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Option Pricing with State-Price Deflators: The Multivariate Exponential Wang Normal Variance: Gamma Asset Pricing Models

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Abstract

Alternatives to the Black-Scholes-Vasicek deflator introduced in [25] are proposed. They are based on the multivariate Wang variance-gamma process considered in [66]. As an application, closed form analytical multiple integral formulas for pricing the European geometric basket option with a deflated multivariate exponential Wang variance-gamma asset pricing model are derived.

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1 Introduction

The concept of state-price deflator or stochastic discount factor, which has been introduced by Duffie [16], p. 23 and 97, is a convenient ingredient of general financial pricing rules. It contains information about the valuation of payments in different states at different points in time. The state-price deflator is a natural extension of the notion of state prices that were introduced earlier and studied by Arrow [2]-[5], Debreu [12], Negishi [55] and Ross [60], a milestone in the history of asset pricing (see Dimson and Mussavian [14]). Although general frameworks for deriving state-price deflators exist (e.g. Milterssen and Persson [53] and Jeanblanc et al. [34]), there are not many papers, which propose explicit expressions for them and their corresponding distribution functions.

We are interested in the construction of alternatives to the multivariate Black-Scholes-Vasicek (BSV) deflator introduced in Hürlimann [25] (see also [26]). A valuable and popular alternative choice to a log-normal distribution for asset pricing is an exponential

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variance-gamma process. It has been introduced in Madan and Seneta [48] and extensively used in financial applications (e.g. Madan and Milne [50], Madan et al. [49], Madan [47], Carr et al. [9], Geman [21], Fiorani [18], Fu et al. [20], Stein et al. [62], Domenig and Vanini [15], etc.). In the present paper we report on state-price deflators of multivariate normal variance-gamma type that are based on the multivariate Wang variance-gamma (WVG) process introduced in Wang [66].

For the interested reader we remark that the article Hürlimann [28] contains an extension of the Black-Scholes deflator to a more general version with interest rates as additional source of randomness. From a mathematical viewpoint it is natural to investigate other generalizations, namely the consideration of alternative asset price processes for use in incomplete financial markets. Indeed, let us assume that asset prices admit no arbitrage. Then, there exists a unique state-price deflator if, and only if, the market is complete. Otherwise, if the market is incomplete, several state-price deflators exist and pricing is a more complex topic (e.g. Munk [54], Theorem 4.2). Therefore, the study of state-price deflators is motivated by one of the main problems of Modern Finance, which consists to understand the pricing and hedging or replication of arbitrary portfolios in incomplete markets. Even if the portfolio is only made of derivatives there is no widely accepted solution to this problem (e.g. Cherny and Madan [10], Section 1).

A short account of the content follows. Section 2 recalls the two main representations of the variance-gamma process. Its generalization to the multivariate Wang variance-gamma (WVG) process is introduced in Section 3. The construction of the multivariate WVG deflator is found in Section 4. Section 5 extends the univariate normal variance-gamma process to its multivariate context and Section 6 derives the corresponding state-price deflator. As an application we derive in Section 7 closed form analytical multiple integral formulas for pricing the European geometric basket option with a deflated multivariate exponential WVG asset pricing model.

2 The Univariate Variance-gamma Process

There are two different representations of the variance-gamma (VG) process. In the original first version, the variance-gamma process is considered as a drifted Brownian motion time changed by an independent gamma process. Viewed from the initial time 0 it is defined by

$$X_t = \theta \cdot G_t + \sigma \cdot W_{G_t}, \quad t > 0, \tag{2.1}$$

where W_t is a standard Wiener process and the independent subordinator $G_t \sim \Gamma(\nu^{-1}t, \nu^{-1})$ is a gamma process with unit mean rate and variance rate ν. Since X_t is a Lévy process, its dynamics is determined by its distribution at unit time. In fact, the random variable $X = X_1 \sim VG(\theta, \sigma^2, \nu)$ follows a three parameter distribution with cumulant generating function (cgf)

$$\begin{aligned} &C_X(u) = \ln E[\exp(uX)] = -\nu^{-1} \cdot \ln\{1 - \nu \cdot (\theta u + \tfrac{1}{2}\sigma^2 u^2)\}, \\ &\sigma, \nu > 0, \quad -\infty < \theta < \infty. \end{aligned} \tag{2.2}$$

Of course, the cgf is only defined over an open interval (use (2.4)-(2.5) below). This formula is obtained from the cgf $C_G(u) = -\nu^{-1} \cdot \ln(1-\nu \cdot u)$ of the gamma random variable $G = G_1$ by conditioning using that $X|G \sim N(\theta G, \sigma^2 G)$ is normally distributed. The increments of the process follow a VG distribution, namely $X_{t+s} - X_s \sim VG(\theta t, \sigma^2 t, \nu / t), \quad 0 \le s < t$. The symmetric case $\theta = 0$ is used in the original asset and option pricing model by Madan and Seneta [48] and Madan and Milne [50].

In the second representation, the VG process is viewed as a bilateral gamma process (e.g. Carr et al. [9], Küchler and Tappe [43]) with the different parameterization

$$X_t =_d \alpha^{-1} \cdot G_t^{(1)} - \beta^{-1} \cdot G_t^{(2)} \sim VG^*(\rho \cdot t, \alpha, \beta), \quad \rho, \alpha, \beta > 0, \tag{2.3}$$

where $G_t^{(i)} \sim \Gamma(\rho t, 1),\ i = 1,2,$ are independent copies of standardized gamma processes with scale parameter 1. The equality in distribution of the formulas (2.1) and (2.3) follows from the fact that the cgf of the independent gamma distributed difference in (2.3) equals

$$C_{X_t}(u) = -\rho t \cdot \ln(1 - (\alpha^{-1} - \beta^{-1})u - (\alpha\beta)^{-1}u^2), \quad -\beta < u < \alpha. \tag{2.4}$$

The two representations are linked by the one-to-one transformation of parameters

$$\begin{aligned} &\nu = \rho^{-1}, \quad \theta = \rho(\alpha^{-1} - \beta^{-1}), \quad \sigma = \sqrt{2(\alpha\beta)^{-1}\rho}, \\ &\rho = \nu^{-1}, \quad \alpha^{-1} = \tfrac{1}{2}(\sqrt{(\nu\theta)^2 + 2\nu\sigma^2} + \nu\theta), \quad \beta^{-1} = \tfrac{1}{2}(\sqrt{(\nu\theta)^2 + 2\nu\sigma^2} - \nu\theta). \end{aligned} \tag{2.5}$$

The VG process has been extensively studied in Madan et al. [49]. It is worthwhile to mention that it is a special case of the CGMY model by Carr et al. [9].

3 The Multivariate Wang Variance-gamma Process

Several multivariate versions of the VG process have been considered so far. Madan and Seneta [48] first introduced a multivariate symmetric VG process by subordinating a multivariate Brownian motion without drift by a common gamma process. The asymmetric version of this model has been developed in Cont and Tankov [11] and Luciano and Schoutens [44]. Generalizing (2.1) these authors consider multivariate Lévy processes with VG components of the type

$$X_t^{(k)} = \theta_k \cdot G_t + \sigma_k \cdot W_{G_t}^{(k)}, \quad k = 1,\dots,n, \tag{3.1}$$

where the $W_t^{(k)}$'s are correlated standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$. This simple model is easy to work with but has some serious drawbacks. For example, linear correlation cannot be fitted once the margins are fixed.

Moreover, the choice of a single parameter ν causes great difficulty in the joint calibration to option prices on the margins as observed by Luciano and Semeraro [45]. To overcome these deficiencies Semeraro [61] and Luciano and Semeraro [45]-[46] consider multivariate subordination to multivariate Brownian motions through the generalized specification

$$X_t^{(k)} = \theta_k \cdot G_t^{(k)} + \sigma_k \cdot W_{G_t^{(k)}}^{(k)}, \quad k = 1,...,n, \tag{3.2}$$

where the $W_t^{(k)}$'s are *independent* standard Wiener processes and $G_t = (G_t^{(1)},...,G_t^{(n)})$ is a multivariate subordinator defined by

$$G_t^{(k)} = Y_t^{(k)} + a_k Z_t, \tag{3.3}$$

with $a_k \geq 0$, and independent gamma processes $Y_t^{(k)} \sim \Gamma(\ell_k t, m_k), Z_t \sim \Gamma(p,q)$. To ensure that the margins (3.2) are VG processes one requires that (3.3) is a gamma process. As observed by Hitaj and Mercuri [24] this is the case under the two alternative choices (i) $a_k = 0$ the independent case or (ii) $a_k = q/m_k$ with $G_t^{(k)} \sim \Gamma((\ell_k + p)t, m_k)$. Wang [66] notes that a closed-form joint characteristic function, which plays a critical role in option pricing and parameter estimation, can only be found in the case of independent Brownian motions in (3.2). In this situation the dependence is mainly due to the drift part, which might be too weak in financial applications. For this reason, Wang [66], Section 2.2, introduces a new multivariate VG process with closed-form joint cgf, called hereafter Wang variance-gamma (WVG) process.

The modelling idea consists to decompose each marginal VG process $X_t^{(k)} = \theta_k G_t^{(k)} + \sigma_k W_{G_t^{(k)}}^{(k)}, G_t^{(k)} \sim \Gamma(\nu_k^{-1} t, \nu_k^{-1}), k = 1,...,n,$ into two independent VG components such that

$$\begin{aligned} X_t^{(k)} &= A_t^{(k)} + Y_t^{(k)}, \quad A_t^{(k)} = \theta_k \tfrac{\nu_k}{\nu_0} G_t + \sigma_k \sqrt{\tfrac{\nu_k}{\nu_0}} W_{G_t}^{(k)}, \quad G_t \sim \Gamma\left(\nu_0^{-1} t, \nu_0^{-1}\right), \\ Y_t^{(k)} &= \theta_k (1 - \tfrac{\nu_k}{\nu_0}) H_t^{(k)} + \sigma_k \sqrt{1 - \tfrac{\nu_k}{\nu_0}} W_{H_t^{(k)}}^{(k)}, \quad H_t^{(k)} \sim \Gamma\left((\nu_k^{-1} - \nu_0^{-1})t, (\nu_k^{-1} - \nu_0^{-1})\right), \end{aligned} \tag{3.4}$$

where $Y_t^{(i)}, Y_t^{(j)}, A_t^{(i)}$ are independent for $i, j = 1,...,n, i \neq j$, and the conditional random vector process $(A_t^{(1)},...,A_t^{(n)} | G_t)$ is multivariate Gaussian with mean vector $\mu_A \cdot t$ and variance-covariance matrix $\Omega_A \cdot t$ given by

$$\mu_A = \left(\theta_1 \tfrac{\nu_1}{\nu_0},...,\theta_n \tfrac{\nu_n}{\nu_0}\right), \quad \Omega_A = \left(\frac{\sqrt{\nu_i \nu_j}}{\nu_0} \rho_{ij} \sigma_i \sigma_j \right) \tag{3.5}$$

The parameters of the VG margins can be arbitrary, but the parameters, which drive the dependence structure, must satisfy the constraint $\nu_0 \geq \max(\nu_1,...,\nu_n)$.

The decomposition of the marginal processes into two components is motivated by the following economic background. The dependent component or systematic part $A_t = (A_t^{(1)},..., A_t^{(n)})$ is interpreted as a systematic factor, which governs the big co-movements of individual assets, while the independent part $Y_t = (Y_t^{(1)},..., Y_t^{(n)})$ represents the individual factors of each asset.

Since the margins are sums of independent processes with known cgf's, the joint cgf of this process can be expressed in closed-form (the case $n = 2$ is Proposition 2.2 in Wang (2009)).

Theorem 3.1 (cgf of the multivariate WVG process). *The joint cgf of the multivariate WVG process* $X_t = (X_t^{(1)},..., X_t^{(n)})$ *with parameters* $\theta_k, \sigma_k, \nu_k, k = 1,..., n, \rho_{ij}, i, j = 1,..., n,$ $\nu_0 \geq \max(\nu_1,..., \nu_n),$ *is determined by* $C_{X_t}(u) = C_{A_t}(u) + \sum_{k=1}^{n} C_{Y_t^{(k)}}(u_k)$ *with*

$$C_{A_t}(u) = -\nu_0^{-1} t \cdot \ln\{1 - \theta^T u - \tfrac{1}{2} u^T \Sigma u\},$$
$$C_{Y_t^{(k)}}(u_k) = -(\nu_k^{-1} - \nu_0^{-1}) t \cdot \ln\{1 - \nu_k \cdot (\theta_k u_k + \tfrac{1}{2}\sigma_k^2 u_k^2)\}, \tag{3.6}$$

and $u = (u_1,..., u_n),\quad \theta = \nu_0 \cdot \mu_A = (\nu_1\theta_1,..., \nu_n\theta_n),\quad \Sigma = \nu_0 \cdot \Omega_A = \left(\sqrt{\nu_i \nu_j}\, \rho_{ij} \sigma_i \sigma_j\right).$

Proof. The dependence assumptions and (2.2) implies first that

$$C_{X_t}(u) = C_{A_t}(u) + \sum_{k=1}^{n} C_{Y_t^{(k)}}(u_k), \text{ with}$$
$$C_{Y_t^{(k)}}(u_k) = -(\nu_k^{-1} - \nu_0^{-1}) t \cdot \ln\{1 - \nu_k \cdot (\theta_k u_k + \tfrac{1}{2}\sigma_k^2 u_k^2)\}.$$

Therefore, it suffices to verify the formula for the cgf of the systematic part. Conditionally on the common gamma subordinator G_t and using the fact that the conditional margins are normally distributed as

$$A_t^{(k)} \big| G_t = \theta_k \tfrac{\nu_k}{\nu_0} G_t + \sigma_k \sqrt{\tfrac{\nu_k}{\nu_0}} W_{G_t}^{(k)} \big| G_t \sim N(\theta_k \tfrac{\nu_k}{\nu_0} G_t, \sigma_k^2 \tfrac{\nu_k}{\nu_0} G_t),$$

one obtains the representation (3.6) from the following calculation

$$C_{A_t}(u) = \ln E[\exp(u^T A_t)] = \ln \mathrm{E}_{G_t}[\mathrm{E}[\exp(u^T A_t) \big| G_t]]$$
$$= \ln \mathrm{E}_{G_t}[\exp(\nu\theta^T u G_t + \tfrac{1}{2} u^T \Sigma u G_t)].\ \Diamond$$

Using a general result about subordination of a Lévy process (e.g. Cont and Tankov [11], Theorem 4.2), it is possible to obtain the Lévy measure of the multivariate WVG process (see Wang [66], Section 2.2). The pairwise linear correlation between the margins $X_t^{(i)}$

and $X_t^{(j)}$ is time-independent and given by

$$\rho\left(X_t^{(i)}, X_t^{(j)}\right) = \frac{\frac{\nu_i \nu_j}{\nu_0}\theta_i\theta_j + \frac{\sqrt{\nu_i \nu_j}}{\nu_0}\rho_{ij}\sigma_i\sigma_j}{\sqrt{\nu_i\theta_i^2 + \sigma_i^2}\cdot\sqrt{\nu_j\theta_j^2 + \sigma_j^2}}. \tag{3.7}$$

A derivation of (3.7) is found in Wang [6], Proposition 2.3. The following facts point out the flexibility of the dependence structure:

(i) $\lim_{\nu_0\to\infty} \rho\left(X_t^{(i)}, X_t^{(j)}\right) = 0$ (asymptotically independent marginal VG processes)

(ii) $\nu_i = \nu_j, \nu_0 = \max(\nu_i, \nu_j), \rho_{ij} = 1$ (maximum dependence between $X_t^{(i)}$ and $X_t^{(j)}$)

(iii) $\nu_i = \nu_j = \nu_0, \rho_{ij} = 1$ (full comonotone dependence between $X_t^{(i)}$ and $X_t^{(j)}$)

4 The Multivariate Wang Variance-gamma Deflator

Consider the class of exponential WVG processes. Given the current prices of $n \geq 1$ risky assets at initial time 0 their future prices at time $t > 0$ are described by exponential VG processes

$$S_t^{(k)} = S_0^{(k)} \exp((\mu_k - \omega_k)t + X_t^{(k)}), \quad k = 1,..., n, \tag{4.1}$$

where μ_k represents the mean logarithmic rate of return of the k-th risky asset per time unit, and the random vector $X_t = (X_t^{(1)},..., X_t^{(n)})$ follows a multivariate WVG process. Using the defining relationship $E[S_t^{(k)}] = S_0^{(k)} \exp(\mu_k t)$ at unit time, one sees that $\omega_k = C_{X^{(k)}}(1) < \infty, k = 1,..., n$, where one assumes that the cgf of $X^{(k)} = X_1^{(k)}$ exists over some open interval, which contains one. Suppose that the multivariate WVG deflator of dimension n has the same form as the price processes in (4.1). For some parameter α and vector $\beta = (\beta_1,..., \beta_n)$ (both to be determined) one sets for it (an Esscher transformed measure)

$$D_t = \exp(-\alpha t - \beta^T X_t), \quad t > 0. \tag{4.2}$$

A simple cgf calculation shows that the defining martingale conditions

$$E[D_t] = e^{-rt}, \quad E[D_t S_t^{(k)}] = S_0^{(k)}, \quad t > 0, \tag{4.3}$$

are equivalent with the system of $n+1$ equations in the $2n+1$ unknowns

$\alpha, \beta_k, \omega_k$ (use that X_t is a Lévy process, hence $C_{X_t}(u) = t \cdot C_X(u)$):

$$r - \alpha + C_X(-\beta) = 0, \quad \mu_k - \omega_k - \alpha + C_X(\beta^{(k)}) = 0,$$
$$\beta^{(k)} = (\beta_1^{(k)}, ..., \beta_n^{(k)}), \quad \beta_j^{(k)} = \delta_j^k - \beta_j, \quad j, k = 1, ..., n. \tag{4.4}$$

Inserting the first equation into the second ones yields the necessary relationships

$$\mu_k - r - \omega_k + C_X(\beta^{(k)}) - C_X(-\beta) = 0, \quad k = 1, ..., n. \tag{4.5}$$

By Theorem 3.1 these equations are equivalent with

$$\mu_k - r - \omega_k + C_A(\beta^{(k)}) - C_A(-\beta) + C_{Y^{(k)}}(1 - \beta_k) - C_{Y^{(k)}}(-\beta_k) = 0, \quad k = 1, ..., n. \tag{4.6}$$

Since the system (4.4) has n degrees of freedom, the unknown ω_k can be chosen arbitrarily, say

$$\omega_k = \mu_k - r + C_{Y^{(k)}}(1 - \beta_k) - C_{Y^{(k)}}(-\beta_k) = C_{X^{(k)}}(1), \quad k = 1, ..., n, \tag{4.7}$$

which is interpreted as the (time-independent) WVG market price of the k-th risky asset. With the made restriction on the cgf, this value is always finite. Inserted into (4.6) shows that the parameter vector β is determined by the equations

$$C_A(\beta^{(k)}) = C_A(-\beta), \quad k = 1, ..., n. \tag{4.8}$$

We are ready to show the following WVG deflator representation.

Theorem 4.1 (WVG deflator of dimension n). *Given are $n \ge 1$ risky assets with exponential VG real-world prices (4.1), where the random vector process $X_t = (X_t^{(1)}, ..., X_t^{(n)})$ follows a multivariate WVG process. Then, the WVG deflator (4.2) is determined by*

$$D_t = \exp(-\alpha t - \sum_{k=1}^{n} \beta_k X_t^{(k)}), \quad t > 0, \quad \textit{with} \tag{4.9}$$

$$\alpha = r + C_X(-\beta), \quad \beta_k \sigma_k^2 = \theta_k + (\tfrac{1}{2} + \gamma_k)\sigma_k^2, \quad \gamma_k = \sum_{j \ne k} \sqrt{\tfrac{\nu_j}{\nu_k}} \rho_{kj} \tfrac{\sigma_j}{\sigma_k}, \quad k = 1, ..., n. \tag{4.10}$$

Moreover, in the univariate case $n = 1$ one has $\gamma_1 = 0$.

Proof. The first equation in (4.4) yields α. Since $C_A(u) = -\nu_0^{-1} \cdot \ln\{1 - \theta^T u - \frac{1}{2} u^T \Sigma u\}$ by Theorem 3.1, it follows that the conditions (4.8) are equivalent with the equations

$$\theta^T \beta^{(k)} - \tfrac{1}{2}\beta^{(k)^T}\Sigma\beta^{(k)} + \theta^T\beta - \tfrac{1}{2}\beta^T\Sigma\beta = 0, \quad k = 1,...,n.$$

A calculation shows that the latter is equivalent with the stated conditions for $\beta_k, k = 1,..,n$, where in case $n = 1$ one has $\gamma_1 = 0$ (empty sum). ◊

5 The Normal Variance-gamma Process and its Multivariate Wang Version

Recall the embedding of the VG process into the *bilateral gamma* (BG) process defined by (e.g. Küchler and Tappe [43], Section 6)

$$X_t =_d \alpha^{-1} \cdot G_t^{(1)} - \beta^{-1} \cdot G_t^{(2)} \sim BG(\gamma \cdot t, \alpha, \delta \cdot t, \beta), \quad \gamma, \alpha, \delta, \beta > 0, \tag{5.1}$$

where $G_t^{(1)} \sim \Gamma(\gamma \cdot t, 1)$ and $G_t^{(2)} \sim \Gamma(\delta \cdot t, 1)$ are independent standardized gamma processes with scale parameter 1 and shape parameters $\gamma \cdot t$ respectively $\delta \cdot t$. Clearly, the VG process (2.3) is the special case $\gamma = \delta = \rho$ of (5.1). For even greater flexibility it is natural to embed the BG process into the six parameter *normal bilateral gamma* (NBG) process defined by

$$\begin{aligned} &X_t =_d \xi \cdot t + \psi \cdot W_t + \alpha^{-1} \cdot G_t^{(1)} - \beta^{-1} \cdot G_t^{(2)} \\ &\sim NBG(\xi \cdot t, \psi^2 \cdot t, \gamma \cdot t, \alpha, \delta \cdot t, \beta), \quad \psi \geq 0, \gamma, \alpha, \delta, \beta > 0, -\infty < \xi < \infty, \end{aligned} \tag{5.2}$$

with $W_t \sim N(0,t)$ a standard Wiener process, $G_t^{(1)}, G_t^{(2)}$ as above and independent of W_t. The cgf of the NBG process is determined by

$$\begin{aligned} &C_{X_t}(u) = \xi \cdot t \cdot u + \tfrac{1}{2}\psi^2 \cdot t \cdot u^2 - \gamma \cdot t \cdot \ln(1 - \alpha^{-1}u) - \delta \cdot t \cdot \ln(1 + \beta^{-1}u), \\ &-\beta < u < \alpha. \end{aligned} \tag{5.3}$$

A few words about this rich class of Lévy processes are in order. The distribution of the NBG random variable $X = X_1 \sim NBG(\xi, \psi^2, \gamma, \alpha, \delta, \beta)$ includes a number of important and increasingly discussed families of distributions (see Hürlimann [31] for a discussion with many references). Especially, it is worthwhile to mention that the *Brownian-Laplace* motion considered in Reed [58] is a re-parameterization of the *normal variance-gamma* (NVG) process obtained from (5.2) by setting $\gamma = \delta = \rho$. In fact, the independent and stationary increments $X =_d X_{t+s} - X_s, 0 \leq s < t,$ of this process follow a *generalized normal Laplace* (GNL) distribution introduced in Reed (2006) and defined by

$$X = \xi \cdot \rho + \psi \cdot \sqrt{\rho} \cdot Z + \alpha^{-1} \cdot G^{(1)} - \beta^{-1} \cdot G^{(2)}$$
$$\sim GNL(\xi, \psi^2, \rho, \alpha, \beta) = NVG(\xi \cdot \rho, \psi^2 \cdot \rho, \rho, \alpha, \beta), \tag{5.4}$$

where $Z, G^{(1)}, G^{(2)}$ are independent with $Z \sim N(0,1)$, and $G^{(1)}, G^{(2)}$ are standardized gamma with scale parameter 1 and shape parameter ρ. From now on, the focus will be restricted to the NVG process and its multivariate version.

The multivariate *Wang normal variance-gamma* (WNVG) process is obtained from the WVG process as the univariate NVG is obtained from the univariate VG process. For convenience, the VG margins are described in terms of the original parameter set $(\theta_k, \sigma_k, \nu_k)$ instead of $(\rho_k, \alpha_k, \beta_k), k = 1,..., n$ (parameter transformation (2.5)). The one-dimensional incremental margins of the WNVG process are described by convolutions $X_t^{(k)} = Y_t^{(k)} + Z_t^{(k)}$ with VG processes $Y_t^{(k)}$ and independent Wiener processes $Z_t^{(k)}$ such that

$$Y_t^{(k)} = \theta_k G_t^{(k)} + \sigma_k W_{G_t^{(k)}}^{(k)}, \quad Z_t^{(k)} = \xi_k t + \psi_k \tilde{W}_t^{(k)}, \quad k = 1,..., n, \tag{5.5}$$

with $\tilde{W}_t^{(k)} \sim N(0,t)$ a standard Wiener process. The dependence structure of the WNVG process is inherited from the multivariate WVG process $Y_t = (Y_t^{(1)},..., Y_t^{(n)})$ defined in Section 3 and the multivariate Wiener process $Z_t = (Z_t^{(1)},..., Z_t^{(n)})$ with mean vector $\xi = (\xi_1,..., \xi_n)$ and covariance matrix $\Psi = \left(\rho_{ij}^N \psi_i \psi_j\right)$. The joint cgf can be expressed in closed-form.

Theorem 5.1 (cgf of the multivariate WNVG process). *The joint cgf of the multivariate WNVG process $X_t = (X_t^{(1)},..., X_t^{(n)})$ with parameters: $\xi_k, \psi_k, \theta_k, \sigma_k, \nu_k, k = 1,..., n, \rho_{ij}^N, \rho_{ij}^{VG}$ $i, j = 1,..., n,$ $\nu_0 \geq \max(\nu_1,..., \nu_n),$ is determined by $C_{X_t}(u) = \xi^T u \cdot t + \frac{1}{2}(u^T \Psi u) \cdot t + C_{A_t}(u) + \sum_{k=1}^{n} C_{Y_t^{(k)}}(u_k)$ with*

$$C_{A_t}(u) = -\nu_0^{-1} t \cdot \ln\{1 - \theta^T u - \tfrac{1}{2} u^T \Sigma u\},$$
$$C_{Y_t^{(k)}}(u_k) = -(\nu_k^{-1} - \nu_0^{-1}) t \cdot \ln\{1 - \nu_k \cdot (\theta_k u_k + \tfrac{1}{2}\sigma_k^2 u_k^2)\}, \tag{5.6}$$

with $\xi = (\xi_1,..., \xi_n), \quad \Psi = \left(\rho_{ij}^N \psi_i \psi_j\right), \quad \theta = \left(\nu_1\theta_1,..., \nu_n\theta_n\right), \quad \Sigma = \left(\sqrt{\nu_i \nu_j}\, \rho_{ij}^{VG} \sigma_i \sigma_j\right).$

Proof. Since X_t is the convolution of Y_t and Z_t, the representation (5.6) is the sum of the joint cgf's of a multivariate Wiener process and the WVG process given in Theorem 3.1. ◊

Important special cases of the WNVG process are the multivariate *normal generalized asymmetric Laplace* (NGAL) process (case $\nu_k = \nu_0,\, k = 1,...,n$) and the multivariate *normal asymmetric Laplace* (NAL) process (case $\nu_k = \nu_0 = 1,\, k = 1,...,n$). The naming of the latter stems from the fact that the VG margins, in this case equal to $Y_t^{(k)} = \theta_k E_t^{(k)} + \sigma_k W_{E_t^{(k)}}^{(k)},\, E_t^{(k)} \sim Exp(t),\, k = 1,...,n$, reduce to asymmetric or skew Laplace processes. Therefore, the incremental margins $X^{(k)} =_d X_{t+s}^{(k)} - X_s^{(k)}$ follow a normal asymmetric Laplace (NAL) distribution, also called *normal Laplace* (NL) distribution by Reed (2006) and Reed and Jorgensen (2004), of the form

$$X^{(k)} = \xi_k + \psi_k \cdot Z + \alpha_k^{-1} \cdot E_1 - \beta_k^{-1} \cdot E_2 \sim NAL(\xi_k, \psi_k, \alpha_k, \beta_k), \tag{5.7}$$

with $Z \sim N(0,1)$, $E_1, E_2 \sim Exp(1)$, (Z, E_1, E_2) independent, and $\alpha_k^{-1} = \frac{1}{2}(\sqrt{\theta_k^2 + 2\sigma_k^2} + \theta_k),\, \beta_k^{-1} = \frac{1}{2}(\sqrt{\theta_k^2 + 2\sigma_k^2} - \theta_k),\, k = 1,...,n$ (cf. (2.5)).
Since the Laplace and normal distributions constitute Laplace's first and second law of errors (e.g. Kotz et al. [36], Chap. 1), it is worthy to consider convolutions of the two error distributions for modelling purposes. A probabilistic genesis of (5.7) is found in Reed and Jorgensen [59]. This distribution arises naturally if a Brownian motion $dX = \mu \cdot dt + \sigma \cdot dW$ with initial state $X_0^{(k)} \sim N(\xi_k, \psi_k^2)$ is observed at an exponentially distributed random time T . If the logarithmic price of a financial asset is assumed to follow a Brownian motion, then its logarithmic price at the time of the first trade on a fixed future date could be expected to follow a distribution close to a normal Laplace (e.g. Reed [57], p. 5). Similarly, a standardized gamma time changed Brownian motion with initial random normal state leads to a normal variance gamma distribution. The empirical fitting capabilities of the normal Laplace have been tested in several case studies. For example, Hürlimann [27] shows that an AR(1) process with NL noise achieves a best goodness-of-fit for the Swiss consumer price index among various competing non-Gaussian noise specifications.
The joint cgf of the multivariate NAL process is determined as follows.

Corollary 5.1 (cgf of the multivariate NAL process). *The joint cgf of the multivariate NAL process* $X_t = (X_t^{(1)},..., X_t^{(n)})$ *with parameters:*
$\xi_k, \psi_k, \theta_k, \sigma_k,\, k = 1,...,n,\, \rho_{ij}^N, \rho_{ij}^{AL}\;\, i, j = 1,...,n,$ *is determined by*

$$C_{X_t}(u) = \xi^T u \cdot t + \tfrac{1}{2}(u^T \Psi u) \cdot t - t \cdot \ln(1 - \theta^T u - \tfrac{1}{2} u^T \Sigma u), \tag{5.8}$$

with $\xi = (\xi_1,...,\xi_n),\quad \Psi = (\rho_{ij}^N \psi_i \psi_j),\quad \theta = (\theta_1,...,\theta_n),\quad \Sigma = (\rho_{ij}^{AL} \sigma_i \sigma_j).$

Proof. This follows from Theorem 5.1 setting $\nu_0 = \nu_k = 1,\, k = 1,...,n$. ◊

The vector of increments $X = (X^{(1)},..., X^{(n)}) \sim NAL(\xi, \Psi, \theta, \Sigma)$ is the convolution of a normal vector $Z=(Z^{(1)},...,Z^{(n)})\sim N(\xi,\Psi)$ and an asymmetric Laplace $Y=(Y^{(1)},...,Y^{(n)})\sim AL(\theta,\Sigma)$. The distribution of the latter vector has been introduced by Kozubowski and Podgorski [40] while its characteristic function appears in Kozubowski [38] and Kozubowski and Panorska [39]. It is studied in the book by Kotz et al. [36] (see also Kotz et al. [37] and Kozubowski et al. [41]). Parameter estimation of the multivariate shifted asymmetric Laplace (SAL) distribution $\xi+Y\sim SAL(\xi,\theta,\Sigma)$ is discussed in Visk [65] and Hürlimann [30].

6 The Multivariate Wang Normal Variance-gamma Deflator

Consider now $n\geq 1$ risky assets, whose real-world prices are described by exponential normal VG processes of the type

$$S_t^{(k)}=S_0^{(k)}\exp((\mu_k-\omega_k)t+X_t^{(k)}),\quad k=1,...,n, \tag{6.1}$$

where μ_k represents the mean logarithmic rate of return of the k-th risky asset per time unit, and the random vector $X_t=(X_t^{(1)},...,X_t^{(n)})$ follows a multivariate WNVG process with cgf (5.6). Clearly, one must have $\omega_k=C_{X^{(k)}}(1)<\infty,\ k=1,...,n$, where one assumes that the cgf of $X^{(k)}=X_1^{(k)}$ exists over some open interval, which contains one. Suppose that the WNVG deflator of dimension n has the same form as the price processes in (6.1). For some parameter α and vector $\beta=(\beta_1,...,\beta_n)$ (both to be determined) it is defined by the Esscher transform

$$D_t=\exp(-\alpha t-\beta^T X_t),\quad t>0. \tag{6.2}$$

The martingale conditions (4.3) lead to the same system of $n+1$ equations (4.4) in the $2n+1$ unknowns α,β_k,ω_k, and (4.5) holds. By Theorem 5.1 the latter equations are equivalent with

$$\begin{aligned}&\mu_k-r-\omega_k+\xi^T\beta^{(k)}+\tfrac{1}{2}\beta^{(k)^T}\Psi\beta^{(k)}+C_A(\beta^{(k)})\\&+\xi^T\beta-\tfrac{1}{2}\beta^T\Psi\beta-C_A(-\beta)+C_{Y^{(k)}}(1-\beta_k)-C_{Y^{(k)}}(-\beta_k)=0,\quad k=1,...,n.\end{aligned} \tag{6.3}$$

Again, the unknown ω_k can be chosen arbitrarily. A convenient appropriate choice, which leads to a simple solution of the system (6.3), consists to set for $k=1,...,n$:

$$\begin{aligned}&\omega_k=\mu_k-r+\xi_k+\tfrac{1}{2}(1-2\beta_k)\Psi_{kk}\\&-\sum_{i=1,i\neq k}^{n}\beta_i\Psi_{ik}+C_{Y^{(k)}}(1-\beta_k)-C_{Y^{(k)}}(-\beta_k)=C_{X^{(k)}}(1),\end{aligned} \tag{6.4}$$

which is interpreted as the (time-independent) WNVG *market price of the* k -th *risky asset*. With the made restriction on the cgf, this value is always finite. Inserted into (6.3) one sees that the parameter vector β is determined by the equations

$$C_A(\beta^{(k)}) = C_A(-\beta), \quad k = 1,...,n. \tag{6.5}$$

To show this, one uses the relationships

$$\xi^T \beta^{(k)} + +\xi^T \beta = \xi_k, \quad \tfrac{1}{2}\beta^{(k)^T}\Psi\beta^{(k)} - \tfrac{1}{2}\beta^T\Psi\beta = \tfrac{1}{2}(1-2\beta_k)\Psi_{kk} - \sum_{i=1, i\neq k}^{n} \beta_i \Psi_{ik}, \quad k = 1,...,n.$$

We are ready to show the following WNVG deflator representations.

Theorem 6.1 (WNVG deflator of dimension n). *Given are* $n \geq 1$ *risky assets with exponential normal VG real-world prices (6.1), where the random vector process* $X_t = (X_t^{(1)},..., X_t^{(n)})$ *follows a multivariate WNVG process. Then, the WNVG deflator (6.2) is determined by*

$$D_t = \exp(-\alpha t - \sum_{k=1}^{n} \beta_k X_t^{(k)}), \quad t > 0, \qquad \text{with} \tag{6.6}$$

$$\alpha = r + C_X(-\beta), \quad \beta_k \sigma_k^2 = \theta_k + (\tfrac{1}{2} + \gamma_k)\sigma_k^2, \quad \gamma_k = \sum_{j\neq k} \sqrt{\tfrac{\nu_j}{\nu_k}} \rho_{ij}^{VG} \tfrac{\sigma_j}{\sigma_k}, \quad k = 1,...,n. \tag{6.7}$$

Moreover, in the univariate case $n = 1$ *one has* $\gamma_1 = 0$.

Proof. The proof of Theorem 4.1 applies. ◊

Corollary 6.1 (NAL deflator of dimension n). *Given are* $n \geq 1$ *risky assets with exponential normal asymmetric Laplace real-world prices (6.1), where the random vector process* $X_t = (X_t^{(1)},..., X_t^{(n)})$ *follows a multivariate NAL process with cgf (5.8). The NAL deflator is determined by*

$$D_t = \exp(-\alpha t - \sum_{k=1}^{n} \beta_k X_t^{(k)}), \quad t > 0, \qquad \textit{with} \tag{6.8}$$

$$\alpha = r + C_X(-\beta), \quad \beta_k \sigma_k^2 = \theta_k + (\tfrac{1}{2} + \gamma_k)\sigma_k^2, \quad \gamma_k = \sum_{j\neq k} \sqrt{\tfrac{\nu_j}{\nu_k}} \rho_{ij}^{AL} \tfrac{\sigma_j}{\sigma_k}, \quad k = 1,...,n. \tag{6.9}$$

Proof. This follows from Theorem 6.1 replacing ρ_{ij}^{VG} by ρ_{ij}^{AL}. ◊

7 Pricing Geometric Basket Options with State-price Deflators

In the literature one distinguishes between two types of basket options. The *arithmetic basket option* is defined on the weighted arithmetic average of asset prices such that

$$S_t = \sum_{k=1}^{n} c_k S_t^{(k)}, \tag{7.1}$$

where the weights (c_k) can be negative, and in this situation it includes *spread options*. The *geometric basket option* is defined on the weighted geometric average of asset prices

$$S_t = \prod_{k=1}^{n} [S_t^{(k)}]^{c_k}, \quad c_k > 0, \quad \sum_{k=1}^{n} c_k = 1. \tag{7.2}$$

Since distribution functions of weighted sums of correlated asset prices can usually not be written in explicit closed form, the pricing of arithmetic basket options is rather challenging. Different and mostly approximate methods to price them have been developed so far by many authors including Turnbull and Wakeman [63], Milevsky and Posner [52], Krekel et al. [42], Carmona and Durrleman [8], Borovka et al. [6], Wu et al. [67], Venkatramanan and Alexander [64], Alexander and Venkatramanan [1], Brigo et al. [7]. The pricing of the geometric basket option is more straightforward.

We illustrate usefulness of the multivariate WVG and WNVG deflators by pricing the geometric basket options. The obtained explicit analytical pricing formulas can be viewed as multivariate generalizations of the Black-Scholes formula.

Consider an European geometric basket call option with maturity date T and exercise price K in the *multivariate WVG market* with $n \geq 1$ risky assets that follow the price process (4.1) and is subject to the WVG deflator (4.9)-(4.10). Its price at initial time 0 is given by

$$C = E[D_T (S_T - K)_+]. \tag{7.3}$$

A straightforward calculation, which takes into account the normalizing choice (4.7), shows that

$$\begin{aligned} D_T S_T &= S_0 \cdot \exp\{-C_X(-\beta)T + \sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n} (c_k - \beta_k) X_T^{(k)}\}, \\ d_k &= C_{Y^{(k)}}(-\beta_k) - C_{Y^{(k)}}(1-\beta_k), \\ D_T K &= K e^{-rT} \cdot \exp\{-C_X(-\beta)T - \sum_{k=1}^{n} \beta_k X_T^{(k)}\}. \end{aligned} \tag{7.4}$$

By the representation (3.4) one has

$$X_T^{(k)} = \theta_k \tfrac{\nu_k}{\nu_0} G_T + \sigma_k \sqrt{\tfrac{\nu_k}{\nu_0}} W_{G_T}^{(k)} + \theta_k (1 - \tfrac{\nu_k}{\nu_0}) H_T^{(k)} + \sigma_k \sqrt{1 - \tfrac{\nu_k}{\nu_0}} W_{H_T^{(k)}}^{(k)},$$

with independent distributed gamma random variables $G_T \sim \Gamma(\gamma_0 T, \gamma_0)$ and $H_T^{(k)} \sim \Gamma((\gamma_k T, \gamma_k)$, $\gamma_0 = \nu_0^{-1}, \gamma_k = \nu_k^{-1} - \nu_0^{-1}$. To evaluate (7.3) we condition (7.4) on the random vector $U_T = (G_T, H_T^{(1)}, ..., H_T^{(n)})$ whose density function is the product of gamma densities of the form

$$f_{U_T}(x_0, x_1, ..., x_n) = f_{G_T}(x_0) \cdot \prod_{k=1}^{n} f_{H_T^{(k)}}(x_k) = \prod_{k=0}^{n} [\gamma_k / \Gamma(\gamma_k)](\gamma_k x_k)^{\gamma_k T - 1} e^{-\gamma_k x_k} .$$

Proceeding this way rewrite (7.3) as multiple integral $C = e^{-C_X(-\beta)T} \cdot \iiint C(w) f_{U_T}(w) dw$ with

$$\begin{aligned} C(w) &= E[(S_0 \cdot \exp\{\sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n} (c_k - \beta_k) X_T^{(k)}\} \\ &- Ke^{-rT} \cdot \exp\{-\sum_{k=1}^{n} \beta_k X_T^{(k)}\})_+ \big| U_T = w]. \end{aligned} \tag{7.5}$$

Each of the two conditional correlated normally distributed sums in (7.5) is normally distributed, and their joint distribution is bivariate normal. Therefore, the distribution of the conditional random couple

$$(\sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n} (c_k - \beta_k) X_T^{(k)}, -\sum_{k=1}^{n} \beta_k X_T^{(k)} \big| U_T = w), \text{ with } \quad w = (w_0, w_1, ..., w_n),$$

is determined by the conditional means

$$\begin{aligned} &E[\sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n} (c_k - \beta_k) X_T^{(k)} \big| U_T = w] = c^T d \cdot T + w^T m^{(1)}, \\ &E[-\sum_{k=1}^{n} \beta_k X_T^{(k)} \big| U_T = w] = w^T m^{(2)}, \\ &c = (c_1, ..., c_n), \quad d = (d_1, ..., d_n), \quad m^{(1)} = (m_0^{(1)}, m_1^{(1)}, ..., m_n^{(1)}), \\ &m^{(2)} = (m_0^{(2)}, m_1^{(2)}, ..., m_n^{(2)}), \quad m_0^{(1)} = \sum_{k=1}^{n} (c_k - \beta_k) \theta_k \tfrac{\nu_k}{\nu_0}, \quad m_0^{(2)} = -\sum_{k=1}^{n} \beta_k \theta_k \tfrac{\nu_k}{\nu_0}, \\ &m_k^{(1)} = (c_k - \beta_k) \theta_k (1 - \tfrac{\nu_k}{\nu_0}), \quad m_k^{(1)} = -\beta_k \theta_k (1 - \tfrac{\nu_k}{\nu_0}), \quad k = 1, ..., n. \end{aligned} \tag{7.6}$$

the conditional variances

$$
\begin{aligned}
&Var[\sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n}(c_k - \beta_k) X_T^{(k)} \big| U_T = w] = w^T s^{(1)}, \\
&s^{(1)} = ([s_0^{(1)}]^2, [s_1^{(1)}]^2, \dots, [s_n^{(1)}]^2), \\
&[s_0^{(1)}]^2 = \sum_{i,j=1}^{n} \rho_{ij} \tfrac{\sqrt{\nu_i \nu_j}}{\nu_0} (c_i - \beta_i)\sigma_i (c_j - \beta_j)\sigma_j, \\
&[s_k^{(1)}]^2 = (1 - \tfrac{\nu_k}{\nu_0})(c_k - \beta_k)^2 \sigma_k^2, \quad k = 1, \dots, n, \\
&Var[-\sum_{k=1}^{n} \beta_k X_T^{(k)} \big| U_T = w] = w^T s^{(2)}, \quad s^{(2)} = ([s_0^{(2)}]^2, [s_1^{(2)}]^2, \dots, [s_n^{(2)}]^2), \\
&[s_0^{(2)}]^2 = \sum_{i,j=1}^{n} \rho_{ij} \tfrac{\sqrt{\nu_i \nu_j}}{\nu_0} \beta_i \sigma_i \beta_j \sigma_j, \quad [s_k^{(1)}]^2 = (1 - \tfrac{\nu_k}{\nu_0}) \beta_k^2 \sigma_k^2, \ k = 1, \dots, n,
\end{aligned}
\tag{7.7}
$$

and the conditional covariance

$$
\begin{aligned}
&Cov[\sum_{k=1}^{n} c_k d_k T + \sum_{k=1}^{n}(c_k - \beta_k) X_T^{(k)}, -\sum_{k=1}^{n} \beta_k X_T^{(k)} \big| U_T = w] = \rho s_0^{(1)} s_0^{(2)} w_0, \\
&\rho s_0^{(1)} s_0^{(2)} = \sum_{i,j=1}^{n} \rho_{ij} \tfrac{\sqrt{\nu_i \nu_j}}{\nu_0} (\beta_i - c_i)\sigma_i \beta_j \sigma_j.
\end{aligned}
\tag{7.8}
$$

Now, let $\Phi(x)$ denotes the standard normal distribution, $\overline{\Phi}(x) = 1 - \Phi(x)$ its survival function, and $\varphi(x) = \Phi'(x)$ its density. The bivariate standard normal density is defined and denoted by

$$
\varphi_2(x, y; \rho) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(x^2 - 2\rho xy + y^2 \right) \right\}.
$$

From (7.5) and the definitions (7.6)-(7.8) one obtains

$$
C(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_0 e^{c^T d \cdot T + w^T m^{(1)} + \sqrt{w^T s^{(1)}} x} - K e^{-rT + w^T m^{(2)} + \sqrt{w^T s^{(2)}} y})_+ \varphi_2(x, y; \rho) dx dy. \tag{7.9}
$$

The expression in the bracket of (7.9) is non-negative provided $x \geq x(y)$ with

$$
x(y) = \frac{\ln(K/S_0) - (r + c^T d)T}{\sqrt{w^T s^{(1)}}} + \frac{w^T (m^{(2)} - m^{(1)})}{\sqrt{w^T s^{(1)}}} + \frac{\sqrt{w^T s^{(2)}}}{\sqrt{w^T s^{(1)}}} y.
$$

Since $\varphi_2(x, y; \rho) = \varphi(y)\varphi((x - \rho y)/\sqrt{1-\rho^2})/\sqrt{1-\rho^2}$ a separation of the double integral yields $C(w) = \int_{-\infty}^{\infty} J(y, w)\varphi(y) dy$ with the inner integral

$$J(y,w) = (1/\sqrt{1-\rho^2}) \cdot \int_{x(y)}^{\infty} \{S_0 e^{c^T d \cdot T + w^T m^{(1)} + \sqrt{w^T s^{(1)}} x} - K e^{-rT + w^T m^{(2)} + \sqrt{w^T s^{(2)}} y}\} \varphi((x-\rho y)/\sqrt{1-\rho^2}) dx . \quad (7.10)$$

A straightforward application of Lemma A1.1 in the Appendix 1 yields

$$J(y,w) = S_0 e^{c^T d \cdot T + w^T m^{(1)} + \rho\sqrt{w^T s^{(1)}} y + \frac{1}{2}(1-\rho^2) w^T s^{(1)}} \Phi\left(\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{(1-\rho^2) w^T s^{(1)}}\right)$$
$$- K e^{-rT + w^T m^{(2)} + \sqrt{w^T s^{(2)}} y} \Phi\left(\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}}\right).$$

To simplify notation rewrite the arguments within the normal distribution functions as

$$\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{(1-\rho^2) w^T s^{(1)}} = a + ey, \quad \tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} = b + ey, \text{ with}$$
$$a = \frac{(r + c^T d)T - \ln(K/S_0) + w^T (m^{(1)} - m^{(2)} + (1-\rho^2) s^{(1)})}{\sqrt{(1-\rho^2) w^T s^{(1)}}},$$
$$b = \frac{(r + c^T d)T - \ln(K/S_0) + w^T (m^{(1)} - m^{(2)})}{\sqrt{(1-\rho^2) w^T s^{(1)}}}, \quad e = \frac{\rho\sqrt{w^T s^{(1)}} - \sqrt{w^T s^{(2)}}}{\sqrt{(1-\rho^2) w^T s^{(1)}}}.$$

Furthermore, one has

$$e^{\rho\sqrt{w^T s^{(1)}} y} \varphi(y) = e^{\frac{1}{2}\rho^2 w^T s^{(1)}} \varphi(y - \rho\sqrt{w^T s^{(1)}}), \quad e^{\sqrt{w^T s^{(2)}} y} \varphi(y) = e^{\frac{1}{2} w^T s^{(2)}} \varphi(y - \sqrt{w^T s^{(2)}}).$$

Now, using twice the Lemma A1.2 of the Appendix 1 one obtains

$$C(w) = S_0 e^{c^T d \cdot T + w^T (m^{(1)} + \frac{1}{2} s^{(1)})} \Phi\left(\tfrac{a + e\rho\sqrt{w^T s^{(1)}}}{\sqrt{1+e^2}}\right) - K e^{-rT + w^T (m^{(2)} + \frac{1}{2} s^{(2)})} \Phi\left(\tfrac{b + e\sqrt{w^T s^{(2)}}}{\sqrt{1+e^2}}\right).$$

Based on the above expressions for the coefficients a, b, e one obtains further

$$\begin{aligned} C(w) &= S_0 e^{c^T d \cdot T + w^T (m^{(1)} + \frac{1}{2} s^{(1)})} \Phi\left(\tfrac{(r + c^T d)T - \ln(K/S_0) + w^T (m^{(1)} - m^{(2)}) + \Omega_1^2(w)}{\Omega(w)}\right) \\ &- K e^{-rT + w^T (m^{(2)} + \frac{1}{2} s^{(2)})} \Phi\left(\tfrac{(r + c^T d)T - \ln(K/S_0) + w^T (m^{(1)} - m^{(2)}) - \Omega_2^2(w)}{\Omega(w)}\right), \\ \Omega_k^2(w) &= w^T s^{(k)} - \rho\sqrt{w^T s^{(1)} \cdot w^T s^{(2)}}, \quad k = 1,2, \\ \Omega(w) &= \sqrt{\Omega_1^2(w) + \Omega_2^2(w)}. \end{aligned} \quad (7.11)$$

Summarizing, we have shown the following main result.

Theorem 7.1 (Geometric basket multivariate WVG market call option formula). *Given is the multivariate exponential WVG process (4.1) subject to the WVG deflator (4.9)-(4.10). Then, in the above notations, one has*

$$C = E[D_T(S_T - K)_+] = e^{-C_X(-\beta)T} \cdot \{S_0 e^{c^T d \cdot T} \cdot \Psi_{U_T}(a_1(w), b_1(w)) \\ - Ke^{-rT} \cdot \Psi_{U_T}(a_2(w), b_2(w))\}, \tag{7.12}$$

$$\Psi_{U_T}(a(w), b(w)) = \iiint e^{a(w)} \Phi(b(w)) f_{U_T}(w) dw, \\ a_k(w) = w^T (m^{(k)} + \tfrac{1}{2} s^{(k)}), \\ b_k(w) = \frac{(r + c^T d)T - \ln(K/S_0) + w^T (m^{(1)} - m^{(2)}) + (-1)^{k-1} \Omega_1^2(w)}{\Omega(w)}, \quad k = 1,2. \tag{7.13}$$

Remarks 7.1. A similar option pricing formula can be derived for *the multivariate WNVG market* with $n \geq 1$ risky assets defined in Section 6 with WNVG deflator (6.6)-(6.7). The cgf in (7.12) must be replaced by the cgf (5.8). Since (4.7) is replaced by (6.4) the quantities d_k in (7.4) must be replaced by $d_k = C_{Y^{(k)}}(-\beta_k) - C_{Y^{(k)}}(1-\beta_k) - \xi_k - \frac{1}{2}(1-2\beta_k)\Psi_{kk} + \sum_{i=1, i \neq k}^{n} \beta_i \Psi_{ik}$. Moreover, $X_T^{(k)}$ is replaced by the equation (5.5), that is $X_T^{(k)} + \xi_k t + \psi_k \widetilde{W}_t^{(k)}$, so that (7.6)-(7.8) must be replaced accordingly (details are left to the reader). In the univariate case $n=1$ the WVG process reduces to the VG process, and the multidimensional pricing formulas (7.12)-(7.13) reduce to the one-dimensional formulas (4.12)-(4.15) in Hürlimann [32]. The numerical evaluation of the multiple integrals of the form (7.13) can be performed using number theoretic methods (e.g. Niederreiter [56], Foglia [19], Fang and Wang [17], etc.). Another possibility is the use of the fast Fourier transform (FFT) to evaluate densities with known characteristic functions (e.g. Hürlimann [31], Appendix 1). The standard FFT of f_{U_T} as a product of $n+1$ gamma densities results in a grid of $N^{n+1} = 2^{(n+1)q}$ points, where $N = 2^q$ is the number of points required for the FFT of each gamma density. This approach has the advantage to be applied for other subordinators than the gamma distribution like an inverse Gaussian or a classical tempered stable distribution. When compared to the simpler multivariate NVM mixture models proposed in Hürlimann [32], which result in a one-dimensional geometric basket option pricing formula, the present multivariate extension is computationally more complex. For example, choosing $q = 10$ in FFT calculation results in more than 1 billion points for evaluation of the pricing formula for a bivariate model.

At this point some important connections with the standard no-arbitrage framework of Mathematical Finance must be mentioned (e.g. Wüthrich et al. [68], Section 2.5, and Wüthrich and Merz [69], Chap. 2). By the Fundamental Theorem of Asset Pricing, the assumption of no-arbitrage (weak form of efficient market hypothesis) is equivalent with the existence of an equivalent martingale measure for deflated price processes. In complete markets, the equivalent martingale measure is unique, perfect replication of contingent claims holds, and straightforward pricing applies. In incomplete markets, an economic model is required to decide upon which equivalent martingale measure is appropriate. Now, let P denotes the real-world measure and P^* an equivalent martingale measure. Then, one can either work under P, where the prices processes are deflated with a state-price deflator. Alternatively, one can work under P^* by discounting the prices processes with the bank account numeraire. Working with financial instruments only, one often works under P^*. But, if additionally insurance liabilities are considered, one works under P (see Wüthrich et al. [68], Remark 2.13). A recent non-trivial example is pricing of the "guaranteed maximum inflation death benefit (GMIDB) option" (equation (5.4) in Hürlimann [26]). Theorem 7.1 demonstrates the practicability of the state-price deflator approach for exponential WVG price processes as applied to the European geometric basket call option. The conditions under which the WVG and WNVG multivariate markets are complete and arbitrage-free, that is there exists a unique equivalent martingale measure and prices are uniquely defined (whether under P^* or under P with state-price deflator), remain to be found. This is a non-trivial problem that has been tackled so far only for the multivariate Black-Scholes model (see Dhaene et al. [13]).

Finally, as a mode of conclusion, let us mention that other multivariate versions of the VG process and generalizations can be found in the recent literature (e.g. Ishwaran and Zarepour [33], Kaishev [35], Guillaume [22]-[23] and Marfè [51]). The construction of state-price deflators and their use in actuarial science and finance for these and other multivariate processes is an interesting topic for future research.

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Appendix

Integral identities of normal type

The crucial identities used in the derivation of Theorem 7.1 are stated and proved.

Lemma A1.1. *For any real numbers* b, c, μ *and* $\sigma > 0$ *one has the identity*

$$\sigma^{-1} \cdot \int_c^\infty e^{bx} \cdot \varphi((x-\mu)/\sigma)dx = e^{b\mu + \frac{1}{2}b^2\sigma^2} \cdot \Phi\left(\frac{\mu - c}{\sigma} + b\sigma\right). \quad \text{(A1.1)}$$

Proof. Consider first the case $\mu = 0, \sigma = 1$. From the relation $e^{bx}\varphi(x) = e^{\frac{1}{2}b^2}\varphi(x-b)$ one gets

$$\int_c^\infty e^{bx} \cdot \varphi(x)dx = e^{\frac{1}{2}b^2} \cdot \int_{c-b}^\infty \varphi(t)dt = e^{\frac{1}{2}b^2} \cdot \Phi(b-c).$$

Using this one obtains by a change of variables

$$\sigma^{-1} \cdot \int_c^\infty e^{bx} \cdot \varphi((x-\mu)/\sigma)dx = \int_{(c-\mu)/\sigma}^\infty e^{b\mu + b\sigma t} \cdot \varphi(t)dt = e^{b\mu + \frac{1}{2}b^2\sigma^2} \cdot \Phi\left(\frac{\mu - c}{\sigma} + b\sigma\right). \quad \Diamond$$

Lemma A1.2. *For any real numbers* a, b, μ *and* $\sigma > 0$ *one has the identity*

$$\sigma^{-1} \cdot \int_{-\infty}^\infty \Phi(a+bx)\varphi((x-\mu)/\sigma)dx = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right). \quad \text{(A1.2)}$$

Proof. Consider the functions $F(z) = \int_{-\infty}^\infty \Phi(z+x)\varphi(x)dx, \quad G_a(z) = \int_{-\infty}^\infty \Phi(a+zx)\varphi(x)dx$. One notes that $F(0) = \int_{-\infty}^\infty \Phi(x)\varphi(x)dx = \frac{1}{2}$ and $F'(z) = \int_{-\infty}^\infty \varphi(z+x)\varphi(x)dx = \frac{\sqrt{2}}{2}\varphi(\frac{z}{\sqrt{2}})$, from which it follows that $F(a) = F(0) + \int_0^a F'(z)dz = \Phi(\frac{a}{\sqrt{2}})$. On the other hand, one has $G_a(1) = F(a) = \Phi(\frac{a}{\sqrt{2}})$ and $G_a^{'}(z) = z \cdot \int_{-\infty}^\infty \varphi(a+zx)\varphi(x)dx = \frac{z}{(1+z^2)^{3/2}}\varphi(\frac{a}{\sqrt{1+z^2}})$, hence $G_a(b) = G_a(1) + \int_1^b G'(z)dz = \Phi(\frac{a}{\sqrt{1+b^2}})$. It follows that

$$\sigma^{-1} \cdot \int_{-\infty}^\infty \Phi(a+bx)\varphi((x-\mu)/\sigma)dx = G_{a+b\mu}(b\sigma) = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right). \quad \Diamond$$