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Valuation of Exit Strategy under Decaying Abandonment Value

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Abstract

We examine the valuation of abandonment decision in a contingent claims model with uncertainty in future market conditions and analyze the effect of determinants on the abandonment value. We find the abandonment value is positively related with the number of abandonment opportunities. The increase in the volatility, variable cost, and facility value increases the expected abandonment value, whereas the increase in the growth rate and depreciation rate reduces the expected abandonment value. The volatility, growth rate, and depreciation rate are negatively related with the exit threshold, whereas the variable cost and facility value are positively related with the exit threshold.

Mathematics Subject Classification: 60G51; 60H05; 60H10; 65D25
Keywords: Real option; Abandonment option; Exit strategy

1 Introduction

Capital budgeting methods based on option pricing theory have recently been developed to incorporate managers' ability to respond to the resolution of uncertainty over time. Conventional capital budgeting procedure presumes that future cash flows and duration of a project are certain and salvage value is the last relevant cash flow. Thus, the only relevant decision is whether to accept or reject the project at time zero. In a real business environment, however, a firm makes production and investment decisions contingent on subsequent estimation of stochastic incremental cash flows. When the value of future production opportunity is less than the value of a project that is abandoned irreversibly, firms may either choose to delay abandonment or exercise abandonment option to collect the abandonment value of a project. Therefore, a firm's flexible strategic behavior to react on the future prospect of investment projects can affect their abandonment value.

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The abandonment value of capital assets is related to the economic depreciation rate which measures the decline in market value of capital assets through time. Since the abandonment value of capital assets is treated as a strike price and can decay at a depreciation rate through time, various abandonment decision criteria can be formulated under market uncertainty. The goal of this paper is to evaluate exit strategy and find abandonment decision criteria under the consideration of decaying abandonment value, flexible strategic behavior, and investment under market uncertainty.

Robichek and Van Horne [1] are the pioneers to recognize the practical importance of the option to abandon a project. They argue that a project is abandoned immediately as soon as the salvage value exceeds the net present value of subsequent expected operating cash flows. Dyl and Long [2] argue that the decision rule proposed by Robichek and Van Horne [1] may be sub-optimal, and that an optimal decision rule must "consider the cases where it may be more profitable to wait and abandon in the future." If a project is not abandoned immediately, a firm retains the option to abandon in the future and the option can be valuable. Thus, an optimal abandonment decision on a project depends on the salvage value of the project and the optimal timing of exercising the abandonment option.

Smit and Trigerogis [3] describe the abandonment option as the opportunity for firms to permanently abandon the current operation and realize the abandonment value of capital equipment and other assets in secondary markets if the market condition severely declines. An example where abandonment is important is research and development (R&D) programs. When results from experiments are not favorable, the line of research should be abandoned. Other examples are that mines can be abandoned and factories can be closed permanently. Whenever maintenance costs for maintaining idle production capacity are too costly, perceptibly decreasing salvage value due to technological progress or natural obsolesce may induce firms to abandon capital investment projects before they reach their terminal dates. Severe market competition or over-pessimistic economic condition can lead firms losing competitive niche and exit the market permanently.

Baker and Powell [4] note that an often-cited decision rule for firms to abandon projects when the abandonment value of capital assets is greater than the present value of all cash flows generated by capital assets beyond the abandonment year, discounted to the abandonment decision point, is technically incorrect primarily because it ignores future abandonment opportunities. In theory, the option to abandon a project is widely recognized as an 'American' put option on a dividend-paying stock, and is valuable because the option allows firms to do 'wait and see'. If firms have the option to abandon investment projects before implementing the projects, the abandonment option can increase the value of the projects. The increase in the number of abandonment opportunities can increase the abandonment value.

Several papers in the real options literature study the abandonment option. Brennan and Schwartz [5] use the techniques of continuous time arbitrage and stochastic control theory to evaluate natural resource projects, and provide the optimal abandonment decision rules at known intervals based on a constant salvage value and price of the underlying commodity. McDonald and Siegel [6] use option-pricing techniques to study the investment problem that a firm has the option to shut down production costlessly and temporarily. Dixit [7] analyzes the 'hysteresis' effect on a firm's entry and exit decision when the firm's output price follows the random walk and the firm's assets depreciates immediately on abandonment. Myers and Majd [8] model abandonment option in an 'American' put option framework and evaluate the abandonment option under assumption of stochastic project value, deterministic declining salvage value and constant payout ratios. In contrast, Dixit and Pindyck [9] study the abandonment option by considering an infinitely lived dividend paying investment with constant salvage value and derive optimal abandonment rules. Clark and Rousseau [10] investigate how abandonment option can be used as a management tool to evaluate the invest/abandon decision and analyze ongoing project management, financial forecasting and the timing of strategic moves. Pfeiffer and Schneider [11] explore how an abandonment option influences the optimal timing of information in a sequential adverse selection capital budgeting model. Wong [12] examines how the presence of an abandonment option affects the timing and intensity of a firm's investment. This paper extends this development of real option approach by focusing on optimal abandonment decisions. We analyze and evaluate the option to abandon a project for its salvage value by developing a multi-period contingent claims model under the assumption of stochastic market demand and projects' decaying salvage value.

Our paper is also related to recent studies in real options incorporating firms' strategic behavior into investment project evaluation. For example, Grenadier [13] develops an equilibrium framework for strategic option exercise games to analyze the timing of real estate development. Kulatilaka and Perotti [14] provide a strategic rationale for growth options under uncertainty and imperfect competition. Childs and Triantis [15] examine dynamic R&D investment policies and the valuation of R&D programs by considering several important characteristics of R&D programs. This paper also considers a firm's strategic behavior in formulating optimal abandonment decision criteria.

The paper is organized as follows. Section 2 describes and formulates mathematical model for abandonment option valuation. Section 3 presents numerical solutions. We conclude in Section 4, and the proof is given in Appendix.

2 Valuation Model

We follow Kulatilaka and Perotti (1998) and assume a firm has monopoly power in both the investment opportunity and the product market. Since future market condition is unknown, we assume that the demand for the product is linear in prices and increasing in the random variable θ. Let P(Q) be the inverse demand function expressing the market price as a function of product quantity Q: $P(\theta, Q) = \theta - Q$ and $\frac{d\theta}{\theta} = \mu dt + \sigma dz$, where θ is the market condition following lognormal distribution with drift term μ and variance σ^2. dz is the standard Wiener process.

The firm produces at a unit cost of v only when the market is profitable. The firm chooses an output level $Q^* = \frac{1}{2}(\theta_T - v)$ and generates operating profits $\pi = \frac{1}{4}(\theta_T - v)^2$. The firm will not produce if $\theta_T < \theta^*(T) \equiv v$. Therefore, the payoff function at time T can be expressed as:

$$\pi_T = \begin{cases} \frac{(\theta_T - v)^2}{4}, if\ \theta_T \geq \theta^*(T) \\ 0 \quad , if\ \theta_T < \theta^*(T) \end{cases} \tag{1}$$

Assuming that the capital investment is irreversible, we discount the firm's expected operating profits at the cost of capital r and express the present value of operating profit

V_0 as:

$$V_0 = e^{-rT}\left[\frac{(\theta_T - v)^2}{4}\right] prob(\theta_T \geq v|\theta_0)$$

$$= \frac{\theta_0^2}{4}\exp\left[2\left(\mu - \frac{r}{2} + \frac{\sigma^2}{2}\right)T\right]N(d_1) - \frac{v\theta_0}{2}\exp[(\mu - r)T]\,N(d_2) + \frac{v^2}{4}e^{-rT}N(d_3) \tag{2}$$

where $d_1 = \frac{\ln\left(\frac{\theta_0}{v}\right) + \left(\mu + \frac{3}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $d_3 = d_1 - 2\sigma\sqrt{T}$.

In contrast, if the firm makes an initial reversible investment of F on a production facility that depreciates at a rate ρ, the downside of payoff function in Equation (1) needs to modify to recognize the value of abandonment option. When exiting the market, the firm can cease the operation, sell the facility, and get the abandonment value in return. The abandonment value at time *t* is $Fe^{-\rho t}$.

We now consider that a firm can only exercise the abandonment option at the end of investment project (terminal). This case is similar to European put options. We can express the payoff at time *T* as:

$$\pi_T = \begin{cases} \frac{(\theta_T - v)^2}{4}, if\ \theta_T \geq \theta^*(T) \\ Fe^{-\rho T}, if\ \theta_T < \theta^*(T) \end{cases} \tag{3}$$

where $\theta^*(T)$ is the criteria at time T and can be expressed as $\theta^*(T) = v + 2\sqrt{Fe^{-\rho T}}$. The criteria show that even though operating profits are higher than unit cost of production, firms may exit the market. The expected payoff at time *T*, $E(\pi_T|\theta_0)$, is:

$$E(\pi_T|\theta_0) = \left[\frac{(\theta_T - v)^2}{4}\right] prob[\theta_T \geq \theta^*(T)|\theta_0] + Fe^{-\rho T} prob[\theta_T < \theta^*(T)|\theta_0]$$

$$= \frac{\theta_0^2}{4}\exp\left[2\left(\mu + \frac{\sigma^2}{2}\right)T\right]N(d_1) - \frac{v\theta_0}{2}e^{\mu T}N(d_2) + \frac{v^2}{4}N(d_3) + Fe^{-\rho T}[1 - N(d_3)]$$

where $d_1 = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu + \frac{3}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $d_2 = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, and $d_3 = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$

Given the expected payoff at time *T*, we can express the present value of expected future payoff as:

$$e^{-rT}E(\pi_T|\theta_0) = \frac{\theta_0^2}{4}\exp\left[2\left(\mu - \frac{1}{2}r + \frac{\sigma^2}{2}\right)T\right]N(d_1) - \frac{v\theta_0}{2}\exp[(\mu - r)T]N(d_2)$$

$$+ \frac{v^2}{4}e^{-rT}N(d_3) + Fe^{-(\rho + r)T}[1 - N(d_3)] \tag{4}$$

Comparing Equation (4) with Equation (2), we can express the expected abandonment value at time *T* as:

$$Expected\ Abandonment\ Value = e^{-rT}E(\pi_T|\theta_0) - V_0 \tag{5}$$

Equation (5) shows that the expected abandonment value is the excess value of an investment project with abandonment option over an investment project without abandonment option.

2.1 The Evaluation of a Project with Two Abandonment Opportunities at a Predetermined Node and Terminal

We now consider that a firm can exercise the abandonment option not only at the terminal but also at a predetermined node t_1. This case is similar to warrants. Figure 1 shows the time line of an investment project with two abandonment opportunities.

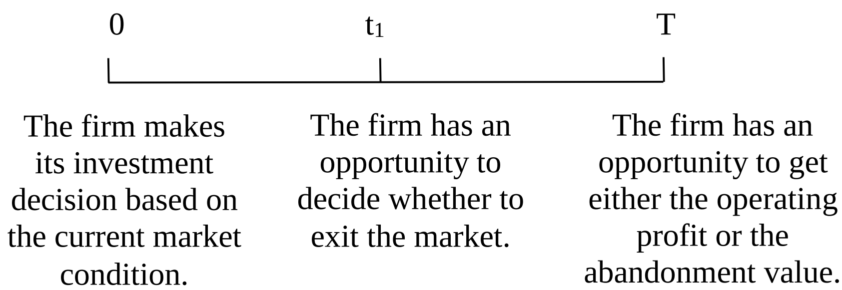

Figure 1: Time Line of an Investment with Two Abandonment Opportunities.

At time t_1, if the firm does not exercise the abandonment option, the firm will continue the operation and receive the present value of the expected payoff at time T. Alternatively, the firm may consider exiting the market if the present value of expected payoff at time T is lower than the abandonment value at time t_1. The payoff function at time t_1 is:

$$\pi_{t_1} = \begin{cases} e^{-r(T-t_1)}E\left(\pi_T \middle| \theta_{t_1}\right), if\theta_t \geq \theta^*(t_1) \\ Fe^{-\rho t_1} \qquad\qquad ,if\theta_t < \theta^*(t_1) \end{cases} \tag{6}$$

where $\theta^*(t_1)$ is the firm's exit threshold. Equation (6) shows that the expected operating profit is an increasing function of θ_{t_1} (the market condition at time t_1). We can express the expected payoff at time t_1 as (see Appendix for derivation)

$$\begin{aligned} E\left(\pi_{t_1}\middle|\theta_0\right) &= e^{-r(T-t_1)}E\left(\pi_T\middle|\theta_{t_1}\right)prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right] \\ &\quad + Fe^{-\rho t_1}prob\left[\theta_{t_1} < \theta^*(t_1)\middle|\theta_0\right] \end{aligned}$$

$$= e^{-r(T-t_1)}\left\{\frac{(\theta_T - v)^2}{4}prob\left[\theta_T \geq \theta^*(T)\middle|\theta_{t_1}\right] + Fe^{-\rho T}prob\left[\theta_T < \theta^*(T)\middle|\theta_{t_1}\right]\right\}prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right]$$

$$+Fe^{-\rho t_1}prob\left[\theta_{t_1} < \theta^*(t_1)\middle|\theta_0\right]$$

$$= \frac{1}{4}(\theta_0)^2 e^{2\left(\mu+\frac{1}{2}\sigma^2\right)T-r(T-t_1)}\int_{-\infty}^{a_1}\int_{-\infty}^{a_2}(2\pi)^{-1}e^{-\frac{(x_1)^2}{2}}e^{-\frac{(x_2)^2}{2}}dx_1dx_2$$

$$-\frac{v}{2}(\theta_0)e^{\mu T-r(T-t_1)}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2}(2\pi)^{-1}e^{-\frac{(y_1)^2}{2}}e^{-\frac{(y_2)^2}{2}}dy_1dy_2$$
$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}-Fe^{-\rho T}\right]\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}(2\pi)^{-1}e^{-\frac{(z_1)^2}{2}}e^{-\frac{(z_2)^2}{2}}dz_1dz_2$$
$$+\left[Fe^{-(\rho+r)T+rt_1}-Fe^{-\rho t_1}\right]\int_{-\infty}^{c_1}(2\pi)^{-\frac{1}{2}}e^{-\frac{(z_1)^2}{2}}dz_1+Fe^{-\rho t_1} \quad (7)$$

where x_i, y_i, and z_i follow standard normal distributions, and a_i, b_i, and c_i are integral upper-boundaries. The definitions are as the following:

$$x_1=\frac{\ln\theta_{t_1}-\left[\ln\theta_0+(\mu+\frac{3}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}},x_2=\frac{\ln\theta_T-\left[\ln\theta_{t_1}+(\mu+\frac{3}{2}\sigma^2)(T-t_1)\right]}{\sigma\sqrt{T-t_1}}$$
$$y_1=\frac{\ln\theta_{t_1}-\left[\ln\theta_0+(\mu+\frac{1}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}},y_2=\frac{\ln\theta_T-\left[\ln\theta_{t_1}+(\mu+\frac{1}{2}\sigma^2)(T-t_1)\right]}{\sigma\sqrt{T-t_1}}$$
$$z_1=\frac{\ln\theta_{t_1}-\left[\ln\theta_0+(\mu-\frac{1}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}},z_2=\frac{\ln\theta_T-\left[\ln\theta_{t_1}+(\mu-\frac{1}{2}\sigma^2)(T-t_1)\right]}{\sigma\sqrt{T-t_1}}$$
$$a_1=\frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right]+(\mu+\frac{3}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}},a_2=\frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right]+(\mu+\frac{3}{2}\sigma^2)(T-t_1)}{\sigma\sqrt{T-t_1}}$$
$$b_1=\frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right]+(\mu+\frac{1}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}},b_2=\frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right]+(\mu+\frac{1}{2}\sigma^2)(T-t_1)}{\sigma\sqrt{T-t_1}}$$
$$c_1=\frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right]+(\mu-\frac{1}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}},c_2=\frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right]+(\mu-\frac{1}{2}\sigma^2)(T-t_1)}{\sigma\sqrt{T-t_1}}$$

Then, we discount the Equation (7) with the cost of capital to get the present value of expected operating profits.

$$e^{-rt_1}E\left(\pi_{t_1}|\theta_0\right)=\frac{\theta_0^2\exp\left[2\left(\mu-\frac{r}{2}+\frac{1}{2}\sigma^2\right)T\right]}{4\sigma^2\sqrt{t_1(T-t_1)}}\int_{-\infty}^{a_1}\int_{-\infty}^{a_2}(2\pi)^{-1}e^{-\frac{(x_1)^2}{2}}e^{-\frac{(x_2)^2}{2}}dx_1dx_2$$
$$-\frac{v\theta_0\exp[(\mu-r)T]}{2\sigma^2\sqrt{t_1(T-t_1)}}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2}(2\pi)^{-1}e^{-\frac{(y_1)^2}{2}}e^{-\frac{(y_2)^2}{2}}dy_1dy_2$$
$$+\frac{\exp(-rT)}{\sigma^2\sqrt{t_1(T-t_1)}}\left[\frac{v^2}{4}-Fe^{-\rho T}\right]\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}(2\pi)^{-1}e^{-\frac{(z_1)^2}{2}}e^{-\frac{(z_2)^2}{2}}dz_1dz_2$$
$$+\left[Fe^{-(\rho+r)T}-Fe^{-(\rho+r)t_1}\right]\int_{-\infty}^{c_1}(2\pi)^{-\frac{1}{2}}e^{-\frac{(z_1)^2}{2}}dz_1+Fe^{-(\rho+r)t_1} \quad (8)$$

Comparing Equation (8) with Equation (4), we find that adding one abandonment opportunity increases the present value of the expected operating profit significantly by the excess of the present value of abandonment value.

2.2 The Evaluation of a Project with *n* Abandonment Opportunities at any Time before Maturity

We now extend our model by considering that a firm can exercise the abandonment option at any time before maturity. This case is similar to American put options. We divided the time to maturity into *n* periods. Each node represents the time that the firm can exercise the abandonment option. Figure 2 shows the time line of an investment with *n* abandonment opportunities.

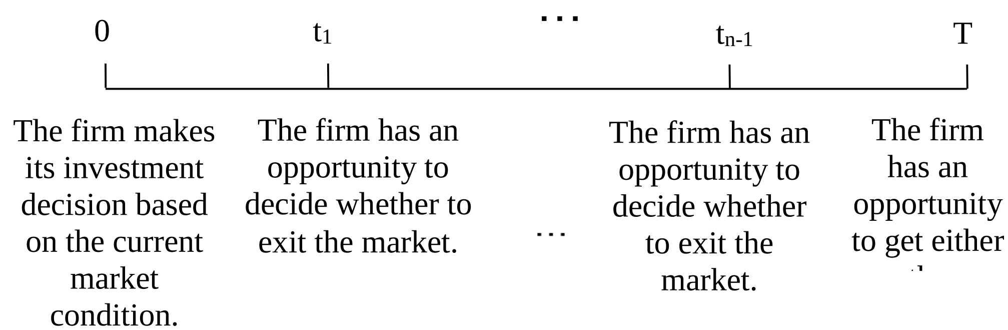

Figure 2: Time Line of an Investment with *n* Abandonment Opportunities

To solve for expected payoff at a period, we can work backwards to estimate expected operating profit at the prior period. Since the abandonment value decreases through time, the decision criteria at each node can be different. We need to know the decision criteria at a determined node first so that we can derive the payoff function. The payoff function at terminal is Equation (3). The payoff function at each node before the terminal can be expressed as:

$$\pi_{t_i} = \begin{cases} e^{-r(t_{i+1}-t_i)} E\left(\pi_{t_{i+1}} \middle| \theta_{t_i}\right), if \theta_{t_i} \geq \theta^*(t_i) \\ Fe^{-\rho t_i} \qquad , if \theta_{t_i} < \theta^*(t_i) \end{cases} for\ i = 1, 2, \dots, n-1 \tag{9}$$

The expected value of π_{t_i} can be expressed as:

$$E\left(\pi_{t_i} \middle| \theta_{t_{i-1}}\right) = e^{-r(t_{i+1}-t_i)} E\left(\pi_{t_{i+1}} \middle| \theta_{t_i}\right) prob\left[\theta_{t_i} \geq \theta^*(t_i) \middle| \theta_{t_{i-1}}\right] + Fe^{-\rho t_i} prob\left[\theta_{t_i} < \theta^*(t_i) \middle| \theta_{t_{i-1}}\right]$$

We use the recursive method to obtain the present value of future expected operating profit at time 0.

$$e^{-rt_1} E\left(\pi_{t_1} \middle| \theta_0\right) = \frac{\theta_0^2 \exp\left[2\left(\mu - \frac{r}{2} + \frac{1}{2}\sigma^2\right) t_n\right]}{4\sigma^n \sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} \cdots \int_{-\infty}^{a_n} (2\pi)^{-\frac{n}{2}} \exp\left[-\sum_{i=1}^{n} \frac{(x_i)^2}{2}\right] dx_1 \cdots dx_n$$

$$-\frac{v\theta_0\exp[(\mu-r)t_n]}{2\sigma^n\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2}\cdots\int_{-\infty}^{b_n}(2\pi)^{-\frac{n}{2}}\exp\left[-\sum_{i=1}^{n}\frac{(y_i)^2}{2}\right]dy_1\cdots dy_n$$
$$+\frac{\exp(-rt_n)}{\sigma^n\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}\left[\frac{v^2}{4}-Fe^{-\rho t_n}\right]\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}\cdots$$
$$\cdot\int_{-\infty}^{c_n}(2\pi)^{-\frac{n}{2}}\exp\left[-\sum_{i=1}^{n}\frac{(z_i)^2}{2}\right]dz_1\cdots dz_n$$
$$+\frac{[Fe^{-(\rho+r)t_n}-Fe^{-(\rho+r)t_{n-1}}]}{\sigma^{n-1}\sqrt{t_1(t_2-t_1)\cdots(t_{n-1}-t_{n-2})}}\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}\cdots\int_{-\infty}^{c_{n-1}}(2\pi)^{-\frac{(n-1)}{2}}\exp\left[-\sum_{i=1}^{n-1}\frac{(z_i)^2}{2}\right]dz_1\cdots dz_{n-1}$$
$$+\frac{[Fe^{-(\rho+r)t_{n-1}}-Fe^{-(\rho+r)t_{n-2}}]}{\sigma^{n-2}\sqrt{t_1(t_2-t_1)\cdots(t_{n-2}-t_{n-3})}}\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}\cdots\int_{-\infty}^{c_{n-2}}(2\pi)^{-\frac{(n-2)}{2}}\exp\left[-\sum_{i=1}^{n-2}\frac{(z_i)^2}{2}\right]dz_1\cdots dz_{n-2}$$
$$\vdots$$
$$+\left[Fe^{-(\rho+r)t_2}-Fe^{-(\rho+r)t_1}\right]\int_{-\infty}^{c_1}(2\pi)^{-\frac{1}{2}}e^{-\frac{(z_1)^2}{2}}dz_1+Fe^{-(\rho+r)t_1} \qquad (10)$$

where x_i, y_i, and z_i follow standard normal distributions, and a_i, b_i, and c_i are integral upper-boundaries. The definitions are as the following:

$$x_i=\frac{\ln\theta_{t_i}-\left[\ln\theta_{t_{i-1}}+\left(\mu+\frac{3}{2}\sigma^2\right)(t_i-t_{i-1})\right]}{\sigma\sqrt{t_i-t_{i-1}}},y_i=\frac{\ln\theta_{t_i}-\left[\ln\theta_{t_{i-1}}+\left(\mu+\frac{1}{2}\sigma^2\right)(t_i-t_{i-1})\right]}{\sigma\sqrt{t_i-t_{i-1}}},$$
$$z_i=\frac{\ln\theta_{t_i}-\left[\ln\theta_{t_{i-1}}+\left(\mu-\frac{1}{2}\sigma^2\right)(t_i-t_{i-1})\right]}{\sigma\sqrt{t_i-t_{i-1}}}\ for\ i=1,2,\ldots,n$$
$$a_i=\frac{\ln\left[\frac{\theta_{i-1}}{\theta^*(t_i)}\right]+\left(\mu+\frac{3}{2}\sigma^2\right)(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}},b_i=\frac{\ln\left[\frac{\theta_{i-1}}{\theta^*(t_i)}\right]+\left(\mu+\frac{1}{2}\sigma^2\right)(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}},$$
$$c_i=\frac{\ln\left[\frac{\theta_{i-1}}{\theta^*(t_i)}\right]+\left(\mu-\frac{1}{2}\sigma^2\right)(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\ for\ i=1,2,\ldots,n$$

Comparing Equation (10) with (8), we find that adding more abandonment opportunities increases the present value of the expected operating profit significantly by the excess of the present value of abandonment value.

3 Numerical Solutions

We apply the numerical method to derive some results to show how the number of abandonment opportunities, volatility of future market condition, variable cost, growth rate, depreciation rate, and facility value can affect the expected abandonment value and how exit thresholds can be varied at different abandonment nodes. Matlab programming is used to plot the simulation results. We select a base set of following parameters for our analysis: growth rate of market condition μ = 15%, cost of capital r = 10%, rate of depreciation ρ = 50%, variable cost per unit v = 5, facility value F = 10,000, volatility σ = 40%, and time to maturity T = 1. To simplify the analysis, we only consider projects with one, two, and three abandonment opportunities.

Figure 3 shows the relationship between expected abandonment value and current market conditions under different number of abandonment opportunities. The figure shows that the more opportunities for firms to exit the market before maturity, the more valuable the abandonment option, and the higher the expected abandonment value. However, the increase in expected abandonment value decreases with the increase in abandonment opportunities. The expected abandonment value is negatively related with the current market condition.

Figure 4 shows the relationship between expected abandonment value and volatility under different levels of market condition. The figure shows that the expected abandonment value is positively related with the volatility of market condition. The change in expected abandonment value given a change in volatility increases as current market conditions become better.

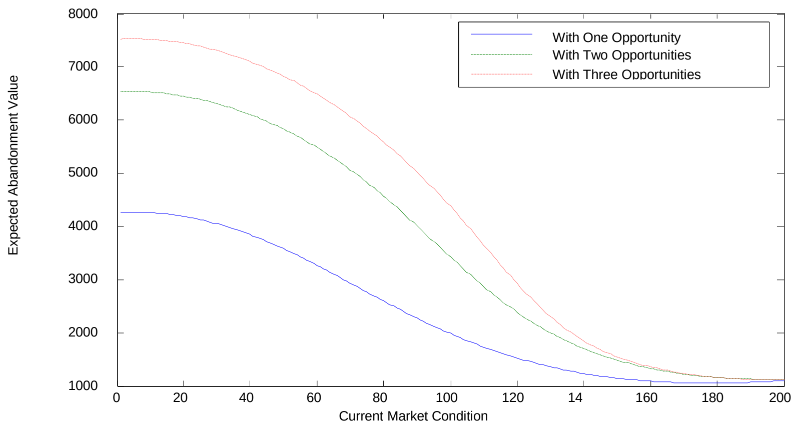

Figure 3: Expected Abandonment Values under Different Number of Abandonment Opportunities.

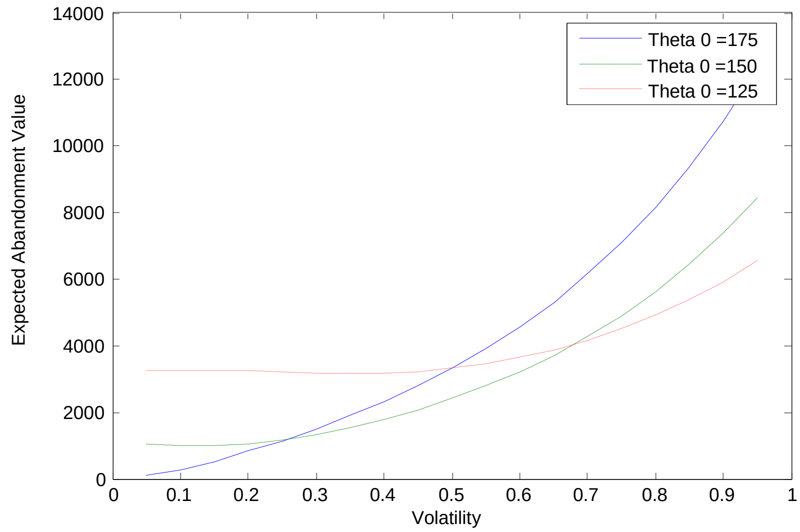

Figure 4: The Relationship between Volatility and Expected Abandonment Value under Different Market Conditions.

Figure 5 shows how exit thresholds at each abandonment node vary under different levels of volatility. At any given abandonment node, the higher the volatility, the lower the exit threshold. At a low volatility level, the longer the time to maturity, the higher the exit threshold, and the exit threshold decreases as the terminal approaches. On the other hand, at a high volatility level, the longer the time to maturity, the lower the exit threshold, and the exit threshold increases as the terminal approaches.

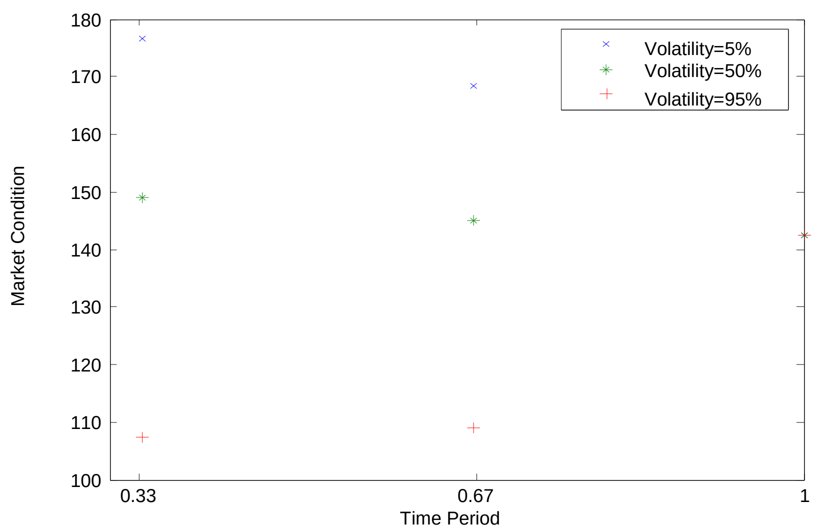

Figure 5: The Effect of Volatility on Exit Threshold.

Figure 6 shows the relationship between expected abandonment value and variable cost under different levels of market condition. The figure shows that the expected abandonment value is positively related with the variable cost. The change in expected abandonment value given a change in variable cost increases as current market conditions become worse.

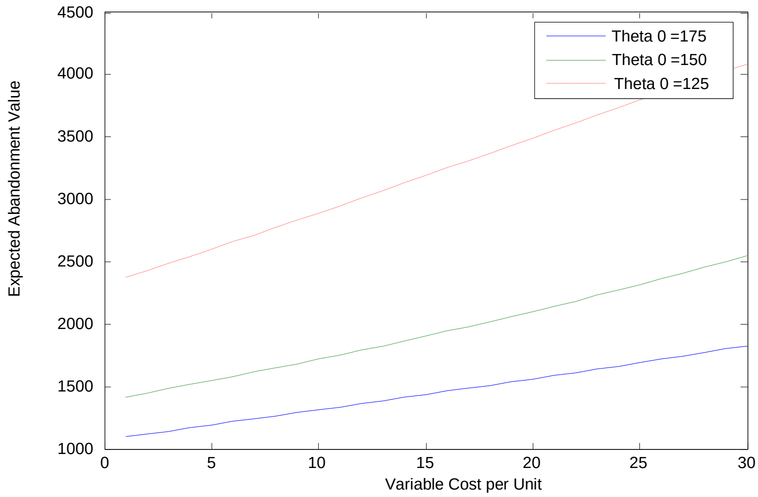

Figure 6: The Relationship between Variable Cost and Expected Abandonment Value under Different Market Conditions.

Figure 7 shows how exit thresholds at each abandonment node vary under different levels of variable cost. At any given abandonment node, the higher the variable cost, the higher the exit threshold. At a low variable cost level, the longer the time to maturity, the higher the exit threshold, and the exit threshold decreases as the terminal approaches. On the other hand, at a high variable cost level, the longer the time to maturity, the higher the exit threshold, and the exit threshold decreases at the beginning and becomes increasing in the middle to the terminal.

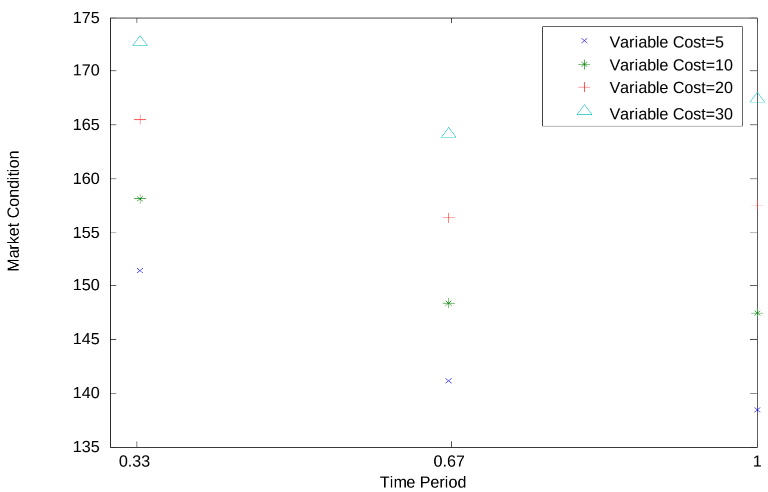

Figure 7: The Effect of Variable Cost on Exit Threshold.

Figure 8 shows the relationship between expected abandonment value and growth rate under different levels of market condition. The figure shows that the expected abandonment value is negatively related with the growth rate. The change in expected abandonment value given a change in growth rate increases as current market conditions become worse.

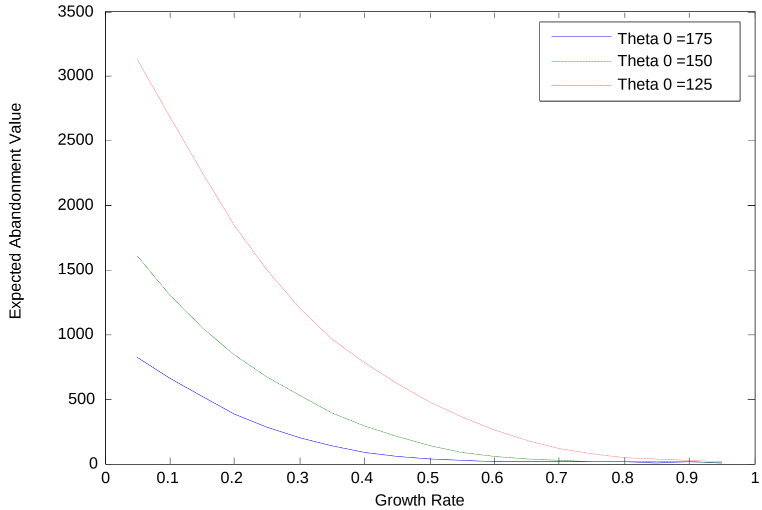

Figure 8: The Relationship between Growth Rate and Expected Abandonment Value under Different Market Conditions.

Figure 9 shows how exit thresholds at each abandonment node vary under different levels of growth rate. At any given abandonment node, the higher the growth rate, the lower the exit threshold. At a low growth rate level, the longer the time to maturity, the higher the exit threshold, and the exit threshold decreases as the terminal approaches. On the other hand, at a high growth rate level, the longer the time to maturity, the lower the exit threshold, and the exit threshold increases as the terminal approaches.

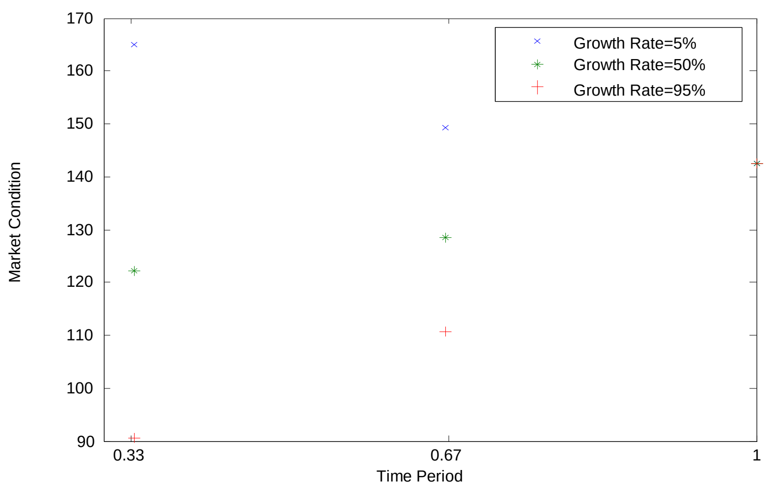

Figure 9: The Effect of Growth Rate on Exit Threshold.

Figure 10 shows the relationship between expected abandonment value and depreciation rate under different levels of market condition. The figure shows that the expected abandonment value is negatively related with the depreciation rate. The change in expected abandonment value given a change in depreciation rate increases as current market conditions become worse.

Figure 11 shows how exit thresholds at each abandonment node vary under different levels of depreciation rate. At any given abandonment node, the higher the depreciation rate, the lower the exit threshold. At a low depreciation rate level, the longer the time to maturity, the lower the exit threshold, and the exit threshold increases as the terminal approaches. On the other hand, at a high depreciation rate level, the longer the time to maturity, the higher the exit threshold, and the exit threshold decreases as the terminal approaches.

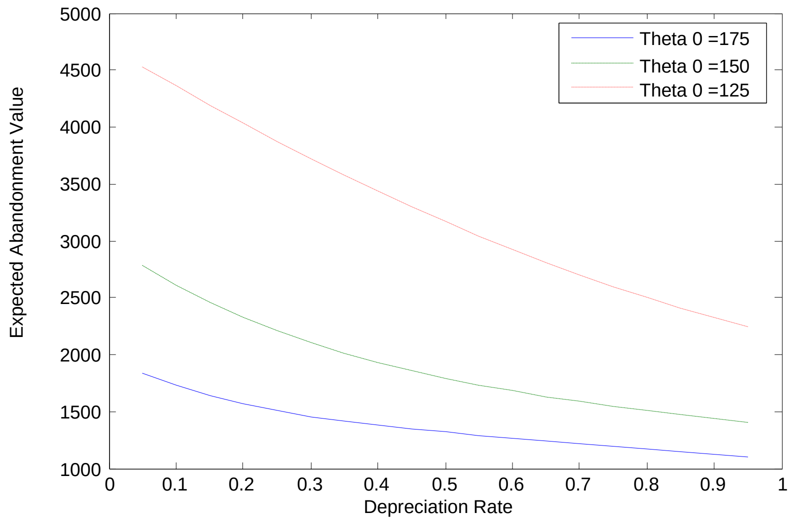

Figure 10: The Relationship between Depreciation Rate and Expected Abandonment Value under Different Market Conditions

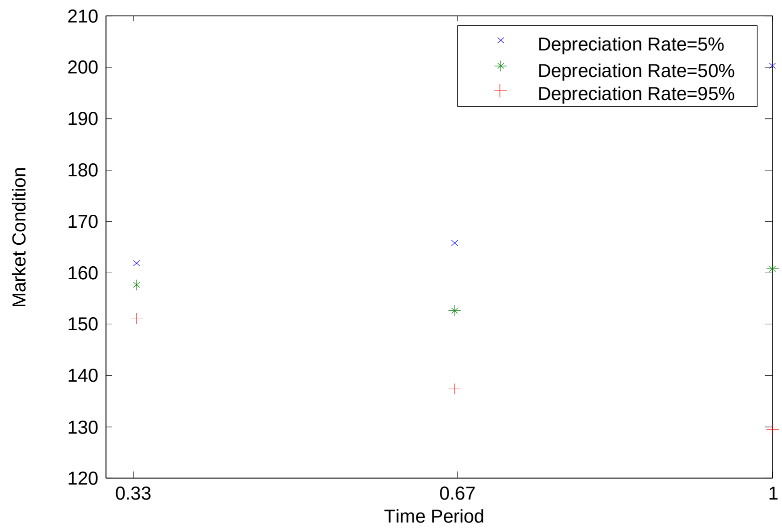

Figure 11: The Effect of Depreciation Rate on Exit Threshold.

Figure 12 shows the relationship between expected abandonment value and facility value under different levels of market condition. The figure shows that the expected abandonment value is positively related with the facility value. The change in expected abandonment value given a change in facility value increases as current market conditions become worse.

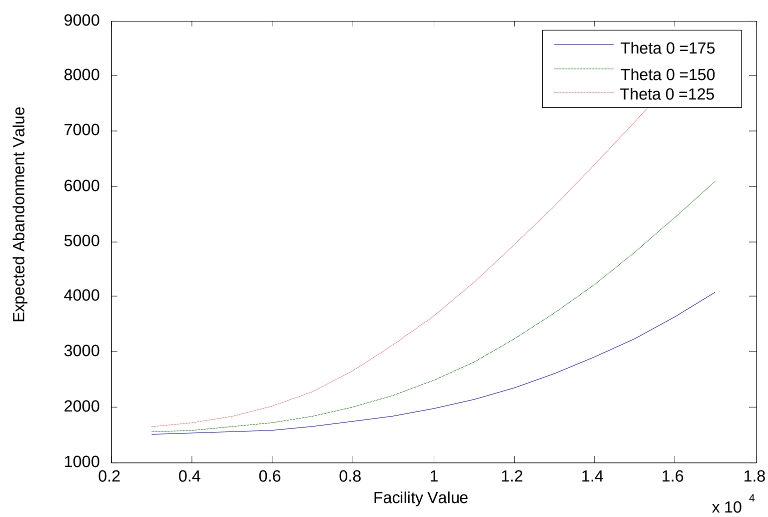

Figure 12: The Relationship between Facility Value and Expected Abandonment Value under Different Market Conditions.

Figure 13 shows how exit thresholds at each abandonment node vary under different levels of facility value. At any given abandonment node, the higher the facility value, the higher the exit threshold. In all levels of facility value, the longer the maturity, the higher the exit threshold, and the exit threshold decreases at the beginning and becomes increasing in the middle to the terminal.

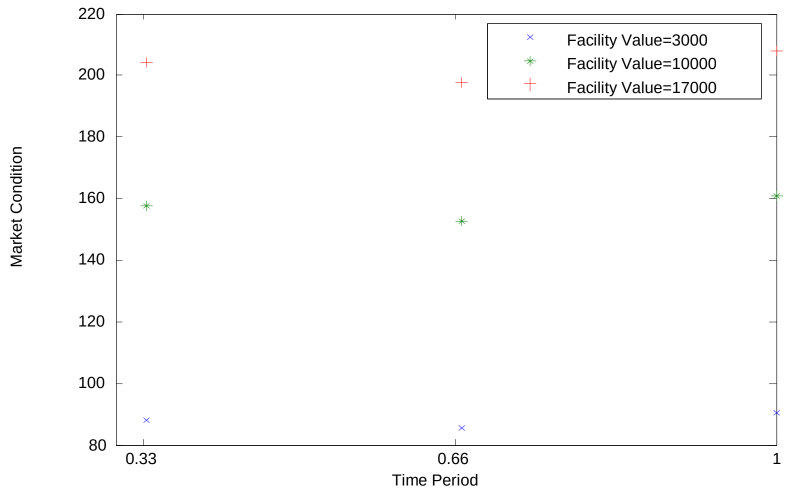

Figure 13: The Effect of Facility Value on Exit Threshold.

4 Conclusion

We analyze the abandonment decision in a contingent claims model that considers future abandonment opportunities, economic depreciation, flexible strategic behavior, and investment under uncertainty. Results from the numerical method show that volatility of future market condition, variable cost, growth rate, depreciation rate, and facility value can affect the abandonment value of investment projects. We also provide the decision criteria under various scenarios.

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Appendix

This Appendix derives the expected value of operating profit. We assume market condition at time T, θ_T, follows a lognormal distribution. The natural logarithm of θ_T has a normal distribution with $E(\ln\theta_T) = \ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T$ and $Var(\ln\theta_T) = \sigma\sqrt{T}$. Then

$$E(\pi_T|\theta_0) = \left[\frac{(\theta_T - v)^2}{4}\right] prob[\theta_T \geq \theta^*(T)|\theta_0] + Fe^{-\rho T} prob[\theta_T < \theta^*(T)|\theta_0]$$

$$= \left[\frac{(\theta_T - v)^2}{4} - Fe^{-\rho T}\right] prob[\theta_T \geq \theta^*(T)|\theta_0] + Fe^{-\rho T}$$

$$= \frac{1}{4}\int_{ln\theta^*(T)}^{\infty} (\theta_T)^2 \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{[ln\theta_T - E(ln\theta_T)]^2}{2\sigma^2 T}} d(ln\theta_T)$$
$$- \frac{v}{2}\int_{ln\theta^*(T)}^{\infty} \theta_T \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{[ln\theta_T - E(ln\theta_T)]^2}{2\sigma^2 T}} d(ln\theta_T)$$
$$+ \left(\frac{v^2}{4} - Fe^{-\rho T}\right)\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{[ln\theta_T - E(ln\theta_T)]^2}{2\sigma^2 T}} d(ln\theta_T) + Fe^{-\rho T}$$

Let $(\theta_T)^2 = e^{2ln\theta_T}$, $\theta_T = e^{ln\theta_T}$ and substitute $E(ln\theta_T) = ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T$ into the equation

$$E(\pi_T|\theta_0) = \frac{1}{4}\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right\}^2 - 4\sigma^2 T ln\theta_T}{2\sigma^2 T}} d(ln\theta_T)$$
$$- \frac{v}{2}\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right\}^2 - 2\sigma^2 T ln\theta_T}{2\sigma^2 T}} d(ln\theta_T)$$
$$+ \left(\frac{v^2}{4} - Fe^{-\rho T}\right)\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right\}^2}{2\sigma^2 T}} d(ln\theta_T)$$
$$+ Fe^{-\rho T}$$

$$= \frac{1}{4}(\theta_0)^2 e^{2\left(\mu - \frac{1}{2}\sigma^2\right)T + 2\sigma^2 T}\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + 2\sigma^2 T\right]\right\}^2}{2\sigma^2 T}} d(ln\theta_T)$$
$$- \frac{v}{2}\theta_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T}\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma^2 T\right]\right\}^2}{2\sigma^2 T}} d(ln\theta_T)$$
$$+ \left(\frac{v^2}{4} - Fe^{-\rho T}\right)\int_{ln\theta^*(T)}^{\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{\left\{ln\theta_T - \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right\}^2}{2\sigma^2 T}} d(ln\theta_T) + Fe^{-\rho T}$$

We change the original probability functions with new expectation value and standardize the new probability functions. In the standard normal distribution, $prob(Z \geq Z_a) = prob(Z \leq -Z_a)$.

$$E(\pi_T|\theta_0) = \frac{1}{4}(\theta_0)^2 e^{2\left(\mu-\frac{1}{2}\sigma^2\right)T+2\sigma^2 T}\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}}dz$$
$$-\frac{v}{2}\theta_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\sigma^2 T}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}}dz$$
$$+\left(\frac{v^2}{4}-Fe^{-\rho T}\right)\int_{-\infty}^{d_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}}dz + Fe^{-\rho T}$$
$$= \frac{1}{4}(\theta_0)^2 e^{2\left(\mu+\frac{1}{2}\sigma^2\right)T}N(d_1) - \frac{v}{2}(\theta_0)e^{\mu T}N(d_2) + \left(\frac{v^2}{4}-Fe^{-\rho T}\right)N(d_3) + Fe^{-\rho T}$$

where

$$d_1 = \frac{-ln\theta^*(T) + \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + 2\sigma^2 T\right]}{\sigma\sqrt{T}} = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu + \frac{3}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{-ln\theta^*(T) + \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma^2 T\right]}{\sigma\sqrt{T}} = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_3 = \frac{-ln\theta^*(T) + \left[ln\theta_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}} = \frac{ln\left[\frac{\theta_0}{\theta^*(T)}\right] + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

We extend the restriction to that there is another abandonment opportunity before terminal. The solution must be derived from time T to time t_1.

$$E\left(\pi_{t_1}\middle|\theta_0\right) = e^{-r(T-t_1)}E\left(\pi_T\middle|\theta_{t_1}\right)prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right]$$
$$+ Fe^{-\rho t_1}prob\left[\theta_{t_1} < \theta^*(t_1)\middle|\theta_0\right]$$
$$= e^{-r(T-t_1)}E\left(\pi_T\middle|\theta_{t_1}\right)prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right] + Fe^{-\rho t_1}\left\{1 - prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right]\right\}$$
$$= e^{-r(T-t_1)}\left\{\left[\frac{(\theta_T - v)^2}{4} - Fe^{-\rho T}\right]prob\left[\theta_T \geq \theta^*(T)\middle|\theta_{t_1}\right]prob[\theta_{t_1}\right.$$
$$\left.\geq \theta^*(t_1)|\theta_0] + Fe^{-\rho T}prob[\theta_{t_1} \geq \theta^*(t_1)|\theta_0]\right\}$$
$$- Fe^{-\rho t_1}prob\left[\theta_{t_1} \geq \theta^*(t_1)\middle|\theta_0\right] + Fe^{-\rho t_1}$$
$$= e^{-r(T-t_1)}\frac{1}{4}\int_{ln\theta^*(t_1)}^{\infty}\left(\theta_T{}^2\right)\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E\left(ln\theta_{t_1}\middle|\theta_0\right)]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{ln\theta^*(T)}^{\infty}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E\left(ln\theta_T\middle|\theta_{t_1}\right)]^2}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$
$$-e^{-r(T-t_1)}\frac{v}{2}\int_{ln\theta^*(t_1)}^{\infty}(\theta_T)\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{ln\theta^*(T)}^{\infty}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E(ln\theta_T)]^2}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$
$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}\right.$$
$$\left.- Fe^{-\rho T}\right]\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{ln\theta^*(T)}^{\infty}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E(ln\theta_T)]^2}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$
$$+\left[e^{-r(T-t_1)}\,Fe^{-\rho T} - \,Fe^{-\rho t_1}\right]\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1}) + Fe^{-\rho t_1}$$
$$= \frac{1}{4}e^{-r(T-t_1)}\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E\left(ln\theta_{t_1}\middle|\theta_0\right)]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E\left(ln\theta_T\middle|\theta_{t_1}\right)]^2+4\sigma^2(T-t_1)ln\theta_{t_1}}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$

$$-e^{-r(T-t_1)}\frac{v}{2}\int_{ln\theta^*(t_1)}^{\infty}(\theta_T)\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E(ln\theta_T)]^2+2\sigma^2(T-t_1)ln\theta_{t_1}}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$

$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}\right.$$

$$\left.-Fe^{-\rho T}\right]\int_{-\infty}^{-ln\theta^*(t_1)}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})\int_{-\infty}^{-ln\theta^*(T)}\frac{1}{\sqrt{2\pi(T-t_1)}\sigma}e^{-\frac{[ln\theta_T-E(ln\theta_T)]^2}{2\sigma^2(T-t_1)}}d(ln\theta_T)$$

$$+\left[e^{-r(T-t_1)}\,Fe^{-\rho T}-\,Fe^{-\rho t_1}\right]\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})+Fe^{-\rho t_1}$$

$$=\frac{1}{4}e^{-r(T-t_1)+2\left(\mu-\frac{1}{2}\sigma^2\right)(T-t_1)+2\sigma^2(T-t_1)}\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E\left(ln\theta_{t_1}|\theta_0\right)]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})(\theta_{t_1})^2\int_{-\infty}^{a_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{{x_2}^2}{2\sigma^2}}dx_2$$

$$-\frac{v}{2}e^{-r(T-t_1)+\left(\mu-\frac{1}{2}\sigma^2\right)(T-t_1)+\frac{1}{2}\sigma^2(T-t_1)}\int_{ln\theta^*(t_1)}^{\infty}\frac{1}{\sqrt{2\pi t_1}\sigma}e^{-\frac{[ln\theta_{t_1}-E(ln\theta_{t_1})]^2}{2\sigma^2 t_1}}d(ln\theta_{t_1})(\theta_{t_1})\int_{-\infty}^{b_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{{y_2}^2}{2\sigma^2}}dy_2$$

$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}-Fe^{-\rho T}\right]\int_{-\infty}^{c_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{{z_1}^2}{2\sigma^2}}dz_1\int_{-\infty}^{c_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{{z_2}^2}{2\sigma^2}}dz_2$$

$$+\left[e^{-r(T-t_1)}\,Fe^{-\rho T}-\,Fe^{-\rho t_1}\right]\int_{-\infty}^{c_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{{z_1}^2}{2\sigma^2}}dz_1+Fe^{-\rho t_1}$$

$$=\frac{1}{4}(\theta_0)^2e^{-r(T-t_1)+2\left(\mu-\frac{1}{2}\sigma^2\right)(T-t_1)+2\sigma^2(T-t_1)+2\left(\mu-\frac{1}{2}\sigma^2\right)t_1+2\sigma^2 t_1}\int_{-\infty}^{a_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{{x_1}^2}{2\sigma^2 t_1}}d(x_1)\int_{-\infty}^{a_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{{x_2}^2}{2\sigma^2}}dx_2$$

$$-\frac{v}{2}(\theta_0)e^{-r(T-t_1)+\left(\mu-\frac{1}{2}\sigma^2\right)(T-t_1)+\frac{1}{2}\sigma^2(T-t_1)+\left(\mu-\frac{1}{2}\sigma^2\right)t_1+\frac{1}{2}\sigma^2 t_1}\int_{-\infty}^{b_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{{y_1}^2}{2\sigma^2 t_1}}d(y_1)\int_{-\infty}^{b_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{{y_2}^2}{2\sigma^2}}dy_2$$

$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}-Fe^{-\rho T}\right]\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}\left(\frac{1}{\sqrt{2\pi}}\right)^2e^{-\frac{{z_1}^2}{2}}e^{-\frac{{z_2}^2}{2}}dz_1dz_2$$

$$+\left[e^{-r(T-t_1)}\,Fe^{-\rho T}-\,Fe^{-\rho t_1}\right]\int_{-\infty}^{c_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{{z_1}^2}{2\sigma^2}}dz_1+Fe^{-\rho t_1}$$

The expected operating profit function can be simplified to the following:

$$E\left(\pi_{t_1}|\theta_0\right)=\frac{1}{4}(\theta_0)^2e^{2\left(\mu+\frac{1}{2}\sigma^2\right)T-r(T-t_1)}\int_{-\infty}^{a_1}\int_{-\infty}^{a_2}(2\pi)^{-1}e^{-\frac{(x_1)^2}{2}}e^{-\frac{(x_2)^2}{2}}dx_1dx_2$$

$$-\frac{v}{2}(\theta_0)e^{\mu T-r(T-t_1)}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2}(2\pi)^{-1}e^{-\frac{(y_1)^2}{2}}e^{-\frac{(y_2)^2}{2}}dy_1dy_2$$

$$+e^{-r(T-t_1)}\left[\frac{v^2}{4}-Fe^{-\rho T}\right]\int_{-\infty}^{c_1}\int_{-\infty}^{c_2}(2\pi)^{-1}e^{-\frac{(z_1)^2}{2}}e^{-\frac{(z_2)^2}{2}}dz_1dz_2$$

$$+\left[Fe^{-(\rho+r)T+rt_1}-Fe^{-\rho t_1}\right]\int_{-\infty}^{c_1}(2\pi)^{-\frac{1}{2}}e^{-\frac{(z_1)^2}{2}}dz_1+Fe^{-\rho t_1}$$

where

$$x_1=\frac{\ln\theta_{t_1}-\left[\ln\theta_0+(\mu+\frac{3}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}},\;x_2=\frac{\ln\theta_T-\left[\ln\theta_{t_1}+(\mu+\frac{3}{2}\sigma^2)(T-t_1)\right]}{\sigma\sqrt{T-t_1}}$$

$$y_1=\frac{\ln\theta_{t_1}-\left[\ln\theta_0+(\mu+\frac{1}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}},\;y_2=\frac{\ln\theta_T-\left[\ln\theta_{t_1}+(\mu+\frac{1}{2}\sigma^2)(T-t_1)\right]}{\sigma\sqrt{T-t_1}}$$

$$z_1 = \frac{\ln\theta_{t_1} - \left[\ln\theta_0 + (\mu - \frac{1}{2}\sigma^2)t_1\right]}{\sigma\sqrt{t_1}}, z_2 = \frac{\ln\theta_T - \left[\ln\theta_{t_1} + (\mu - \frac{1}{2}\sigma^2)(T - t_1)\right]}{\sigma\sqrt{T - t_1}}$$

$$a_1 = \frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right] + (\mu + \frac{3}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}}, a_2 = \frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right] + (\mu + \frac{3}{2}\sigma^2)(T - t_1)}{\sigma\sqrt{T - t_1}}$$

$$b_1 = \frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right] + (\mu + \frac{1}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}}, b_2 = \frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right] + (\mu + \frac{1}{2}\sigma^2)(T - t_1)}{\sigma\sqrt{T - t_1}}$$

$$c_1 = \frac{\ln\left[\frac{\theta_0}{\theta^*(t_1)}\right] + (\mu - \frac{1}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}}, c_2 = \frac{\ln\left[\frac{\theta_{t_1}}{\theta^*(T)}\right] + (\mu - \frac{1}{2}\sigma^2)(T - t_1)}{\sigma\sqrt{T - t_1}}$$