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Optimal Option Pricing via Esscher Transforms with the Meixner Process

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Abstract

The Meixner process is a special type of Levy process. It originates from the theory of orthogonal polynomials and is related to the Meixner-Pollaczek polynomials by a martingale relation. In this paper, we apply instead the Meixner density function for option hedging. We make use of the decomposed Meixner and applied the Esscher transform to obtain the optimal option hedging strategy. We further obtain the option price by solving the parabolic partial differential equation which arises from the Meixner-OU process.

Mathematics Subject Classification : 91B28, 91G10

Keywords: Meixner process, Option pricing, Esscher transform, Hedging strategy.

1. Introduction

It has been widely appreciated for some time that fluctuations in financial data show consistent excess kurtosis indicating the presence of large fluctuations not

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predicted by Gaussian models. The need for models that can describe these large events has never greater, with the continual growth in the derivatives industry and the recent emphasis on better risk management.

Option valuation is one of the most important topics in financial mathematics. Black and Scholes [1] derived a compact pricing formula for a standard European call option by formulating explicitly the model on the risk-neutral measure, under a set of assumptions.

The accurate modeling of financial price series is important for the pricing and hedging of financial derivatives such as options. Research on option theory with alternative pricing models has tended to focus on the pricing issue. It is well known that non- Gaussian pricing models lead to the familiar volatility smile effect caused by that ‘fat’ tails of the non-Gaussian PDF’s.

To price and hedge derivatives securities it is crucial to have a good modeling of the probability distribution of the underlying product. The most famous continuous time model used is the calibrated Black-Scholes model. It uses the normal distribution to fit the Log-returns of the underlying; the price process of the underlying is given by the geometric Brownian motion.

$$X_t = X_0 \exp(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t)$$

Where $\{B_t, t \geq 0\}$ is a standard Brownian motion ie. B_t follows a normal distribution with mean 0 and variance t. Its key property is that it is complete (ie a perfect hedge is an idealized market in theory possible).

It is however known that the Log-returns of most financial assets have lean actual kurtosis that is higher than that of the normal distribution. As a result of the kurtosis being higher than that of the normal distribution we look into another distribution that will fit in the data in most perfect way. Empirical evidence has shown that the normal distribution is a very poor model to fit real life data. In order to achieve a better fit we replace the Brownian motion by a special Levy process called the Meixner process. Several authors proposed similar process models. For example Eberlein and Keller [2] proposed the Hyperbolic Models and their generalizations. Barndorff – Nielsen [3] proposed the Normal Inverse Gaussian (NIG) Levy process. Luscher [4] used the NIG to price synthetic Collateralized Debt Obligations (CDO). Osu et al [5] applied the same model as a tool to investigate the effect in future, the economy of a developing nation with poor financial policy.

Our aim in this paper is to apply instead the Meixner density function for option hedging. We make use of the decomposed Meixner$(x;\, \alpha, \beta, \delta, m)$ with the application of the Esscher transform to obtain the optimal option hedging strategy. Furthermore, we obtain the option price by solving the parabolic partial differential equation which arises from the Meixner-OU process.

2. The Meixner Process

The Meixner distribution belongs to the class of the infinitely divisible distributions and as such give rise to a Levy process. The Meixner process is very flexible, has simple structure and leads to analytically and numerically tractable formulas. The Meixner process originates from the theory of orthogonal polynomials and was proposed to serve a model of financial data.

The density of the Meixner distribution Meixner (a, b, d, m) is given by [6]

$$f(x; a, b, d, m) = \frac{(2\cos(\frac{b}{2}))^{2d}}{2a\pi\Gamma(2d)} \exp(\frac{b(x-m)}{a})|\Gamma(d + \frac{i(x-m)}{a})|^2,$$

where $a > 0, -\pi < b < \pi, d > 0 \; and \; m \in \mathbb{R}$

2.1. Moments

Moments of all order of this distribution exist and is given (and compared to the Normal distribution) below as

	Meixner (a, b, d, m)	**Normal**(μ, σ^2)	**Meixner** $(\boldsymbol{a}, \boldsymbol{0}, \boldsymbol{d}, \boldsymbol{m})$
Mean	$m + a\tan(b/2)$	μ	m
Variance	$\frac{a^2 d}{2}(\cos^{-2}(b/2))$	σ^2	$\frac{a^2 d}{2}$
Skewness	$\sin\left(b/2\right)\sqrt{2/d}$	0	0
Kurtosis	$3 + \frac{3-2\cos^2(b/2)}{d}$	3	$3 + \frac{1}{d}$

We can clearly see that the kurtosis of the Meixner distribution is always greater than the normal Kurtosis and stationary increments and where the distribution of (Meixner process) is given by the Meixner distribution Meixner(a, b, dt, mt). Figure 1 below compares the performance of the Normal and Meixner distributions with fictitious financial data.

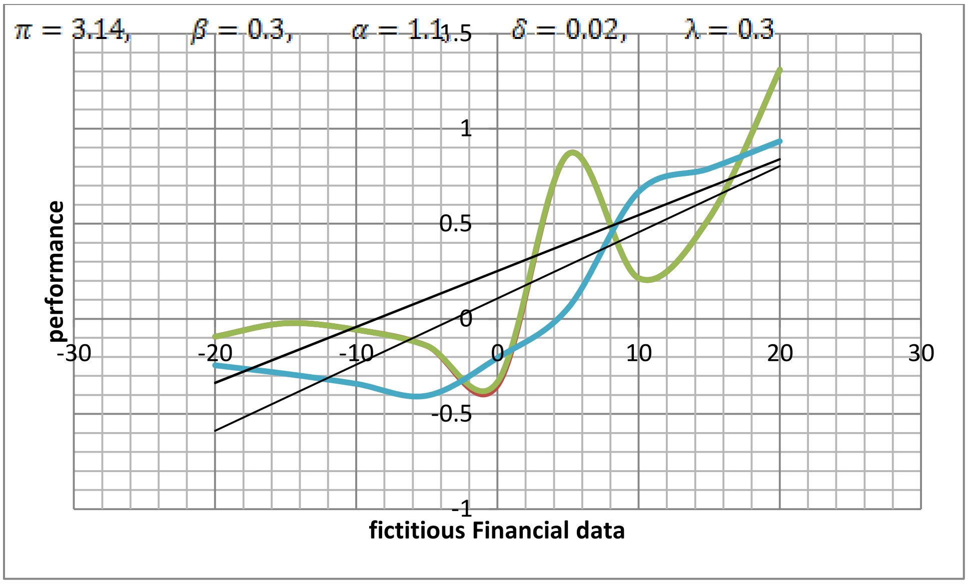

Figure 1: The Microsoft excel plot of the fictitious market data using the Normal distribution (blue) and Meixner distribution (green) with the trend (black).

2.1.1. Levy Triple

The Meixner process $\{M_t\}_t$ has a triplet of Levy character$(\tilde{a}, o, v)$, where

$$\tilde{a} = ad\tan\left({}^{b}/_{2}\right) - 2d\int_{1}^{\infty} \frac{\cosh\left(\frac{bx}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)} dx + M$$

$$= \psi'(0) - \int_{|x|>1} X \ V(dx).$$

In general a Levy process consists of three independent parts a lower deterministic part, a Brownian part, and a pure jump part. It can be shown that the Meixner process $\{M_t, t \geq 0\}$ has no Brownian part and a pure jump part governed by the Levy measure

$$V(dx) = d\frac{exp(\frac{bx}{a})}{x\sinh(\frac{\pi x}{a})}dx.$$

The characteristics function of the Meixner(a, b, d, m) distribution is given by

$$E[exp(iuM_t)] = \left(\frac{cos\left(\frac{b}{2}\right)}{cosh\left(\frac{au-ib}{2}\right)}\right)^{2d} exp(imu)$$

2.1.2. Semi Heaviness of Tails

The Meixner (a, b, d, m) distribution has semi-Levy tails [7], which means that the tails of the density function behave as

$$f(x: a, b, d, m) \sim C_- |x|^{\rho-} \exp(-\sigma_- |x|) \qquad \text{as} \quad x \to +\infty$$

$$f(x: a, b, d, m) \sim C_+ |X|^{\rho+} \exp(-\sigma_+ |x|)) \qquad \text{as} \quad x \to +\infty,$$

for some $\rho_-, \rho_+ \in \mathbb{R}\ and\ C_-, C_+, \sigma_-, \sigma_+ \geq 0$.For some $C_-, C_+ \geq 0$,

$\rho_- = \rho_+ = (2d - 1)$, $\sigma_- = (\Pi - b)/a$, $\sigma_+ = (\Pi + b)/a$.

The Levy measure is not finite

$$\lim_{x \to \infty} v(dx) = \infty$$

The process has an infinite number of jumps.

3. Esscher Transform Method

The Esscher transform [8] was developed to approximate a distribution around a point of interest, such that the new mean is equal to this point. In actuarial science, it is a well - known tool in the risk theory literature. In the content of [9], the Esscher transform becomes an efficient technique for financial options, and other derivatives, valuation. That is, if the log of the underlying asset prices follows a stochastic process with stationary and independent increments and given the assumption of risk neutrality, the risk-neutral probabilities associated with a model can be calculated.

For a probability distribution function (pdf), $f(x)$, let h be a real number such that

$$M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)dx$$

exists. As a function in x,

$$f(x,h) = \frac{e^{hx} f(x)}{m(h)}$$

is a probability density function and it is called the Esscher Transform of the original distribution.

3.1. Risk-Neutral Esscher Transform

Let X_t be a random variable with an infinitely divisible distribution. Thus, its cumulative distribution function and moment generating function are given by

$$F(x,t) = p_r[X_t \leq X]$$

and

$$m(z,t) = E[e^{zX_t}].$$

By assuming that $m(z,t)$ is continuous at t=0, it can be proved that

$$m(z,t) = [m(z,1)]^t. \tag{3.1}$$

The density function of this random variable is given by

$$f(x,t) = \frac{dF(x,t)}{dx}, \qquad t > 0.$$

Then,

$$m(z,t) = \int_{-\infty}^{\infty} e^{zx} f(x,t)dx.$$

Let h be a real number such that $m(h,t)$ exists. Gerber and Shiu[9], then introduced the Esscher transform with parameter h, of the stochastic process $\{X_t\}$. This process has stationary and independent increments. Thus, the new pdf of X_t , $t > 0$ is

$$f(x,t;h) = \frac{e^{hx} f(x,t)}{\int_{-\infty}^{\infty} e^{hx} f(x,t) dx} = \frac{e^{hx} f(x,t)}{m(h,t)} \tag{3.2}$$

The new moment generating function is;

$$m(z,t;h) = \int_{-\infty}^{\infty} e^{zx} f(x,t;h) dx = \frac{m(z+h.t)}{m(h,t)} . \tag{3.3}$$

By equation (1),

$$m(z,t;h) = [m(z,1,h)]^t \tag{3.4}$$

To have a risk neutral transform, we see $h = h^*$, such that the asset pricing X_t discounted at the risk-free, r is a Martingale with respect to the probability measure corresponding to h^*. That is

$$X_0 = E^Q[e^{-rt} X_t] = e^{-rt} E^Q[X_t] \tag{3.5}$$

and

$$X_t = X_0 \, e^{S_t}. \tag{3.6}$$

Where X_t is the continuously compounded rate of return over t periods. Using (3.6) into (3.5), the parameter h^* is the solution of the equation

$$e^{rt} = m[1,t;h^*]^t. \tag{3.7}$$

Thus, we have a value for h^* depending 0n the probability distribution of $X_{t.}$ by equation (3.4), the solution does not depend on t so we may set $t = 1$.

$$e^r = m[1,1,h^*] = \frac{m(1+h^*,1)}{m(h^*,1)} \tag{3.8}$$

$$r = ln[m(1,1,h^*)]. \tag{3.9}$$

The Esscher transform of parameter h^* is called the risk-neutral Esscher transform, and the corresponding equivalent Martingale measure is called the risk-neutral Esscher measure. Although the risk-neutral Esscher measure is unique, there may be other equivalent Martingale measure.

3.1.1. European Call Option Valuation Using Esscher Transform.

Developing the methodology, [9] assumed the same assumption made by [1]; the risk-free interest rate is constant; the market is frictionless and trading is continuous; there are no taxes; no transaction cost; and no restriction on borrowing or short sales; all assets are perfectly divisible; there are no arbitrage opportunities; and the assets do not distribute dividends. Harrison and Kreps [10] showed that the condition of no arbitrage is intimately related to the existence of an equivalent Martingale measure. The risk-neutral probability measure will be given by the risk-neutral Esscher transform. Thus, for a European call option, we have

$$C = e^{-rt} \int_{-\infty}^{\infty} \max(S_t - K, 0)\, f(x, t; h) dx. \tag{3.10}$$

Assuming that the stock prices are log-normally distributed, let the stochastic process $\{X_t\}$ be a Weiner process with mean per unit time μ and variance per unit time δ^2.

Then,

$$F(x, t) = N(x, \mu t, \delta^2 t),$$

and

$$m(z, t) = e^{\left[\mu z + \frac{1}{2}\delta^2 z^2\right]}. \tag{3.11}$$

Thus from (3.3)

$$m(z,t;h) = e^{\{\left[(\mu+h\delta^2)z+\frac{1}{2}\delta^2z^2\right]t}. \tag{3.12}$$

Hence the Esscher transform of parameter h of the Weiner process is again a Weiner process, with modified mean per unit time and unchanged variance per unit time. Thus,

$$f(x,t;h) = N(x;(\mu+h\delta^2)t\,,\delta^2 t\,)$$

Using the modified distribution in equation (3.2) and equation (3.6) into (3.10), we have;

$$C = e^{-rt}\int_{x^*}^{\infty}(X_0e^x - k)\frac{e^{hx}}{m(h,t)}\,f(x,t)dx\,. \tag{3.13}$$

Note that the lower bound of the integral is given by $x^* = \ln\left(\frac{K}{X_0}\right)$.

That is to price call options, we only need the rate of returns that produce values equal or greater than the exercise price. By equation (3.11)

$$m(\,h,t) = E[e^{hx}] = e^{ht\left(\mu+\frac{1}{2}h\delta^2\right)}\,. \tag{3.14}$$

Rewriting the call option, using equation (13) and (12), we have;

$$C = \frac{e^{-rt}}{e^{h(\mu+\frac{1}{2}h\delta^2)}}\left[S_0\int_{x^*}^{\infty}e^{(1+h)x}f(x,t)dx - k\int_{x^*}^{\infty}e^{hx}f(x,t)dx\right]. \tag{3.15}$$

To solve the expected value of a truncated normal random variable, we apply the method in [11]. Thus;

$$C = e^{-r}S_0N\left(\frac{\mu-\ln\left(\frac{K}{S_0}\right)}{\delta}+(1+h)\delta\right)e^{\mu+h\delta^2+\frac{1}{2}\delta^2} - e^{-r}k\left(\frac{\mu-ln\left(\frac{K}{S_0}\right)}{\delta}\right)+(1+h)\delta. \tag{3.16}$$

We can find h^* for a random variable normally distributed X_t^2, thus;

$$h^* = \frac{r-\mu}{\delta^2} - \frac{1}{2}.$$

Replacing this in equation (3.16), we obtain,

$$C = S_0 N \frac{-\ln\left(\frac{K}{S_0}\right) + \left(r + \frac{1}{2}\delta^2\right)}{\delta} - e^{-r} K N \left(\frac{-ln\left(\frac{K}{S_0}\right) + (r - \frac{1}{2}\delta^2}{\delta}\right). \qquad (3.17)$$

Thus, from the risk-neutral Esscher transform, we obtain the traditional Black-Scholes formula for pricing a European call option. Note that the expected rate of return μ, which represents the preference of investors does not appear in the final formula.

3.1.2. Equivalent Martingale Measure

According to the fundamental theorem of asset pricing the arbitrage free price V_t of the derivative at time $t \in [0, T]$ is given by

$$V_{\boldsymbol{t}} = E_Q\left[e^{-r(T-t)}\, G(\{S_u, 0 \le u \le T\})\middle|\mathcal{F}_t\right],$$

Where Q is an equivalent Martingale measure ,$\{f_t\}_t$ is the natural filtration of ,$\{X_t\}_t$. An equivalent Martingale measure is a probability measure which is equivalent (it has the same null-sets) to the given (historical) probability measure and under which the discounted process $\{e^{-rt}X_t\}$ is a Martingale. Unfortunately for most models, the more realistic ones, the class equivalent measure is rather large and often covers the full no-arbitrage interval.

In this perspective the Black-Scholes model, where there is a unique equivalent Martingale measure ism exceptional. Models with more than one equivalent measure are called incomplete.

Meixner model is such an incomplete model so called Esscher transform easily find at least one equivalent Martingale measure, which we will use in the sequel for the valuation of derivatives securities. The choice of the Esscher measure may be justified by a utility maximizing argument.

The model which provides exactly Meixner(a, b, d, m) daily log-returns for the stock is that which replaces the Brownian motion process in the BS-model by a Meixner process given by

$$X_{\boldsymbol{t}} = X_0 \exp(|M_t|)$$

To choose an equivalent Martingale measure we use Esscher transform.

$$\frac{dQ_\theta}{dP} |f_t = \frac{\exp(\theta X_t)}{E_p(\exp(\theta X_t))}$$

Then choose $\theta\ s.t\ e^{-rt} X_t$ is a Martingale under Q.We know that $\frac{M(1+\theta,1)}{M(\theta,1)} = e^r$ (Martingale condition) when $M(\theta, 1) - f(\exp(\theta, X_1))$

$$\frac{\int exp(\theta+1)x exp\left(\frac{b(x-m)}{a}\right)|\Gamma(d+i\frac{(x-m)}{a})|^2 dx}{\int exp(\theta x) exp\left(\frac{b(x-m)}{a}\right)|\Gamma(d+i\frac{(x-m)}{a})|^2 dx} = exp\{r\}.$$

With $y = \frac{x-m}{a}$ we get that

$$exp\{m-r\} = \frac{\int exp(\theta a+b)y|\Gamma(d+iy)|^2 dy}{\int exp(\theta a+a+b)y|\Gamma(d+iy)|^2 dy} \left(\frac{\cos\left(\frac{\theta a+a+b}{2}\right)}{\cos\left(\frac{\theta a+b}{2}\right)}\right)^{2d} .$$

This gives

$$exp\left\{\frac{m-r}{2d}\right\} = \frac{\cos\left(\frac{\theta a+b}{2}\right)\cos\left(\frac{a}{2}\right) - \sin\left(\frac{\theta a+b}{2}\right)\sin\left(\frac{a}{2}\right)}{\cos\left(\frac{\theta a+b}{2}\right)}$$

$$= \cos\left(\frac{a}{2}\right) - \tan\left(\frac{\theta a+b}{2}\right)\sin\left(\frac{a}{2}\right),$$

so that

$$\theta = -\frac{1}{a}\left\{b + 2\arctan\left(\frac{exp\left\{\frac{m-r}{2d}\right\} - \cos\left(\frac{a}{2}\right)}{\sin\left(\frac{a}{2}\right)}\right)\right\}.$$

The equivalent Martingale measure is given by

$$Q = \exp(\theta x)\, v dx.$$

And the density equals

$$f(x;\theta) = \frac{exp\{\theta x\} f(x)}{\int exp\{\theta y\} f(y) dy}$$

$$= \frac{exp\left\{\theta x + b\left(\frac{x-m}{a}\right)\right\} |\Gamma(d + i\frac{x-m}{a})|^2}{a exp(\theta m) 2\pi \Gamma(2d)\left(\frac{1}{2\cos(\frac{a\theta+b}{2})}\right)^{2d}} \left(\frac{\left(2\cos\left(\frac{a\theta+b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)} exp\left\{\theta x + b\left(\frac{x-m}{a}\right)\right\} |\Gamma(d + i\frac{(x-m)}{a})|^2\right).$$

The equivalent Martingale measure Q follows again a Meixner $(a, a\theta, +b, d, m)$ distribution.

4. Optimal Option Hedging with the Meixner Process

The advantage of the Meixner model over the other Levy model is that all crucial formulas are exactly given so that it is not depending on computationally demanding numerical inversion proceeds. This numerical advantage can be important when a big number of prices or related quantities have to be completed simultaneously.

The process

$$dx_t = -\lambda x_t dt + dz_t, x_0 > 0 \quad , \tag{4.1}$$

where the process z_t is a subordinator; more precisely, it is a Levy process with no Brownian part, non-negative drift and only positive increments. The processes

$\{z_t, t \geq 0\}$ is called Ornstein-Uhlenbeck(OU) processes [12]. The rate parameter λ is arbitrary positive and $z = \{z_t, t \geq 0\}$ is the Background Driving Levy Process (BDLP). The process z is an increasing process and $x_0 > 0$, it becomes clear that the process x is strictly positive and bounded from below by the deterministic function $x_0 e^{-\lambda t}$.

The Meixner$(x;\ \alpha, \beta, \delta, m)$ is self-decomposable [13]. Therefore we have

$$w(x) = \delta\lambda(\pi - \beta)exp\left(\left(\frac{\beta+\pi}{\alpha}\right)x\right) + (\pi + \beta)exp\left(\left(\frac{\beta-\pi}{\alpha}\right)x\right)\left(sinh\left(\frac{\pi}{\alpha}x\right)\right)^{-2}, \quad (4.2)$$

with cumulant function of the self- decomposable law given as;

$$K(u) = \alpha\delta\lambda u\ tan\left(\left(\frac{\alpha u-\beta}{2}\right)\right) - \lambda m. \quad (4.3)$$

The Meixner-OU process is not driven by a BDLP that is a subordinator. The BDLP has a Levy density that lives over the whole real line. This means that the Meixner-OU process (and its BDLP) can jump upwards and downwards.

Consider the price $C(X_0,\ E, t)$ of a European call option at current time $t = 0$, with exercise price E due to expire in a time t. When the time to expiry is small the returns and interest rate can be neglected [14]. The option price is then very well approximated by

$$C(X_0,\ E, t) \cong \langle max(X - E, 0)\rangle = \int_E^\infty (X - E)P(X\,, t|X_0, 0)dX \quad , \quad (4.4)$$

where $P(X\,, t|X_0, 0)$ is the PDF of the underlying asset price X_t. Bouchaud and Sornette [15] and [16] in their approach to option and hedging found that the wealth variation between times 0 and t can be written as;

$$\Delta w|_0^t = C(X_0,\ E,t) - max(X-E,0) + \int_0^t \emptyset(X_\tau)\dot{X}(\tau)d\tau\ , \tag{4.5}$$

where the first term is the option premium received at $t = 0$, the second term describes the payoff at expiry t and the third term describes the effect of the trading where $\emptyset(X_\tau)(X_\tau = X(\tau))$ is the amount of stock held.

Giving now the decomposed Meixner$(x;\ \alpha,\beta,\delta,m)$ PDF of (4.2) we define its expected value (or in this case the optimal hedging strategy) as;

$$E\big(w(x)\big) = \varphi^*(x_0,E,t)\ = \int_E^\infty w(x)dx$$

$$= \int_E^\infty \delta\lambda(\pi-\beta)exp\left(\left(\frac{\beta+\pi}{\alpha}\right)x\right) + (\pi+\beta)exp\left(\left(\frac{\beta-\pi}{\alpha}\right)x\right)\left(sinh\left(\frac{\pi}{\alpha}x\right)\right)^{-2} dx$$

$$= \int_E^\infty \delta\lambda(\pi-\beta)exp\left(\left(\frac{\beta+\pi}{\alpha}\right)x\right) + (\pi+\beta)exp\left(\left(\frac{\beta-\pi}{\alpha}\right)x\right)\left(csch\left(\frac{\pi}{\alpha}x\right)\right)^{2} dx$$

$$\varphi^*(x_0,E,t) = x_0\left[\frac{\alpha\delta\lambda(\pi-\beta)}{(\beta+\pi)}\ exp\left\{\left(\frac{\pi+\beta}{\alpha}\right)x\right\} + \frac{\alpha}{\pi}exp\left\{-\left(\frac{\pi-\beta}{\alpha}\right)x\right\}\coth\left(\frac{\pi}{\alpha}x\right)\right]_E^\infty. \tag{4.6}$$

Figure 2 below shows expected value (or the optimal hedging strategy) given (4.6).

The pricing for a European call option with respect to the equivalent Martingale measure equals

$$C(x_0,E,t) = x_0P(X_T >; \theta+1) - e^{-rt}kp(X_T > K\colon\theta)$$

$$= x_0\int_E^\infty f(x\colon\theta+1)dx - e^{-rt}K\int_E^\infty f(x\colon\theta)dx$$

$$C(x_0,E,t) = x_0\left[\frac{\alpha\delta\lambda(\pi-\beta)}{(\beta+\pi)}\ exp\left\{\left(\frac{\pi+\beta}{\alpha}\right)x\right\}\right.$$

$$\left. + \frac{\alpha}{\pi}exp\left\{-\left(\frac{\pi-\beta}{\alpha}\right)x\right\}\coth\left(\frac{\pi}{\alpha}x\right)\right]_E^\infty$$

$$-K\left[\frac{\alpha\delta\lambda(\pi-\beta)}{(\beta+\pi)}\,exp\left\{\left(\frac{\pi+\beta}{\alpha}\right)x\right\}+\frac{\alpha}{\pi}exp\left\{-\left(\frac{\pi-\beta}{\alpha}\right)x\right\}coth\left(\frac{\pi}{\alpha}x\right)\right]_E^{\infty} \tag{4.7}$$

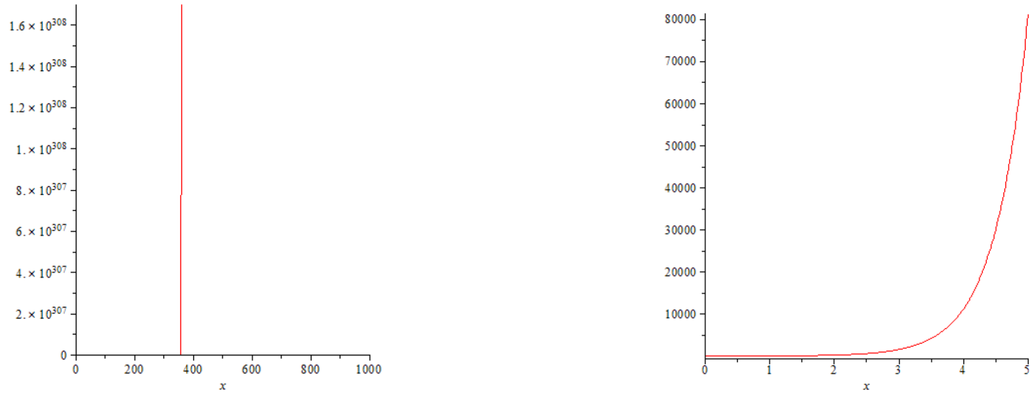

Figure 2: The expected values for high and small values of x under the following assumptions: $\alpha = 2.7$,$\beta = 2.2$,$\sigma = 1.8$,$\lambda = 2.0$,$h = 1.6$ $x_0 = 2.4$ $k = 2.1$(given the fictitious market data and (4.6)) to the extent that Maple can display in the interval of the values of x.

Harrison and kreps[10] established a mathematical foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using probability theory. Gerber and Shiu [8] used the Esscher transform to obtain an equivalent martingale measure which is the risk-neutral probability distribution. Rewriting the call option, using equation (3.13), we now solve for the expected value giving now the decomposed Meixner$(x;\ \alpha,\beta,\delta,m)$ PDF of (4.2).We apply the method in [11] to get;

$$C(x_0,E,t) = x_0\left[\frac{\alpha\delta\lambda(\pi-\beta)}{(\beta+\pi+1+h)}\,exp\left\{\left(\frac{\pi+\beta+1+h}{\alpha}\right)x\right\}+\frac{\alpha}{\pi}exp\left\{-\left(\frac{\pi-\beta-1-h}{\alpha}\right)x\right\}\coth\left(\frac{\pi}{\alpha}x\right)\right]_E^{\infty}$$

$$-K\left[\frac{\alpha\delta\lambda(\pi-\beta)}{(\beta+\pi+h)}\,exp\left\{\left(\frac{\pi+\beta+h}{\alpha}\right)x\right\}+\frac{\alpha}{\pi}exp\left\{-\left(\frac{\pi-\beta-h}{\alpha}\right)x\right\}coth\left(\frac{\pi}{\alpha}x\right)\right]_E^{\infty}\ . \tag{4.8}$$

Equation (4.8) is the approximate wealth variation or the option price whose behaviour for the large and little values of x is as in figure 3.

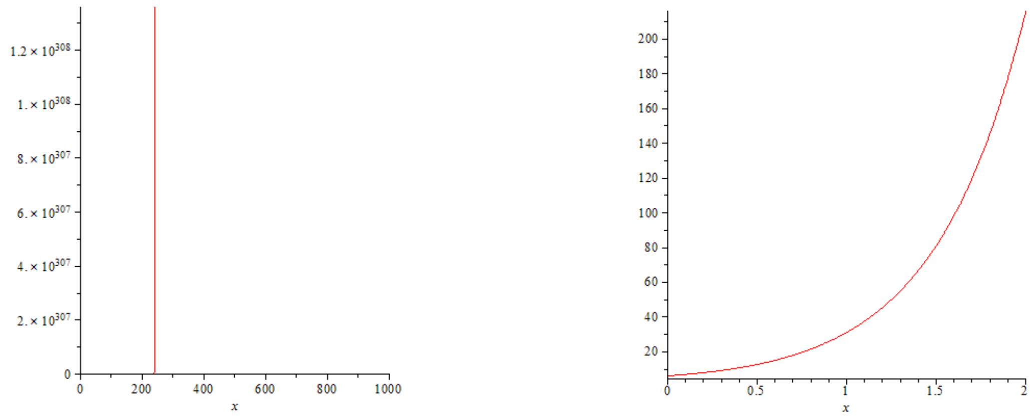

Figure 3: The option price for the large and little values of x to the extent that Maple can display in the interval of the values of x under the following assumptions: $\alpha = 2.7$ $\beta = 2.2$ $\sigma = 1.8,$ $\lambda = 2.0,$ $h = 1.6,$ $x_0 = 2.4,$ $k = 2.1.$

Assume now $x_t$follows instead the Orntsein-Uhlenbeck process as in (4.1)

$$dx_t = -\lambda x_t dt + dz_t, x_0 > 0,$$

with explicit solution

$$x_t = e^{-\lambda t} x_t + e^{-\lambda t} \int_o^t e^{-aS} \, dz_S. \tag{4.9}$$

Applying the Duhammel principle, equation (4.9) has a Gaussian distribution with mean $e^{-at}x_0$ and variance given by

$$\sigma^2 \, (t) = e^{-2at} \int_0^t e^{2aS} \, dS$$

$$= \frac{e^{-2at}}{2a} [e^{2aX} + 1]_0^t$$

$$= \frac{1}{2a}[1 + e^{-2at}]. \tag{4.10}$$

Hence (4.10) has a markov process with stationary transition probability densities

$$F(t,x,y) = \frac{1}{\sigma(t)\sqrt{2\pi}} exp \frac{[-(y-e^{-at}x]}{2\sigma^2(t)} \ . \tag{4.11}$$

This is particularly interesting for $a > 0$, which is the stable case and

$$\alpha = \lim_{t\to\infty} \sigma^2(t) = \frac{1}{2a}, \tag{4.12}$$

and

$$\lim_{t\to\infty} F(t,x,y) = \frac{1}{\sqrt{2\alpha\pi}} \exp\left(\frac{-y^2}{2\alpha}\right). \tag{4.13}$$

Thus as $t \to \infty, x_t \xrightarrow{d} N(0, \frac{1}{2a})$.

The price evolution of risky assets are usually modelled as the trajectory of a diffusion process defined on some underlying probability space $(\Omega, \mathfrak{F}, P)$, with the geometric Brownian motion process the best candidate used as the canonical reference model. It had been shown in [7] that the geometric Brownian motion can indeed be justified as the rational expectation equilibrium in a market with homogenous agents. But the evolution of the stock price process is well known to be described by the dynamics

$$dx_t = \alpha(t)x_t dt + \sigma(t)x_t dW(t), \tag{4.14}$$

with unique solution known to be (α and σ are the drift and volatility respectively, assumed continuous functions of time)

$$x_t = x_o exp\left\{\int_0^t \sigma(u)dW(u) + \int_0^t (\alpha(u) - \frac{1}{2}\sigma^2(u))du\right\}. \tag{4.15a}$$

Given equation (4.12), it is not difficult to see that (4.15a) becomes

$$x_t = x_o exp\left\{\int_0^t \sigma(u)dW(u)\right\}. \qquad (4.15b)$$

By (4.12), we mean that the drift parameter α and future price of an option depend on volatility σ.

Ito's formula on (4.14) gives;

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}V(x,t) + rx\frac{\partial}{\partial x}V(x,t) - rV(x,t) = \frac{\partial}{\partial t}V(x,t)$$

$$\forall(x,t) < (0,\infty) \times (0,T, \qquad (4.16)$$

which is the famous Black-Scholes parabolic partial differential equation.$V = V(x,t)$is the value of option(s) or the portfolio value given different option values with different prices. We shall now solve the PDE (4.16) for stock which are already priced in the market for the option price. If the volatility follows the generic process $V(x,t)$ (where Vmay be stochastic), the option price will be given by

$$C = \int_0^\infty [xN(d_1(V)) - Ke^{-r\tau}N(d_2(V))]\, V_m dV \quad , \qquad (4.17)$$

where V_m is the probability distribution function for the mean of the volatility (which is a delta function for a deterministic process) and $d_1(V)$ and $d_2(V)$ are the same variables. Let (for the deterministic case)

$$V = V_0 exp\{\mu t\},\ 0 \le t \le T. \qquad (4.18)$$

In this case, the probability distribution function of the mean of the volatility is given by

$$V_m = \delta\left(V - \frac{exp\{\mu T - 1\}}{\mu T}V_0\right) \quad , \qquad (4.19)$$

given the Black-Scholes result where $\sqrt{\frac{exp\{\mu T-1\}}{\mu T}V_0}$ replaces σ.

Consider now a stochastic volatility process where Q represents white noise so that;

$$dV = \zeta Q dt, V(0) = V_0, 0 \leq t \leq T \quad . \tag{4.20}$$

The distribution of the mean of V during the time interval is given by

$$V_m \approx N\left(V_0, \frac{T\zeta^2}{3}\right). \tag{4.21}$$

Therefore, the option price is given by

$$C = (3(2\pi\zeta^2 T)^{-1})^{0.5} \int_0^\infty [xN(d_1(V)) - Ke^{-r\tau}N(d_2(V))] \, exp\left(\frac{3(V-V_0)^2}{2\zeta^2 T}\right) dV. \tag{4.22}$$

5. Conclusion

In option theory a major disincentive for using non-Gaussian based models is the absence of a riskless hedge [14]. This makes it to apply the Black-Scholes option framework in anything other than an ad hoc way. In this paper we have further demonstrated the fact that the self-decomposed Meixner density function can be used to hedge a financial derivative. In solving (4.16) for the price of option, we have made use of Merton's theorem that the solution for a deterministic volatility process is the Black-Scholes price with the volatility variable replaced by the average volatility.

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