Δ portion of stock. At the maturity, the up and down values of portfolio are

$$\begin{cases} \pi_u = f_u - \Delta S_0 u \\ \pi_d = f_d - \Delta S_0 d \end{cases}$$

Hull (2008) set $\pi_d = \pi_u$ and uses the no arbitrage arguments. Here, we suppose that π_u is positive and π_d is negative. Therefore, the downside risk (bad part of portfolio) is π_d . To this end, it is enough to assume that

$$\frac{f_d}{d} \le \Delta S_0 \le \frac{f_u}{u}$$

Thus, it is assumed that $\pi_d = 0$. Therefore,

$$\Delta = \frac{f_d}{S_0 d} \text{ and } \pi_u = f_u - \frac{u}{d} f_d .$$

This is referred as downside delta measure. Let

$$K = \frac{u}{d} f_d \text{ and } \pi_T = \max(f_T - K, 0),$$

like the price of an call option written on the f_T as the underlying asset. It is seen that if $f_T = f_d$, then $\pi_T = \pi_u = 0$ and if $f_T = f_u$, then $\pi_T = f_u - \frac{u}{d} f_d = \pi_u$. To obtain π_0 , the value of π_T at time zero, the risk neutral valuation method is applied. The risk neutral probabilities are $q = \frac{e^{rT} - d}{u - d}$, $1 - q = \frac{u - e^{rT}}{u - d}$. Therefore,

$$\pi_0 = e^{-rT} E_Q(\pi) = e^{-rT} (f_u - \frac{u}{d} f_d) \frac{e^{rt} - d}{u - d} .$$

However, $f_0 = \pi_0 - \Delta S_0 = e^{-rT} (f_u - \frac{u}{d} f_d) \frac{e^{rt} - d}{u - d} - \frac{f_d}{d}$. This is the value of option assuming the downside risk is zero. It is surprising that again is the risk neutral valuation formula, i.e.,

$$f_0 = e^{-rT} (q f_u + (1-q) f_d) .$$

That is the price of derivative under the no arbitrage assumption and downside risk neutral arguments are the same.

Remark 1. Here, some extensions are presented. Assume that it is interested to control the downside risk by defining $\pi_d = g(\pi_u)$, where g is an one to one function, for example,

$$g(x) = ax + b .$$

Then, a solution like Δ^* for the above equation is found. Then,

$$\pi^* = f - \Delta^* S$$

follows a nor arbitrage argument. Therefore, $e^{-rT} E(\pi^*) = \pi_0$ which implies that

$$e^{-rT} E(f) = f_0 .$$

That is, in the presence of no arbitrage the choice of Δ is not important. One suggestion may be the value which minimizes the variance of portfolio π. It is easy to see that this value is given by

$$\Delta = \frac{\mathrm{cov}(f, S)}{\mathrm{var}(S)} .$$

3 Delta-Gamma Hedging in Continuous Time

The second part considers the delta hedging arguments in continuous time dynamic of price. Hull (2008) constructed a portfolio as $\pi = f - aS$ and applied the delta hedging method. Here, an additional term bS^2 is added to the theoretical portfolio and the delta-gamma hedging method is applied. Thus, let the portfolio be

$$\pi = f - aS - bS^2 .$$

The simultaneous delta and gamma neutrality of portfolio implies that

$$\frac{\partial \pi}{\partial S} = 0 \text{ and } \frac{\partial^2 \pi}{\partial S^2} = 0 .$$

It is seen that $b = \frac{\Gamma}{2}$, $a = \Delta - \Gamma S$, where Δ and Γ are the delta and gamma of derivative f. Using the Ito lemma, it is seen that $d\pi = r\pi dt$. Therefore,

$$\frac{\partial f}{\partial t} = r(f - aS - bS^2) .$$

The following PDE is the partial differential equation for delta and gamma neutrality:

$$\frac{\partial f}{\partial t} + rS(\Delta - \Gamma S) + \frac{r\Gamma}{2} S^2 = rf .$$

Here, another version of delta hedging is considered. Define the portfolio as $\pi = Sf - \Delta$. Then

$$d\pi = fdS + Sdf + dSdf - \Delta .$$

Then, if it is assumed that

$$\frac{\partial \Delta}{\partial S} = f - S\frac{\partial f}{\partial S},$$

then it can be concluded that

$$\mu Sf + S\mu_f = \mu_\Delta ,$$

Here, $f + S\mu_f = \mu_\Delta = \frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial S}\mu S + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \Delta}{\partial S^2}$ and μ_f is defined, analogously. Now suppose that a functional form is known for Δ. For example, let $\Delta = aS$.

Then, $a = f - S\frac{\partial f}{\partial S}$. Therefore,

$$\frac{\partial f}{\partial S} = \frac{f}{S} + \frac{a}{S}$$

where the functional form for derivative f is found.

4 Dynamic Delta Hedging

Here, it is interested to study the dynamic hedging. The dynamic hedge comes from allowing the delta to be the function of S and t. The usual static hedging assumes the constancy of delta for a short period of time. Then, let

$$\pi = f - \Delta(S,t)S .$$

One can see that

$$\begin{cases} df = \mu_f dt + \sigma S\frac{\partial f}{\partial S}dB \\ d\Delta = \mu_\Delta + \sigma S\frac{\partial \Delta}{\partial S}dB \\ d(s\Delta) = sd\Delta + \Delta ds + d\Delta ds \end{cases} .$$

Also, under the no arbitrage assumption, $d\pi = r\Delta dt$, then it is conclude that

$$\mu_f - [S\mu_D + \Delta\mu S + \frac{\sigma^2 S^2}{2}\frac{\partial \Delta}{\partial S}] = rf - rS\Delta ,$$

if, it is assumed that

$$\frac{\partial f}{\partial S} = \Delta + S\frac{\partial \Delta}{\partial S}$$

which is the no arbitrage assumption. Now, suppose that f is a forward contract,

that is $f = S - Ke^{-r(T-t)}$, then $\Delta + S\frac{\partial \Delta}{\partial S} = 1$, that is

$$\frac{\partial \Delta}{\partial S} = \frac{1}{S} - \frac{\Delta}{S}.$$

Now suppose that f is the payoff function of a call option contract. Then $\frac{\partial f}{\partial S} = N(d_1)$, where $N(.)$ is the distribution function of normal law and d_1 is defined in Hull (2008).

Again consider the equation $\frac{\partial f}{\partial S} = \Delta + \frac{\partial \Delta}{\partial S}$. Rewrite it as

$$\frac{\partial \Delta_{dy}}{\partial S} = \frac{\Delta_{st}}{S} - \frac{\Delta_{dy}}{S},$$

where Δ_{dy} and Δ_{st} stand for the delta in dynamic (dy) and static (st) cases. This first order differential equation is solved as follows

$$\Delta_{dy}(S) = \frac{\Delta_{dy}(0) + \int_0^S u\Delta_{st}(u)du}{S}.$$

5 The Best Derivative For Hedging

Suppose that f is a complex derivative and we search for the best simple derivative such as g for delta hedging of f. Consider the portfolio

$$\pi = f - g.$$

Then $\frac{\partial \pi}{\partial S} = 0$ implies that $\frac{\partial f}{\partial S} = \frac{\partial g}{\partial S}$. Using the Ito lemma and the no arbitrage arguments, it is seen that

$$\frac{\partial (f-g)}{\partial t}dt + \frac{\partial^2 (f-g)}{\partial S^2}(dS)^2 = r(f-g)dt.$$

Since the no arbitrage argument holds therefore

$$\frac{\partial f}{\partial t}dt + \frac{\partial^2 f}{\partial S^2}(dS)^2 = rfdt.$$

Thus, it is seen that

$$\frac{\partial g}{\partial t}dt+\frac{\partial^2 g}{\partial S^2}(dS)^2=rgdt\,.$$

For example, for a special case, suppose that $\frac{\partial g}{\partial t}=0$, then g is solved form equation

$$\frac{\partial^2 g}{\partial S^2}(\sigma S)^2=rg\,.$$

To complete the idea, now a transformation is done on the Black schools partial differential equation, that is

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}=rf\ .$$

Let $g=e^{rt}f$. It can be seen that

$$\frac{\partial g}{\partial t}+rS\frac{\partial g}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 g}{\partial S^2}=0\,.$$

For example, when $\frac{\partial g}{\partial t}=0$, then

$$g=g_0+\frac{S^{1-\frac{2r}{\sigma^2}}}{1-\frac{2r}{\sigma^2}}\,.$$

6 An Approximated Formulae for Price

Here, an approximation is proposed for the price of a derivative whose payoff function is twice differentiable. Assume that S_t is the price of a stock at time t (discrete time setting) and R_t denotes the rate of return from $t-1$ to t . Therefore,

$$S_t=S_{t-1}(1+R_t)\,.$$

The discounted price is $S_t^* = \frac{S_t}{(1+r)^t}$. To remove the arbitrage opportunity, it suffices to assume that S_t^* is a martingale. The necessary and sufficient condition to this end is

$$E_Q(R_t \mid f_t) = r$$

where f_t is the information available up to time t and Q is the risk-neutral probability measure. An special case, is to assume that R_t is an independent and identically distributed with mean r and variance σ^2 . Now, suppose that a derivative with payoff function F exists which F is twice differentiable. Define

$$G(x) = F(S_0 e^x)\ .$$

Also, note that

$$\ln(1+R_i) \approx R_i\ , E(\sum_{i=1}^{t} R_i) = rt$$

And

$$\mathrm{var}(\sum_{i=1}^{t} R_i) = \sigma^2 t\ .$$

Then,

$$F(S_t) = F(S_0 e^{\sum_{i=1}^{t} \ln(R_i+1)}) \approx F(S_0 e^{\sum_{i=1}^{t} \ln R_i}) = G(\sum_{i=1}^{t} R_i)\ .$$

Using the Taylor expansion of G about rt , the following approximated formulae for price if derived

$$e^{-rt} E_Q(F(S_t)) = e^{-rt} G(rt) + e^{-rt} \frac{\sigma^2}{2} G^n(rt)\ .$$

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