Communications in Mathematical Finance, vol.2, no.1, 2013, 29-64
ISSN: 2241-1968 (print), 2241-195X (online)
Scienpress Ltd, 2013

Hedging with a portfolio of Interest Rate Swaps

H. Jaffal[1], **Y. Rakotondratsimba**[2] **and A. Yassine**[1,3]

Abstract

Despite the importance role played by Interest Rate Swaps, as in debt structuring, regulatory requirements and risk management, sounding analyzes related to the hedging of portfolios made by swaps are not clear in the financial literature.
To partially fill this lack, we provide here the study corresponding to a parallel shift of the interest rate. The suitable swap sensitivities to make use in hedging and risk management are obtained here as a byproduct of our analyses. They may be seen as the analogue of the well known bond duration and convexity in the swap framework.
Our present results might provide a support for practitioners, using portfolio of swaps and/or bonds, in their hedge decision-making.

JEL Classification : G11, G12.
Keywords: hedging, optimization, bond, zero-coupon, swap

[1] Laboratoire de Mathématiques Appliquées du Havre (LMAH), Université du Havre, 25 rue Philippe Lebon, B.P. 540, 76058 Le Havre cedex, France, e-mail: jaffalhanan@yahoo.com and adnan.yassine@univ-lehavre.fr
[2] ECE Paris Graduate School of Engineering, 37 quai de Grenelle CS71520 75 725 Paris 15, France, e-mail: w_yrakoto@yahoo.com
[3] Institut Supérieur d'Etudes Logistiques (ISEL), Université du Havre, Quai Frissard, B.P. 1137, 76063 Le Havre cedex, France, e-mail: adnan.yassine@univ-lehavre.fr

Article Info: *Received* : December 14, 2012. *Revised* : January 19, 2013
Published online : March 15, 2013

1 Introduction

Interest Rate Swaps (IRS) appear to be instruments largely used by market participants (companies, local governments, financial institutions, traders, ...) for many purposes including debt structuring, regulatory requirements and risk management.

According to the BIS June 2011 statistics, the Interest Rate Swap (IRS) represents 78.25% of OTC derivatives while the corresponding equity part is just about 0.97%. After the 2007-2009 crisis, many of the Over-The-Counter (OTC) products are now traded under collateralization, and the interest rate practice has moved to multiple curve valuation such that there is a differentiation between discounting and forwarding (see for instance [1]). Nevertheless the single-curve approach, as developed in many textbooks and used in pre-crisis practice, remains to be conceptually important and a benchmark under the conservative hypothesis of tight spread between the LIBOR and OIS. The work under consideration here still belongs to the classical single-curve framework, though our present approach may open a way for the new interest rate environment as we will perform in a future project.

Despite this market importance played by IRS, it appears that sounding analyzes related to the hedging of portfolios made by swaps is not clear in the financial literature. To partially fill this lack, we provide here the study corresponding to a Parallel Shift (referred to as (PS)) of the interest rate. Though such an underlying assumption is little bit less realistic, both practical and theoretical reasons lead to grant a consideration to this particular situation. Some of the arguments are presented in our (lengthy) working paper [2], where we have already analyzed the portfolio hedging using swaps and bonds. Parts of our findings are summarized and reported here. In our numerical illustrations we consider the hedge of a swap portfolio by another swap portfolio, a case we have not considered before. The suitable swap sensitivities to make use in hedging and risk management are obtained here as a byproduct of our analyses. They may be seen as generalizing the well known bond duration and convexity in the swap framework. These obtained sensitivities are in line with the bond situation, for which the need to take into account both the passage of time and horizon hedging are analyzed in [3] and [4].

Our aim in writing this paper is to provide a theoretical support which sheds light practitioners in their decision-making related to the hedge of a position sensitive to interest rate and by using a portfolio made by swaps (and/or bonds). For the time being, there are various broker advertisements and leaflets about switching to alternative instruments (as VIX futures, inverse ETF, Swap future . . .) for the hedging purpose instead of just using a classical bond portfolio. However the arguments used in these leaflets are essentially based on (particular) numerical situations which are certainly informative but unfortunately do not reflect all other cases arising in reality. Systematic analysis of the portfolio hedging mechanism, as performed here, allows to build a tool to better appreciate and judge the statements conveyed through various commercial claims.

Our present project is essentially focused on the hedge of a position sensitive to the interest rate by a portfolio of swaps. The use of a bond portfolio as a hedging instrument has been investigated in [2]. It may be noted that the hedge with a bond future was previously studied in [6] and empirically investigated in [7].

Here we perform systematic analyses of the hedging mechanism in the sense that they are essentially based on the portfolio instrument characteristics. And, in contrast with various academical papers and commercial leaflets related to hedging, we do not lean on particular historical data. Our results provide an approach and formulas which may be directly implemented in order to get the suitable hedge ratio and corresponding hedging error estimates for any given portfolio of swaps. However the interest rate curve, at the hedge horizon, is assumed here to have made a parallel shift belonging to some closed finite interval. Though this appears to be a restrictive assumption, any realistic interest rate curve movement is always inside some band which may be determined based on the market view. It means that we have derived here some sort of robust hedging approach in the sense that it avoids to use involved dynamical stochastic model for the interest rate.

The main technical results related to the hedge of a position (sensitive to the interest rate) by a generic portfolio are introduced in Section 2.

So we start in Subsection 2.1 to present the swap features and analyze the change values associated with a portfolio of swaps. Next in Subsection 2.2, we explain how the sensitivities associated with the portfolio to hedge and covering instruments should be combined and interacted in order to realize the hedging operation purpose. Here we formulate the expression of Profit&loss associated with the covered portfolio to consider.

As is seen in Subsection 2.3, this gives rise to some integer and non-linear optimization problem, for which a standard method of resolution seems not available. Therefore in Proposition 2.1 of this Subsection 2.3, by exploring a linearization technique previously introduced in [8], we state that the minimization problem, linked to the hedging issue, is reduced to a Mixed Integer Linear Problem (MILP). It is well-admitted now that a MILP may be solved by making use of standard solvers.

In Subsection 2.4, we present the Swap portfolio sensitivities involved in the hedging operation.

Our numerical application will be presented in Section 3.

First, in Subsection 3.1, we start with the introduction of yield curve used in the subsequent applications.

Two cases of hedging operation are illustrated. In Subsection **??**, we consider the hedge of a portfolio of swaps by another portfolio of the same nature. In this part, estimates for the portfolio future change value and the remainder term related to the approximation are numerically considered before the hedging illustration itself. Next the same topics are also examined in Subsection 3.3 for the case of hedging a bond portfolio by a portfolio of swaps.

2 Results

Our analysis of a swap position is based on the swap features (for a single swap and a swap portfolio) as presented in Subsection 2.1.

The main idea of the hedging problem is based on interaction between the portfolio to hedge and the hedging instruments, such that the resulting Profit& Loss for the covered portfolio is displayed in Subsection 2.2.

Then the optimization problem linked to such an immunization operation is analyzed in Subsection 2.3.

Finally, we present in Subsection 2.4 the sensitivities and decomposition both for a swap in single position and for a swap portfolio.

2.1 Swap features

A plain vanilla Interest Rate Swap (IRS) is an Over-The-Counter (OTC) contract between two counterparties A and B. The first of them, let us say A agrees, during a given period of time, to pay to B, regularly a cash flow equal to the interest rate corresponding to a predetermined fixed rate on the contractual notional principal. In return, A receives interest at floating rate on the same notional principal for the same period of time.

To quantitatively explicit this exchange, let us consider

$$0 \leq t_0 \leq t < t + \delta \leq t_1 < \ldots < t_i < \ldots < t_M \tag{1}$$

which represent times.

Here t is the current time. The remaining cash-flow times payment of the considered IRS, with maturity t_M, should take place at times $t_1, \ldots, t_i, \ldots, t_M$.

The time t-value of the zero-coupon bond $P(t, t_i)$, having maturity t_i is defined as

$$P(t, t_i) = \exp\Big[-y(t, t_i)\tau(t, t_i)\Big] \tag{2}$$

where it is set that $P(t, 0) \equiv 1$. The nonnegative quantity $y(t, t_i)$ is refereed to as a yield and corresponds to the continuous interest rate which applies at time t during the time-period

$$\tau(t, t_i) = t_i - t.$$

The yield curve at time t is defined by the mapping

$$\tau \in (0, \infty) \longmapsto y(t; \tau) \equiv y(t, t + \tau). \tag{3}$$

In reality $y(t; \tau)$ is not completely given for the whole points in the semi-real axis $(0, \infty)$. We have only at our disposal some (discrete) yield-to-maturities $y(t; \widetilde{\tau}_1), \ldots, y(t; \widetilde{\tau}_j), \ldots, y(t; \widetilde{\tau}_m)$. Facing to this lack of data, interpolations and models are very often used. For instance the Nelson-Siegel-Svensson model [9], (used below in the numerical illustrations in Section 3) is given by

$$y(t; \tau) = \beta_{t;1} + \beta_{t;2} b_2(\tau\lambda) + \beta_{t;3} b_3(\tau\lambda) \tag{4}$$

with

$$b_2(u) = \frac{1 - \exp(-u)}{u} \qquad \text{and} \qquad b_3(u) = b_2(u) - \exp(-u).$$

Here $\beta_{t;1}$, $\beta_{t;2}$ and $\beta_{t;3}$ depend on time t but not on the time-to-maturity τ. The present time-t shape of the yield curve is driven by expression (4), which is very often seen by market participants as a suitable interpolation mean of available rates.

The swap marked-to-market value at time t is defined by

$$\textbf{value\_Swap}_t \equiv \textbf{notional} \times \Bigg(P(t,t_1)\Big\{ y(t_0,t_1) - \textbf{rate\_Swap} \Big\} \tau(t,t_1) + \sum_{i=2}^{M} P(t,t_i)\Big\{ F(t;t_{i-1},t_i) - \textbf{rate\_Swap} \Big\} \tau(t_{i-1},t_i) \Bigg) \quad (5)$$

By **rate_Swap** we mean the contractual predetermined rate, such that at the contract time inception t^*, with $t^* \leq t_0$, the swap has a zero market value. That is

$$\textbf{value\_Swap}_{t^*} = 0.$$

When time passes, the swap market value $\textbf{value\_Swap}_t$ at any time t before the maturity is given by (5), and there is no reason that it is equal to zero. It may take positive or negative value.
With expression (5), at time t_1 the payments are just related to the reduced time-period (t, t_1), such that the reference floating rate is the yield-to-maturity $y(t_0, t_1)$. The value of this last, known from the fixing-time t_0 (and consequently at time t), is defined by

$$y(t_0,t_1) = \frac{1}{\tau(t_0,t_1)} \Big(\frac{1}{P(t_0,t_1)} - 1 \Big). \quad (6)$$

For the other times t_i, with $2 \leq i \leq M$, the exchange payments are related to the full period (t_{i-1}, t_i) and with respect to the yield $y(t_{i-1}, t_i)$. Unfortunately this last quantity is not known at time t, so for the market valuation it is common to replace it by the forward rate

$$F(t;t_{i-1},t_i) \equiv \frac{1}{\tau(t_{i-1},t_i)} \left(\frac{P(t,t_{i-1})}{P(t,t_i)} - 1 \right). \quad (7)$$

The point here is that this last quantity is deterministically known at time t under the availability of the interest rate curve $t_i \longmapsto P(t, t_i)$. Therefore using

(7) then one has

$$\begin{aligned}
&\textbf{value_Swap}_t = \\
&\quad \textbf{notional} \times \Bigg(P(t,t_1)\Big\{ y(t_0,t_1) - \textbf{rate_Swap} \Big\} \tau(t,t_1) \\
&\qquad + \Big\{ P(t,t_1) - P(t,t_M) \Big\} - \textbf{rate_Swap} \times \sum_{i=2}^{M} P(t,t_i)\tau(t_{i-1},t_i) \Bigg)
\end{aligned} \tag{8}$$

The swap market value, as in (8) is one-thing but, for the position management and hedging, the change of the market value matters. Therefore for the (future) time-period $(t, t+\delta)$ let us set

$$\textbf{change\_value\_Swap}_{t,t+\delta}(\cdot) \equiv \textbf{value\_Swap}_{t+\delta}(\cdot) - \textbf{value\_Swap}_t. \tag{9}$$

It should be emphasized that we have assumed that $t+\delta < t_1$ such that no payment takes place during $(t, t+\delta)$. When such an assumption is not satisfied then at least an effective cash-flow is paid or received, such that the analysis becomes little bit complicated. The assumption used here relies on the fact that in practice the horizon under consideration is preferably short enough in order the associated projected view to be more and less credible. However the real hedging horizon may be long, and consequently it is usual among the practitioners to roll their hedging positions. It means that it is important to have at a disposal an accurate analysis for the single-period hedging. This is exactly our main focus in this paper.

The explicit value of the change_value_Swap may be written as

$$\begin{aligned}
&\textbf{change_value_Swap}_{t,t+\delta}(\cdot) \\
&\quad = \textbf{notional} \times \Bigg\{ - \Big\{ y(t_0,t_1) - \textbf{rate_Swap} \Big\} P(t,t_1)\delta \\
&\quad + \Big(1 + \Big\{ y(t_0,t_1) - \textbf{rate_Swap} \Big\} \tau(t+\delta,t_1) \Big) \Big(P(t+\delta,t_1)(\cdot) - P(t,t_1) \Big) \\
&\quad - \Big(P(t+\delta,t_M)(\cdot) - P(t,t_M) \Big) \\
&\quad - \textbf{rate_Swap} \times \sum_{i=2}^{M} \Big(P(t+\delta,t_i)(\cdot) - P(t,t_i) \Big) \tau(t_{i-1},t_i) \Bigg\}
\end{aligned} \tag{10}$$

With this last expression the swap market value change during the time-period $(t, t+\delta)$ arises as a linear combination of changes of zero-coupon bonds with various maturities t_i's. It means that any zero-coupon change as

$$P(t+\delta, t_i)(\cdot) - P(t, t_i)$$

has to be analyzed. For such a purpose, a model for the future evolution of the interest rate is needed. In this paper we will focus on the common hypothesis of parallel shift (PS) of the yield curve at the future time $t+\delta$, which can be translated as

$$y(t+\delta;\tau)(\cdot) = y(t;\tau) + \varepsilon(\cdot) \tag{11}$$

where $\varepsilon(\cdot) \equiv \varepsilon(\cdot; t, \delta)$. In this last, we mean that the shift ε depends on the present time t and horizon δ. The strong fact here (and likely less realistic) is that the shift does not depend on the maturity τ. Nevertheless, the assumption (11) has been introduced and used both in literature and practice. It should be important to note that (11) makes only a sense whenever the shift $\varepsilon(\cdot)$ is not too negative as

$$-y(t;\tau) < \varepsilon(\cdot). \tag{12}$$

Let us denote by $\mathcal{S}_t$ the present time-t value of a portfolio made by fixed leg payer/receiver swaps. So it is assumed that there are I^{**} types of payer swaps $S^{**}_{\cdot;i^{**}}$ and I^* types of receiver swaps $S^*_{\cdot;i^*}$. Of course I^{**} and I^* stand for positive integer numbers.

The swaps

$$S^{**}_{\cdot;i^{**}} \quad \text{and} \quad S^*_{\cdot;i^*}, \quad \text{for} \quad i^{**} \in \{1, \dots, I^{**}\} \quad \text{and} \quad i^* \in \{1, \dots, I^*\}$$

are respectively assumed to have notional amounts

$$\mathbf{notional}(i^{**}) \quad \text{and} \quad \mathbf{notional}(i^*),$$

fair-rate prices

$$\mathbf{rate\_Swap}(i^{**}) \quad \text{and} \quad \mathbf{rate\_Swap}(i^*)$$

maturities

$$t^{**}_{M^{**}(i^{**})}(i^{**}) \quad \text{and} \quad t^*_{M^*(i^*)}(i^*)$$

and having the ordered payment times

$$\mathcal{T}^{**}_{\cdot;i^{**}} = \Big(t^{**}_1(i^{**}), \dots, t^{**}_{j^{**}}(i^{**}), \dots, t^{**}_{M^{**}(i^{**})}(i^{**})\Big), \quad j^{**} \in \big\{1, 2, \cdots, M^{**}(i^{**})\big\}$$

$$\mathcal{T}^*_{\cdot;i^*} = \Big(t^*_1(i^*), \dots, t^*_{j^*}(i^*), \dots, t^*_{M^*(i^*)}(i^*)\Big), \quad j^* \in \big\{1, 2, \cdots, M^*(i^*)\big\}.$$

The time- t value of such a portfolio may be written as

$$\mathcal{S}_t = \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} S^{**}_{t;i^{**}} - \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} S^{*}_{t;i^{*}}. \tag{13}$$

Therefore there are $n^{**}_{i^{**}}$ swaps of type i^{**} each worth $S^{**}_{t;i^{**}}$, and $n^{*}_{i^{*}}$ swaps of type i^{*} each worth $S^{*}_{t;i^{*}}$. It may be noted that each of the $S^{**}_{t;i^{**}}$ and $S^{*}_{t;i^{*}}$'s is given by an expression as (8).

We consider the perspective of an investor holding such a swap portfolio and willing to reduce the risk exposure (i.e. negative impact) at the future time-horizon $t+\delta$, for some nonnegative real number. It should be emphasized that the point here is to try to maintain the level of the portfolio but not to make a profit, though the market evolution may be favorable for that.

The future time $t+\delta$ corresponds to the horizon for which she has a more and less clear view about a possible movement of the market. But another reason is that it may be consistent with the instruments available for the hedging.

To ease the study, in this paper we will just focus on the case where δ is sufficiently close to t such that δ satisfies the restrictions

$$t < t+\delta < \min\Big\{t^{**}_1(1), \ldots, t^{**}_1(i^{**}), \ldots, t^{**}_1(I^{**}), t^{*}_1(1), \ldots, t^{*}_1(i^{*}), \ldots, t^{*}_1(I^{*})\Big\} \tag{14}$$

and

$$\max\Big\{t^{**}_0(1), \ldots, t^{**}_0(i^{**}), \ldots, t^{**}_0(I^{**}), t^{*}_0(1), \ldots, t^{*}_0(i^{*}), \ldots, t^{*}_0(I^{*})\Big\} \le t. \tag{15}$$

The restrictions (14) and (15) mean that no payment takes place during the hedging period $(t, t+\delta)$. The future portfolio value $\mathcal{S}_{t+\delta}(\cdot)$ is unknown at time t, and depend on the yield curve shape at time $t+\delta$.

The swap portfolio change during $(t, t+\delta)$ is given by

$$\begin{aligned}\textbf{change\_value\_port\_Swap}_{t,t+\delta}(\cdot) &\equiv \mathcal{S}_{t+\delta}(\cdot) - \mathcal{S}_t \\ &= \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} \Big\{S^{**}_{t+\delta;i^{**}}(\cdot) - S^{**}_{t;i^{**}}\Big\} - \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} \Big\{S^{*}_{t+\delta;i^{*}}(\cdot) - S^{*}_{t;i^{*}}\Big\}. \end{aligned} \tag{16}$$

The swap changes $S^{**}_{t+\delta;i^{**}}(\cdot) - S^{**}_{t;i^{**}}$ and $S^{*}_{t+\delta;i^{*}}(\cdot) - S^{*}_{t;i^{*}}$ are actually given by expressions as in (10).

2.2 The hedging mechanism and sensitivities

Our main purpose in this Subsection is to show how useful are the high order sensitivities with respect to the parallel shift of the interest rate model when hedging a portfolio made by swaps and/or bonds.

First we formulate the change expressions linked to both the portfolio to hedge and the hedging instruments. After presenting the interaction between the two portfolios, we display the exact Profit&Loss for a covered portfolio and the cost related to the hedging operation.

Next, the sensitivities associated with the portfolio to hedge and the covering instruments and their offsetting effects are introduced. Therefore a combination between these sensitivities enables us to realize the hedging operation purpose.

Finally, we formulate and analyze the optimization problem to consider when searching the numbers of hedging instruments.

Let us denote by V_t the present time t-value of a portfolio assumed to be sensitive to the interest rate which is made by swaps or/and bonds.
At the future time horizon $t+\delta$, with δ is some nonnegative real number, this portfolio may suffer from a loss, in the sense that $V_{t+\delta}(\cdot) < V_t$. To try to maintain the (future) value $V_{t+\delta}(\cdot)$ to be close to V_t, the portfolio manager has to put in place a hedging technique.

The idea underlying the hedging relies on using another portfolio, referred to as a hedging portfolio (or instrument) in the sequel, such that this last would lead to a nonnegative profit compensating the loss on the initial portfolio. Therefore instead of the absolute change

$$V_{t+\delta}(\cdot) - V_t \equiv \mathbf{P\&L\_naked\_portfolio}_{t,t+\delta}(\cdot)$$

associated with the initial naked portfolio, at the horizon $t+\delta$, the change for the covered portfolio is given by

$$\begin{aligned}&\mathbf{P\&L\_covered\_portfolio}_{t,t+\delta}(\cdot)\\&\qquad\equiv \{V_{t+\delta}(\cdot) - V_t\} + \mathbf{P\&L\_hedging\_instrument}_{t,t+\delta}(\cdot).\end{aligned} \tag{17}$$

The hedging portfolio H is assumed at time-t to have the value

$$H_t = \sum_{i^{**}=1}^{I^{**}} H^{**}_{t;i^{**}} n^{**}_{i^{**}} - \sum_{i^{*}=1}^{I^{*}} H^{*}_{t;i^{*}} n^{*}_{i^{*}}. \tag{18}$$

It means that H is made by I^{**} types of instruments $H^{**}_{;i^{**}}$ in long positions and I^{*} types of instruments $H^{*}_{;i^{*}}$ in short positions. For a given type i^{**} (resp. i^{*}), we make use of $n^{**}_{i^{**}}$ (resp. $n^{*}_{i^{*}}$) number of instruments $H^{**}_{;i^{**}}$ (resp. $H^{*}_{;i^{*}}$). The Profit&Loss corresponding to the use of the hedging instrument is (roughly) given by

$$\mathbf{P\&L\_hedging\_instrument}_{t,t+\delta}(\cdot) = \Big\{ H_{t+\delta}(\cdot) - H_t \Big\} - \mathbf{cost\_H}_t \tag{19}$$

such that

$$\begin{aligned} &\mathbf{P\&L\_covered\_portfolio}_{t,t+s}(\cdot) = V_{t+\delta}(\cdot) - V_t \\ &+ \sum_{i^{**}=1}^{I^{**}} \Big\{ H^{**}_{t+\delta;i^{**}}(\cdot) - H^{**}_{t;i^{**}} \Big\} n^{**}_{i^{**}} - \sum_{i^{*}=1}^{I^{*}} \Big\{ H^{*}_{t+\delta;i^{*}}(\cdot) - H^{*}_{t;i^{*}} \Big\} n^{*}_{i^{*}} - \mathbf{cost\_H}_t \end{aligned} \tag{20}$$

where

$$\begin{aligned} \mathbf{cost\_H}_t &= \Big\{ \frac{1}{P(t,t+\delta)} - 1 \Big\} \\ &\times \Big\{ \sum_{i^{**}=1}^{I^{**}} \Big\{ \nu_0^{**} + \nu^{**} |h^{**}_{t;i^{**}}| \Big\} \mathcal{N}^{**}_{i^{**}} n^{**}_{i^{**}} + \sum_{i^{*}=1}^{I^{*}} \Big\{ \nu_0^{*} + \nu^{*} |h^{*}_{t;i^{*}}| \Big\} \mathcal{N}^{*}_{i^{*}} n^{*}_{i^{*}} \Big\} \end{aligned} \tag{21}$$

with ν_0^{**}, ν^{**}, ν_0^{*} and ν^{*} are fixed constants such that $0 \leq \nu_0^{**}, \nu_0^{*} < 1$ and $0 < \nu^{**}, \nu^{*} < 1$. The numerical values of these constants depend on the market practice under consideration. In (21), we have used the fact that the instrument value $H^{**}_{t;i^{**}}$ is the product of its notional $\mathcal{N}^{**}_{i^{**}}$ with its one unit value $h^{**}_{t;i^{**}}$. For an instrument satisfying $h^{**}_{u;i^{**}} \neq 0$ during its life-time, as in the case of a (risk credit free) bond for example, the corresponding cost at time t is very often defined as $\nu^{**} H^{**}_{t;i^{**}}$; so that here one can take $\nu_0^{**} = 0$. The introduction of ν_0^{**} and ν_0^{*} relies on the fact that for some instruments as a swap, one can have that the corresponding market value satisfies $H^{**}_{t;i^{**}} = 0$. In this case, practitioners [10] take as a base for fees the corresponding notional $\mathcal{N}^{**}_{i^{**}}$ such that the cost is rather $\nu_0^{**} \mathcal{N}^{**}_{i^{**}}$ since the term $\nu_0^{**} H^{**}_{t;i^{**}}$ vanishes.

The hedging problem for the initial portfolio V is reduced to suitably choose the financial instruments with values

$$H^{**}_{;1}, \ldots, H^{**}_{;i^{**}}, \ldots, H^{**}_{;I^{**}} \quad \text{and} \quad H^{*}_{;1}, \ldots, H^{*}_{;i^{*}}, \ldots, H^{*}_{;I^{*}}$$

and the corresponding security numbers

$$n_1^{**}, \dots, n_{;i^{**}}^{**}, \dots, n_{;I^{**}}^{**} \quad \text{and} \quad n_1^{*}, \dots, n_{;i^{*}}^{*}, \dots, n_{;I^{*}}^{*}$$

such that the value of

$$\Big| \, \mathbf{P\&L\_covered\_portfolio}_{t,t+\delta}(\cdot) \Big|$$

should be small as possible. The difficulty here is linked to the fact that the future values of the hedging instruments at time $t+\delta$ are unknown at the present time t where the hedge strategy is built.

The choice of the hedging instruments is dictated by the willing that the resultant effect of their change variations would roughly offset (i.e. going in the opposite direction) the change of the portfolio V to hedge. Then, the problem is reduced to a minimization problem of finding suitable allocation for the security numbers $n_1^{**}, \dots, n_{;i^{**}}^{**}, \dots, n_{;I^{**}}^{**}$ and $n_1^{*}, \dots, n_{;i^{*}}^{*}, \dots, n_{;I^{*}}^{*}$.

Under PS or (11) the point is to assume that for any nonnegative integer p one has the approximation

$$\mathcal{V}_{t+\delta}(\cdot) - \mathcal{V}_t \approx \mathbf{Sens}(0; t, \delta, \mathcal{V}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \mathbf{Sens}(k; t, \delta, \mathcal{V}) \varepsilon^k(\cdot) \tag{22}$$

where $\mathcal{V}$ is one of V, $H_{;i^{**}}^{**}$ and $H_{;i^{*}}^{*}$. In (22) the notations

$$\mathbf{Sens}(0; t, \delta, \mathcal{V}) \qquad \text{and} \qquad \mathbf{Sens}(k; t, \delta, \mathcal{V})$$

are used respectively to refer to as the zero and k-th sensitivities order of the considered financial instrument $\mathcal{V}$, computed at time t and for the horizon δ. Without further indication, by sensitivity we always mean the sensitivity of the instrument under consideration with respect to the PS (11) of the yield curve. A main point on the efficiency of (22) in the hedging operation relies on the suitable choice of the integer p such that the approximation-error

$$\mathcal{R}(\cdot) = \left| \Big(\mathcal{V}_{t+\delta}(\cdot) - \mathcal{V}_t \Big) - \left(\mathbf{Sens}(0; t, \delta, \mathcal{V}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \mathbf{Sens}(k; t, \delta, \mathcal{V}) \varepsilon^k(\cdot) \right) \right| \tag{23}$$

is small from the perspective of the hedger, as $\mathcal{R}(\cdot) \leq 10^{-12}$ for example. Such a strong requirement may be useful since very often in practice one has to deal with positions having large notional size as $n\mathcal{V}_t$ with $n = 10^7$.

Making use of the exact version of (22) for $\mathcal{V} = V$, $\mathcal{V} = H^{**}$ and $\mathcal{V} = H^{*}$, and taking (20) and (21) into account, then it arises that

$$\begin{aligned}
&\mathbf{P\&L_covered_portfolio}_{t,t+s}(\cdot) = \\
&\left(\Theta_0^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{0;i^{*}}^{*} n_{i^{*}}^{*}\right) \\
&\quad + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \left[\Theta_k^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{k;i^{*}}^{*} n_{i^{*}}^{*}\right] \varepsilon^k(\cdot) \\
&\quad + \frac{1}{(p+1)!} \left[\Theta_{p+1}^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{p+1}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{p+1}^{**} n_{i^{*}}^{*}\right] \varepsilon^{p+1}(\cdot) \qquad (24)
\end{aligned}$$

where

$$\Theta_0^V \equiv \mathbf{Sens}(0; t, \delta, V) \qquad (25)$$

$$\Theta_{0;i^{**}}^{**} \equiv \mathbf{Sens}(0; t, \delta, H_{;i^{**}}^{**}) - \left\{\nu_0^{**} + \nu^{**} |h_{t;i^{**}}^{**}|\right\} \mathcal{N}_{i^{**}}^{**} \qquad (26)$$

$$\Theta_{0;i^{*}}^{*} \equiv \mathbf{Sens}(0; t, \delta, H_{;i^{*}}^{*}) + \left\{\nu_0^{*} + \nu^{*} |h_{t;i^{*}}^{*}|\right\} \mathcal{N}_{i^{*}}^{*} \qquad (27)$$

$$\Theta_k^V \equiv \mathbf{Sens}(k; t, \delta, V), \qquad (28)$$

$$\Theta_{k;i^{**}}^{**} \equiv \mathbf{Sens}(k; t, \delta, H_{;i^{**}}^{**}), \quad \Theta_{k;i^{*}}^{*} \equiv \mathbf{Sens}(k; t, \delta, H_{;i^{*}}^{*}) \qquad (29)$$

$$\Theta_{p+1}^V \equiv \mathbf{Sens}(p+1; t, \delta, V; \rho), \qquad (30)$$

$$\Theta_{p+1;i^{**}}^{**} \equiv \mathbf{Sens}(p+1; t, \delta, H_{;i^{**}}^{**}; \rho), \quad \Theta_{p+1;i^{*}}^{*} \equiv \mathbf{Sens}(p+1; t, \delta, H_{;i^{*}}^{*}; \rho) \qquad (31)$$

for all $k \in \{1, \ldots, p\}$, $i^{**} \in \{1, \ldots, I^{**}\}$ and $i^{*} \in \{1, \ldots, I^{*}\}$.

These sensitivities will be fully detailed below in the case of a portfolio of swaps, and the case for bonds may be seen in [2]. Nevertheless it should be noted here that ρ is a real number not clearly defined but depends on ε.

In some places of this paper, we refer to as a view on the interest rate shift $\varepsilon(\cdot)$, the hypothesis that there are nonnegative real numbers $\varepsilon^{\bullet}$ and $\varepsilon^{\bullet\bullet}$ for which

$$-\varepsilon^{\bullet} \leq \varepsilon(\cdot) \leq \varepsilon^{\bullet\bullet}. \qquad (32)$$

Though $\varepsilon(\cdot)$ is a random quantity, not known at the present time t, with historical data on zero-coupon prices, it is not hard for the practitioner to get the deterministic values of $\varepsilon^{\bullet}$ and $\varepsilon^{\bullet\bullet}$ corresponding to the available past prices. But she can also incorporate her view for the situation at the considered future horizon δ. Starting from (24), and using the view (32) then an upper bound of $\left|\mathbf{P\&L\_covered\_portfolio}_{t,t+s}(\cdot)\right|$ is readily given by

$$\begin{aligned} F\Big(& n_1^{**},\ldots,n_{i^{**}}^{**},\ldots,n_{I^{**}}^{**},n_1^{*},\ldots,n_{i^*}^{*},\ldots,n_{I^*}^{*};\varepsilon^{\bullet},\varepsilon^{\bullet\bullet}) \\ &\equiv \left|\Theta_0^V+\sum_{i^{**}=1}^{I^{**}}\Theta_{0;i^{**}}^{**}n_{i^{**}}^{**}-\sum_{i^*=1}^{I^*}\Theta_{0;i^*}^{*}n_{i^*}^{*}\right| \\ &+\sum_{k=1}^{p}\frac{1}{k!}\left|\Theta_k^V+\sum_{i^{**}=1}^{I^{**}}\Theta_{k;i^{**}}^{**}n_{i^{**}}^{**}-\sum_{i^*=1}^{I^*}\Theta_{k;i^*}^{*}n_{i^*}^{*}\right|\max\{\varepsilon^{\bullet};\varepsilon^{\bullet\bullet}\}^k \\ &+\frac{1}{(p+1)!}\left(\Upsilon_{p+1}^V+\sum_{i^{**}=1}^{I^{**}}\Upsilon_{p+1;i^{**}}^{**}n_{i^{**}}^{**}+\sum_{i^*=1}^{I^*}\Upsilon_{p+1;i^*}^{*}n_{i^*}^{*}\right)\max\{\varepsilon^{\bullet};\varepsilon^{\bullet\bullet}\}^{p+1} \end{aligned} \tag{33}$$

which may be seen as the objective function associated with a minimization problem and related to the hedging issue presented above.
The quantities Θ_0^V, Θ_0^{**}, Θ_0^*, Θ_k^V, Θ_k^{**}, Θ_k^* are given as above from (25) to Moreover Υ_{p+1}^V, Υ_{p+1}^{**} and Υ_{p+1}^* are suitable nonnegative constants which depend on p, $\varepsilon^{\bullet}$ and $\varepsilon^{\bullet\bullet}$, whose explicit expressions will be clarified below in the case of swap portfolio. For a choice of sufficiently large value of the order p, it is expected that the terms Υ_{p+1}^V, Υ_{p+1}^{**} and Υ_{p+1}^* would have small sizes (see our numerical illustrations below) and consequently

$$\frac{1}{(p+1)!}\left(\Upsilon_{p+1}^V+\sum_{i^{**}=1}^{I^{**}}\Upsilon_{p+1;i^{**}}^{**}n_{i^{**}}^{**}+\sum_{i^*=1}^{I^*}\Upsilon_{p+1;i^*}^{*}n_{i^*}^{*}\right)\epsilon^{p+1}$$

can be removed practically from the function to minimize.

In the common immunization approach, the idea is reduced to match the sensitivities of the portfolio to hedge with those of the corresponding hedging instrument. It means that, with (33), one has to consider the following equations

$$\Theta_0^V+\sum_{i^{**}=1}^{I^{**}}\Theta_{0;i^{**}}^{**}n_{i^{**}}^{**}-\sum_{i^*=1}^{I^*}\Theta_{0;i^*}^{*}n_{i^*}^{*}=0 \tag{34}$$

$$\Theta_k^V+\sum_{i^{**}=1}^{I^{**}}\Theta_{k;i^{**}}^{**}n_{i^{**}}^{**}-\sum_{i^*=1}^{I^*}\Theta_{k;i^*}^{*}n_{i^*}^{*}=0 \quad \text{for all} \quad k\in\{1,\ldots,p\}. \tag{35}$$

Actually (34) and (35) can be viewed as a linear system of $(p+1)$-equations with (I^*+I^{**})-unknowns, which are the $n_1^{**},\ldots,n_{I^{**}}^{**},n_1^{*},\ldots,n_{I^*}^{*}$'s. Typically the frequent situation is $(p+1) \leq (I^* + I^{**})$. Even for the particular case $(p+1) = (I^* + I^{**})$ and if the system admits a solution, a difficulty arises since the variables defined by n^{**} and n^* are restricted to the integer numbers. For $(p+1) < (I^* + I^{**})$ the usual approach is to consider all $n_1^{**}, \ldots, n_{I^{**}}^{**}$ and $n_1^*, \ldots, n_{I^*}^*$ which minimize the square sum

$$\left(\Theta_0^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{0;i^*}^{*} n_{i^*}^{*}\right)^2 + \sum_{k=1}^{p} \left(\Theta_k^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{k;i^*}^{*} n_{i^*}^{*}\right)^2.$$

However this would not the right way to follow. Indeed, by so doing we loose both the control of the maximum hedging loss size and the attenuator effect brought by the term $\frac{1}{k!}\epsilon^k$. Therefore we have to cope directly with the minimization problem with the objective function presented as in (33).

2.3 Optimization

According to the above Subsection, in a generic form, the hedge of portfolio V by portfolio H is reduced to the minimization problem

$$(P_0): \quad (\mathbf{n}^{**}, \mathbf{n}^*) = \mathbf{argmin}\Big\{ F(n^{**}, n^*; \epsilon) \big| \qquad (n^{**}, n^*) \in D_0 \Big\} \tag{36}$$

with the constraint D_0 defined as the set of $n^{**} = (n_1^{**}, \ldots, n_{I^{**}}^{**})$ and $n^* = (n_1^*, \ldots, n_{I^*}^*)$ such that

$$\frac{1}{P(t,t+\delta)} - 1\Big\} \Bigg(\sum_{i^{**}=1}^{I^{**}} \big\{\nu_0^{**} + \nu^{**}|h_{t;i^{**}}^{**}|\big\} \mathcal{N}_{i^{**}}^{**} n_{i^{**}}^{**} + \sum_{i^*=1}^{I^*} \big\{\nu_0^{*} + \nu^{*}|h_{t;i^{*}}^{*}|\big\} \mathcal{N}_{i^{*}}^{*} n_{i^{*}}^{*} \Bigg) \leq D \tag{37}$$

and where D is the amount allowed by the investor not to be exceed in the hedging operation.

Actually it is taken here that $\epsilon = \max\{\varepsilon^{\bullet}; \varepsilon^{\bullet\bullet}\}$, and for convenience we can deal with vectorial notations such that the constraint (37) may be written as

$$a^{**} \cdot n^{**} + a^* \cdot n^* \leq D \tag{38}$$

where

$$a^{**} = \Big(\{\nu_0^{**} + \nu^{**}|h_{t;1}^{**}|\}\mathcal{N}_1^{**}, \ldots, \{\nu_0^{**} + \nu^{**}|h_{t;i^{**}}^{**}|\}\mathcal{N}_{i^{**}}^{**}, \ldots, \{\nu_0^{**} + \nu^{**}|h_{t;I^{**}}^{**}|\}\mathcal{N}_{I^{**}}^{**} \Big)$$

and

$$a^* = \big(\{\nu_0^* + \nu^*|h_{t;1}^*|\}\mathcal{N}_1^*, \dots, \{\nu_0^* + \nu^*|h_{t;i^*}^*|\}\mathcal{N}_{i^*}^*, \dots, \{\nu_0^* + \nu^*|h_{t;I^*}^*|\}\mathcal{N}_{I^*}^*\big).$$

For the function F as defined in (33), (P_0) is an integer optimization problem defined by integer linear constraints. The objective function is both non-linear, non-convex and non-differentiable at the origin. To overcome these difficulties we make use of a linearization technique as introduced in [8] and which consists to replace the initial problem (P_0) by an equivalent linear problem (P_1).

Assuming that ϵ is a fixed given constant, we introduce the following function

$$G\big(x, n^{**}, n^*; \epsilon\big) = \sum_{k=0}^{p} x_k + \frac{1}{(p+1)!}\Big\{\Upsilon_{p+1}^V + \Upsilon_{p+1}^{**}.n^{**} + \Upsilon_{p+1}^{*}.n^*\Big\}\epsilon^{p+1} \qquad (39)$$

where the components x_l's of x are real variables. The point here is to remove the non-linearity by setting

$$x_k = \frac{1}{k!}\Big|\Theta_k^V + \Theta_k^{**}.n^{**} - \Theta_k^*.n^*\Big|\epsilon^k \qquad \text{for all } k \in \{0, 1, \dots, p\}. \qquad (40)$$

Therefore we obtain the following result.

Lemma 2.1. *The problem (P_0) is equivalent to the following minimization problem*

$$(P_1): \qquad (\mathbf{x}, \mathbf{n}^{**}, \mathbf{n}^*) = \boldsymbol{argmin}\Big\{G(x, n^{**}, n^*; \epsilon)\Big| \qquad (x, n^{**}, n^*) \in D_1\Big\} \qquad (41)$$

where D_1 is defined as the set of triplets (x, n^{**}, n^*) satisfying the constraints

$$a^{**}.n^{**} + a^*.n^* \le D \qquad (42)$$

$$0 \le x_k + \frac{1}{k!}\Big\{\Theta_k^V + \Theta_k^{**}.n^{**} - \Theta_k^*.n^*\Big\}\epsilon^k \qquad \textit{for all } k \in \{0, 1, \dots, p\} \qquad (43)$$

$$0 \le x_k - \frac{1}{k!}\Big\{\Theta_k^V + \Theta_k^{**}.n^{**} - \Theta_k^*.n^*\Big\}\epsilon^k \qquad \textit{for all } k \in \{0, 1, \dots, p\} \qquad (44)$$

with the restrictions that

$$0 \le x = (x_k)_k, \quad n^{**} \in \mathbb{N}^{I^{**}} \quad \textit{and} \quad n^* \in \mathbb{N}^{I^*}.$$

In this Proposition, by the equivalence between (P_0) and (P_1), we mean that if an optimal solution (n^*, n^{**}) to (P_0) does exist, then (P_1) admits an optimal solution (x, n^*, n^{**}) [4], and conversely if (x, n^*, n^{**}) is an optimal solution to (P_1) then (P_0) admits (n^*, n^{**}) as an optimal solution. Therefore, with the above result, we are lead to solve problem (P_1) instead of (P_0).

Observe that both the objective function and constraints associated with (P_1) are given by linear transformations, with mixed integer and real coefficients. The problem (P_1), commonly referred to as a Mixed Integer Linear Problem (MILP), is recognized as to be an NP-hard problem due to the non-convexity of the domain and the number of possible combinations of the variables. For small dimensions, MILP can be solved by exact methods that provide an exact optimal solutions. In this case the most of available exact methods are Branch and Bound, Branch and Cut, Branch and Price [11]. However the complexity of MILP exponentially increases with the number of variables and these mentioned methods can fail. To overcome this inconvenience, meta-heuristics methods (see for instance [12]). Usually there are various solvers which yield exact solution to the MILP for a moderate number of variables less than 1 500 which is largely enough for our purpose. Therefore we make use here the commercial CPLEX solver 9.0. Details and references related to this an application are freely available on the web as www.iro.umontreal.ca/ gendron/IFT6551/CPLEX/HTML/.

2.4 Swap portfolio sensitivities and hedging

In this part, we plan to apply our generic finding related to portfolio hedging in the previous Subsection 2.3 in the setting of swap portfolio.

To perform an accurate analysis of the swap portfolio change value, as written in (16), we first introduce the residual term representing the passage of time by

$$\mathbf{Res}(t, \delta, \mathcal{S}) \equiv \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} \mathbf{Res\_Swap}(t, \mathcal{T}^{**}_{\cdot;i^{**}}, \delta) - \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} \mathbf{Res\_Swap}(t, \mathcal{T}^{*}_{\cdot;i^{*}}, \delta) \tag{45}$$

[4] with $x = (x_l)_l$ and x_l defined from n^* and n^{**} as in (40)

where $\mathbf{Res\_Swap}(t, \mathcal{T}^{**}_{\cdot;i^{**}}, \delta)$ is defined by

$$
\begin{aligned}
&\mathbf{Res\_Swap}(t, \mathcal{T}^{**}_{\cdot;i^{**}}; \delta) \equiv \mathbf{Res\_Swap}\Big(t, \mathcal{T}^{**}_{\cdot;i^{**}}; \delta; \mathbf{notional}(i^{**}); \mathbf{rate\_Swap}(i^{**})\Big) \\
&= \mathbf{notional}(i^{**}) \times \Bigg\{ - \Big\{ y\big(t_0^{**}(i^{**}), t_1^{**}(i^{**})\big) - \mathbf{rate\_Swap}(i^{**}) \Big\} P\big(t, t_1^{**}(i^{**})\big)\delta \\
&\quad + \Big[1 + \Big\{ y\big(t_0^{**}(i^{**}), t_1^{**}(i^{**})\big) - \mathbf{rate\_Swap}(i^{**}) \Big\} \tau\big(t+\delta, t_1^{**}(i^{**})\big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{Res\_ZC}\big(t, t_1^{**}(i^{**}); \delta\big) \\
&\quad - \mathbf{Res\_ZC}\big(t, t^{**}_{M^{**}(i^{**})}(i^{**}); \delta\big) \\
&\quad - \mathbf{rate\_Swap}(i^{**}) \times \sum_{j^{**}=2}^{M^{**}(i^{**})} \mathbf{Res\_ZC}(t, t^{**}_{j^{**}}(i^{**}); \delta)\tau\big(t^{**}_{j^{**}-1}(i^{**}), t_{j^{**}}(i^{**})\big) \Bigg\}.
\end{aligned}
\tag{46}
$$

The term $\mathbf{Res\_Swap}\big(t, \mathcal{T}^{*}_{\cdot;i^{*}}\delta\big)$ is symmetrically given as $\mathbf{Res\_Swap}(t, \mathcal{T}^{**}_{\cdot;i^{**}}, \delta)$. In (46) we make use of the residual term **Res_ZC** for the zero-coupon which is defined by

$$
\mathbf{Res\_ZC}(t, T; \delta) = \exp\Big[-y\big(t; \tau(t+\delta, T)\big)\tau(t+\delta, T)\Big] - \exp\Big[-y\big(t; \tau(t, T)\big)\tau(t, T)\Big] \tag{47}
$$

for any t, $\delta > 0$ and T with $t + \delta < T$.

Before planing to hedge a given swap portfolio it would be natural first to ask whether such an operation makes a sense to be performed. Indeed the hedge is only justified whenever the holder of position thinks that the potential loss linked to the interest rate movement would be significantly high when compared with the cost involved when putting in place the hedge. It may be emphasized that a loss may be occurred though a hedging operation is used. However the point is that the magnitude of this loss should be very small in comparison to the potential loss or gain obtained with the naked position. The hedging operation is not intended to get any profit but rather to try to maintain the position roughly as its initial level.

Now to provide some elements allowing the investor/hedger to take the decision to hedge or not, the simple idea we explore is just to determine the extreme effect of the interest rate PS on the portfolio change value.
The corresponding analysis can be done under the particular view that the shift is contained in some given interval $[-\varepsilon^{\bullet}, \varepsilon^{\bullet\bullet}]$ as considered in (32).

Actually we can state the following:

Theorem 2.2. *Let us consider the present time t and horizon δ, such that the swap portfolio under consideration satisfies the restrictions (14) and (15).*

Assume the interest rate has done a PS movement following the view (32) for some nonnegative $\varepsilon^\bullet$ and $\varepsilon^{\bullet\bullet}$ such that $\varepsilon^\bullet$ is sufficiently small in the sense that

$$\varepsilon^\bullet < \min\Big\{ y\Big(t; \tau\big(t+\delta, t^*_{j^*(i^*)}(i^*)\big)\Big) \,;\, y\Big(t; \tau\big(t+\delta, t^{**}_{j^{**}(i^{**})}(i^{**})\big)\Big)\Big\} \tag{48}$$

where the minimum is done for $i^* \in \{1, \ldots, I^*\}$, $i^{**} \in \{1, \ldots, I^{**}\}$, $j^*(i^*) \in \{1, \ldots, M^*(i^*)\}$ and $j^{**}(i^{**}) \in \{1, \ldots, M^{**}(i^{**})\}$.

Suppose all the swap rates satisfy the conditions

$$0 \leq \left(1 + \Big\{ y\big(t_0^{**}(i^{**}), t_1^{**}(i^{**})\big) - \mathbf{rate\_Swap}(i^{**}) \Big\} \tau\big(t+\delta, t_1^{**}(i^{**})\big)\right)$$

$$\text{and} \quad 0 \leq \left(1 + \Big\{ y\big(t_0^{*}(i^{*}), t_1^{*}(i^{*})\big) - \mathbf{rate\_Swap}(i^{*}) \Big\} \tau\big(t+\delta, t_1^{*}(i^{*})\big)\right) \tag{49}$$

for all $i^{**} \in \{1, \ldots, I^{**}\}$ and $i^* \in \{1, \ldots, I^*\}$.

Then the swap portfolio change value during the time-period $(t, t+\delta)$ is deterministically bounded below and above as follows

$$\begin{aligned} &\mathbf{change\_value\_port\_Swap}_{min}(\varepsilon^\bullet, \varepsilon^{\bullet\bullet}) \\ &\qquad \leq \mathbf{change\_value\_port\_Swap}_{t,t+\delta}(\cdot) \leq \\ &\qquad\qquad \mathbf{change\_value\_port\_Swap}_{max}(\varepsilon^\bullet, \varepsilon^{\bullet\bullet}) \end{aligned} \tag{50}$$

where

$$\begin{aligned} &\mathbf{change\_value\_port\_Swap}_{min}(\varepsilon^\bullet, \varepsilon^{\bullet\bullet}) \equiv \\ &\qquad \mathbf{Res}(t, \delta, \mathcal{S}) + \min\Big\{ \omega(\varepsilon) \big| -\varepsilon^\bullet \leq \varepsilon \leq \varepsilon^{\bullet\bullet} \Big\} \end{aligned} \tag{51}$$

and

$$\begin{aligned} &\mathbf{change\_value\_port\_Swap}_{max}(\varepsilon^\bullet, \varepsilon^{\bullet\bullet}) \equiv \\ &\qquad \mathbf{Res}(t, \delta, \mathcal{S}) + \max\Big\{ \omega(\varepsilon) \big| -\varepsilon^\bullet \leq \varepsilon \leq \varepsilon^{\bullet\bullet} \Big\}. \end{aligned} \tag{52}$$

The function $\varepsilon \longmapsto \omega(\varepsilon)$ is defined by

$$\omega(\varepsilon) = \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} \vartheta^{**}_{i^{**}}(\varepsilon) - \sum_{i^*=1}^{I^*} n^*_{i^*} \vartheta^*_{i^*}(\varepsilon) \tag{53}$$

and where

$$\vartheta^{**}_{i^{**}}(\varepsilon) = \mathrm{N}^{**}_{i^{**}}\Bigg\{$$
$$\Big(1+\{y^{**}_{01}(i^{**}) - r^{**}_{i^{**}}\}\widetilde{\tau}^{**}_1(i^{**})\Big)\exp\big[-\widetilde{y}^{**}_1(i^{**})\widetilde{\tau}^{**}_1(i^{**})\big]\big\{\exp\big[-\varepsilon\widetilde{\tau}^{**}_1(i^{**})\big]-1\big\}$$
$$-\exp\big[-\widetilde{y}^{**}_{M^{**}_{i^{**}}}(i^{**})(\widetilde{\tau}^{**}_{M^{**}(i^{**})}(i^{**}))\big]\big\{\exp\big[-\varepsilon\widetilde{\tau}^{**}_{M^{**}(i^{**})}(i^{**})\big]-1\big\}$$
$$-r^{**}_{i^{**}}\sum_{j^{**}=2}^{M^{**}(i^{**})}\exp\big[-\widetilde{y}^{**}_{j^{**}}(i^{**})\widetilde{\tau}^{**}_{j^{**}}(i^{**})\big]\big\{\exp\big[-\varepsilon\widetilde{\tau}^{**}_{j^{**}}(i^{**})\big]-1\big\}\tau^{**}_{j^{**}}(i^{**})\Bigg\}. \tag{54}$$

For convenience we have used the following short notations

$$\mathrm{N}^{**}_{i^{**}} \equiv \textbf{notional}(i^{**}), \quad , r^{**}_{i^{**}} \equiv \textbf{rate\_Swap}(i^{**}),$$
$$\widetilde{\tau}^{**}_{j^{**}}(i^{**}) \equiv \tau\big(t+\delta, t^{**}_{j^{**}}(i^{**})\big), \quad \tau^{**}_{j^{**}}(i^{**}) \equiv \tau\big(t^{**}_{j^{**}-1}(i^{**}), t^{**}_{j^{**}}(i^{**})\big)$$
$$\widetilde{y}^{**}_{j^{**}}(i^{**}) \equiv y\big(t, \widetilde{\tau}^{**}_{j^{**}}(i^{**})\big), \quad y^{**}_{01}(i^{**}) \equiv y\big(t^{**}_0(i^{**}), t^{**}_1(i^{**})\big).$$

The definition of $\vartheta^{*}_{i^{*}}(\varepsilon)$ is similarly defined by taking one star instead of double star in the above notations.

In order to decide to hedge or not, the holder of a swap portfolio position having a PS view of the interest rate as (32) has to take care about the maximum loss magnitude

$$\max\Big\{-\textbf{change\_value\_port\_Swap}_{min}(\varepsilon^{\bullet}, \varepsilon^{\bullet\bullet}); 0\Big\}$$

or the maximum profit

$$\max\Big\{\textbf{change\_value\_port\_Swap}_{max}(\varepsilon^{\bullet}, \varepsilon^{\bullet\bullet}); 0\Big\}.$$

It may be emphasized that this last Proposition goes in the direction of the swap portfolio position stress-testing. In general people make use of a well defined interest rate PS (as 1% for instance) and re-compute the corresponding value of portfolio. Here we extend this usual approach by being able to measure the effect of a PS inside any interval $[-\varepsilon^{\bullet}, \varepsilon^{\bullet\bullet}]$.

Assumption (48) is useful as it says that the interest rate shift size should not more than the yield with lowest level. To simplify, the hypothesis (49) is chosen since it is empirically satisfied for many practical situations. It is also possible to derive estimates results in case where such a hypothesis is not satisfied, but we have not included here the statement for shortness.

From now it is assumed the need to put in place some hedge operations. Recall that a common market practice is to roll over one-period hedging positions. Therefore our focus in this paper is to consider and analyze a given one-period portfolio hedging by a portfolio of swaps.

The suitable k-th order sensitivity for the swap portfolio change value is defined as

$$\mathbf{Sens}(k;t,\delta,\mathcal{S}) \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**}\mathbf{Sens\_Swap}(k;t,\mathcal{T}_{\cdot;i^{**}}^{**},\delta) - \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*}\mathbf{Sens\_Swap}(k;t,\mathcal{T}_{\cdot;i^{*}}^{*}\delta) \tag{55}$$

for all nonnegative integers k, such that $\mathbf{Sens\_Swap}(k;t,\mathcal{T}_{\cdot;i^{**}}^{**};\delta)$ is given by

$$\begin{aligned}\mathbf{Sens\_Swap}\big(k;t,\mathcal{T}_{\cdot;i^{**}}^{**};\delta\big) &\equiv \mathbf{Sens\_Swap}\Big(k;t,\mathcal{T}_{\cdot;i^{**}}^{**};\delta;N_{i^{**}}^{**};r_{i^{**}}^{**}\Big)\\ &= N_{i^{**}}^{**}\times\Bigg\{\Big(1+\{y_{01}^{**}(i^{**})-r_{i^{**}}^{**}\}\widetilde{\tau}_1^{**}(i^{**})\Big)\times\mathbf{Sens\_ZC}\big(k;t,t_1^{**}(i^{**});\delta\big)\\ &\qquad-\mathbf{Sens\_ZC}\big(k;t,t_{M^{**}(i^{**})}^{**}(i^{**});\delta\big)\\ &\qquad-r_{i^{**}}^{**}\sum_{j^{**}=2}^{M^{**}(i^{**})}\mathbf{Sens\_ZC}\big(k;t,t_{j^{**}}^{**}(i^{**});\delta\big)\tau_{j^{**}}^{**}(i^{**})\Bigg\}.\end{aligned} \tag{56}$$

where

$$\mathbf{Sens\_ZC}(k;t,T;\delta) = \big\{\tau(t+\delta,T)\big\}^k\exp\Big[-y\big(t;\tau(t+\delta,T)\big)\tau(t+\delta,T)\Big]. \tag{57}$$

The expression for $\mathbf{Sens\_Swap}(k;t,\mathcal{T}_{\cdot;i^{*}}^{*};\delta)$ is similarly defined. There is also the need to introduce the swap portfolio remainder term as

$$\begin{aligned}\mathbf{Rem}\big(p+1;t,\delta,\mathcal{S};\rho\big) \equiv& \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**}\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{**}}^{**},\delta;\rho\big)\\ &-\sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*}\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{*}}^{*},\delta;\rho\big).\end{aligned} \tag{58}$$

For shortness, the expressions for

$$\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{**}}^{**},\delta;\rho\big) \quad \text{and} \quad \mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{*}}^{*},\delta;\rho\big)$$

are not reported since it is sufficient to mimic things from those of $\mathbf{Res\_Swap}\big(t,\mathcal{T}^{**}_{\cdot;i^{**}};\delta\big)$ and $\mathbf{Sens\_Swap}\big(k;t,\mathcal{T}^{**}_{\cdot;i^{**}};\delta\big)$ by introducing for all maturities $t^{**}_{j^{**}}(i^{**})$ and $t^{*}_{j^{*}}(i^{*})$ the following expressions :

$$\begin{aligned}&\mathbf{Rem\_ZC}(p+1;t,t^{**}_{j^{**}}(i^{**});\delta)\\&\quad=\frac{(-1)^{p+1}}{(p+1)!}\exp\big[-\rho\widetilde{\tau}^{**}_{j^{**}}(i^{**})\big]\mathbf{Sens\_ZC}(p+1;t,t^{**}_{j^{**}}(i^{**});\delta)\end{aligned}\tag{59}$$

and

$$\begin{aligned}&\mathbf{Rem\_ZC}(p+1;t,t^{*}_{j^{*}}(i^{*});\delta)\\&\quad=\frac{(-1)^{p+1}}{(p+1)!}\exp\big[-\rho\widetilde{\tau}^{*}_{j^{*}}(i^{*})\big]\mathbf{Sens\_ZC}(p+1;t,t^{*}_{j^{*}}(i^{*});\delta)\end{aligned}\tag{60}$$

From the above expressions, it appears that to get $\mathbf{Res}(t,\delta,\mathcal{S})$, $\mathbf{Sens}(k,t,\delta,\mathcal{S})$ and $\mathbf{Rem}(p+1;t,\delta,\mathcal{S};\rho)$ we need to compute all zero-coupon sensitivities as

$$\mathbf{Res\_ZC}\big(t,t^{**}_{j^{**}}(i^{**});\delta\big),\quad \mathbf{Res\_ZC}\big(t,t^{*}_{j^{*}}(i^{*});\delta\big)$$

$$\mathbf{Sens\_ZC}\big(k;t,t^{**}_{j^{**}}(i^{**});\delta\big),\quad \mathbf{Sens\_ZC}\big(k;t,t^{*}_{j^{*}}(i^{*});\delta\big)$$

for all $j^{**}\in\{1,\ldots,M^{**}(i^{**})\}$, $j^{*}\in\{1,\ldots,M^{*}(i^{*})\}$, $i^{**}\in\{1,\ldots,I^{**}\}$, $i^{*}\in\{1,\ldots,I^{*}\}$ and $k\in\{1,\ldots,p\}$.

In the case of a single swap, we have emphasized in (12) the care to grant when a negative PS of the interest rate is considered. Similarly for a portfolio of swaps, we need to consider an analogous restriction which takes the form

$$\begin{aligned}\max\Big\{&\Big(-y\big(t;\widetilde{\tau}^{**}_{j^{**}}(i^{**})\big)\Big)_{j^{**}\in\{1,\ldots,M^{**}(i^{**})\},\quad i^{**}\in\{1,\ldots,I^{**}\}};\\&\Big(-y\big(t;\widetilde{\tau}^{*}_{j^{*}}(i^{*})\big)\Big)_{j^{*}\in\{1,\ldots,M^{*}(i^{*})\},\quad i^{*}\in\{1,\ldots,I^{*}\}}\Big\}<\varepsilon\end{aligned}\tag{61}$$

Our result related to the three-parts decomposition of the swap portfolio change (16) can be now stated.

Theorem 2.3. *Assume that the interest rate curve has done a PS at time-$(t+\delta)$, as described in (11) for some $\varepsilon(\cdot)\neq 0$ satisfying the restriction (61). Let p be a nonnegative integer. Then a real number $\rho=\rho(\varepsilon,p)$ satisfying $0<\rho<\varepsilon$ or $\varepsilon<\rho<0$, does exist such that the* $\mathbf{change\_value\_Swap}_{t,t+\delta}(\cdot)$

during the time-period $(t, t+\delta)$ *is given by the sum of the following three terms*

$$\begin{aligned}\textbf{change\_port\_value\_Swap}_{t,t+\delta} = \\ &\textbf{Res}(t,\delta;\mathcal{S}) \\ &+\sum_{l=1}^{p}\frac{(-1)^l}{l!}\textbf{Sens}\big(l;t,\delta;\mathcal{S}\big)\varepsilon^k(\cdot) \\ &+\textbf{Rem}\big(p+1;t,\delta;\mathcal{S},\rho(\cdot)\big)\varepsilon^{p+1}(\cdot)\end{aligned} \tag{62}$$

where $\textbf{Res}(t,\delta;\mathcal{S})$, $\textbf{Sens}\big(l;t,\delta;\mathcal{S}\big)$ *and* $\textbf{Rem}\big(p+1;t,\delta;\mathcal{S},\rho(\cdot)\big)$ *are respectively defined in (45), (55) and (58).*

According to this last Proposition, the swap portfolio change value $\mathcal{S}_{t+\delta}(\cdot)-\mathcal{S}_t$ may be decomposed into three parts. The first term $\textbf{Res}(t,\delta,\mathcal{S})$ corresponds to the passage of time, as

$$\textbf{change\_value\_port\_Swap}_{t,t+\delta}\Big|_{\varepsilon=0} = \textbf{Res}(t,\delta,\mathcal{B}).$$

It means that if at the future time horizon $t+\delta$, the interest rate remains as the same as the one at the present time t, then the portfolio varies due to the passage of time and its value is exactly given by this residual term.

The second term, written in the right part of (62) is a stochastic term since it depends on the future shift $\varepsilon(\cdot)$ of the interest rate curve, and appears to be a polynomial expression whose the coefficients are given by the various sensitivities of the swap portfolio. As our below numerical experiments show, this second term carries most of the information about the portfolio change at the considered horizon. With these previous two terms, the following portfolio change value approximation can be written:

$$\begin{aligned}\textbf{change\_value\_port\_Swap}_{t,t+\delta}(\cdot) \approx \\ \textbf{Res}(t,\delta;\mathcal{S})+\sum_{k=1}^{p}\frac{(-1)^k}{k!}\textbf{Sens}\big(k;t,\delta;\mathcal{S}\big)\varepsilon^k(\cdot).\end{aligned} \tag{63}$$

So the related error approximation is

$$\begin{aligned}\textbf{error\_approx\_port\_change\_Swap}_{t,\delta}(\cdot) &\equiv \textbf{change\_value\_port\_Swap}_{t,t+\delta}(\cdot) \\ &-\Big\{\textbf{Res}(t,\delta;\mathcal{S})+\sum_{k=1}^{p}\frac{(-1)^k}{k!}\textbf{Sens}\big(k;t,\delta;\mathcal{S}\big)\varepsilon^k(\cdot)\Big\} \\ &\equiv \textbf{Rem}\Big(p+1;t,\delta,\mathcal{S},\rho(\cdot)\Big)\varepsilon^{p+1}(\cdot).\end{aligned} \tag{64}$$

With the expression (58), this **error_approx_port_change_Swap** is a linear combination of the swaps notional values, which in general has a big size as 10^7 or more. The coefficients which are involved in this combination depend on the numbers of considered swaps. It means that some care should be granted before using an approximation as (63). It is not obvious that the error-approximation as (64) has actually a size admitted to be small following the perspective of the investor. This implies that the knowledge of the magnitude of such error approximation is of importance. The corresponding analysis is performed under the view (32) about the PS of the interest rate.

Lemma 2.4. *Under the hypothesis (49), the view (32) and with the restriction (61) then a deterministic estimates of the swap portfolio remainder* **Rem** *is given by*

$$\mathbf{Rem}\Big(p+1;t,\delta,\rho;\mathcal{S}\Big) \leq \max\Big\{\big|\Phi(\rho)\big|; -\varepsilon^{\bullet} < \rho < \varepsilon^{\bullet\bullet}\Big\} \\ \leq \max\Big\{\big|\Psi(\rho)\big|; -\varepsilon^{\bullet} < \rho < \varepsilon^{\bullet\bullet}\Big\} \quad (65)$$

where

$$\Phi(\rho) = \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} \mathbf{Rem\_Swap}^{**}_{i^{**}}\left(p+1;t,\mathcal{T}^{**}_{.;i^{**}},\delta;\rho\right) \\ - \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} \mathbf{Rem\_Swap}^{*}_{i^{*}}\left(p+1;t,\mathcal{T}^{*}_{.;i^{*}},\delta;\rho\right) \quad (66)$$

and

$$\Psi(\rho) = \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} \Big|\mathbf{Rem\_Swap}^{**}_{i^{**}}\left(p+1;t,\mathcal{T}^{**}_{.;i^{**}},\delta;\rho\right)\Big| \\ + \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} \Big|\mathbf{Rem\_Swap}^{*}_{i^{*}}\left(p+1;t,\mathcal{T}^{*}_{.;i^{*}},\delta;\rho\right)\Big|. \quad (67)$$

The terms $\mathbf{Rem\_Swap}\big(p{+}1;t,\mathcal{T}^{**}_{.;i^{**}},\delta;\rho\big)$ *and* $\mathbf{Rem\_Swap}\big(p{+}1;t,\mathcal{T}^{*}_{.;i^{*}},\delta;\rho\big)$ *are the remainder of each payer swap* $S^{**}_{.;i^{**}}$ *and receiver swap* $S^{*}_{.;i^{*}}$.

As in the case of a single swap position, the assumption (49) seems to be satisfied in various practical situations.

According to estimate (65), then the portfolio swap price change decomposition (62) should be performed at some order p, where p is the first nonnegative integer for which the right member of this estimate is less than an amount tolerance level ψ (as for instance $\psi = 10^{-8}$) which would be acceptable by the swap hedger.

3 Numerical Application

Before we present some numerical illustration related to swap hedging, let us first define the zero-coupon yield curve used in this Section.

3.1 Zero-coupon yield curve

The interest rate curve used here is assumed to be given by the Nelson-Siegel-Svenson model as defined in (4).

As in Diebold-Li [8], for all $\tau \geq 0$, the model is calibrated as

$$\beta_{t;1} = 0.0758, \quad \beta_{t;2} = -0.02098, \quad \beta_{t;2} = -0.00162, \quad \lambda = 0.609$$

Features linked to the hedging operation framework is summarized in Table 1.

Table 1: Situation under consideration.

δ	ν_0^{**}	ν_0^{*}	$\varepsilon^{\bullet\bullet}$	$-\varepsilon^{\bullet}$	ϵ	p
90 days	20%	20%	−3%	3%	3%	12

In this table: δ represents the hedging time-horizon, ν_0^{**}, ν_0^{*} are the deposit rates linked to holding (either for payer or receiver) the swaps instruments. Here $\varepsilon^{\bullet}, \varepsilon^{\bullet\bullet}$ are defined in equation (32) and, p is the maximum order to use in the computations of sensitivities. The choice $p = 12$ is chosen in order to insure the small sizes of the remainder terms, and consequently leading to efficient approximations. The notations with tilde $\tilde{()}$ are used in the sequel to refer the portfolio to hedge.

Two cases of hedging operations are considered here:

- Case1: The portfolio to cover is made by five types of payer swaps and three types of receiver swaps. The Hedging portfolio is made by one type of payer swap and three types of receiver swaps.

- Case2: The portfolio to cover is made by five types of bonds in long positions and three types of bonds in short positions. The Hedging portfolio is made by five types of payer swaps and three types of receiver swaps.

3.2 Case1:Hedging a swap portfolio by a swap portfolio

In this Subsection, we are interested to cover a swap portfolio by another swap portfolio.

The portfolio to hedge is made by five types of payer swaps $\widetilde{S_1}^{**}$, $\widetilde{S_2}^{**}$, $\widetilde{S_3}^{**}$, $\widetilde{S_4}^{**}$ and $\widetilde{S_5}^{**}$, and three types of receiver swaps $\widetilde{S_1}^{*}$, $\widetilde{S_2}^{*}$ and $\widetilde{S_3}^{*}$. All of their characteristics are summarized in Table 2.

Table 2: Bounds of the portfolio swap to hedge

type	number	maturity	frequency	rate_swap
$\widetilde{S_1}^{**}$	$\widetilde{n}_1^{**} = 100$	3years	6 months	6.6490%
$\widetilde{S_2}^{**}$	$\widetilde{n}_2^{**} = 200$	4 years	6 months	6.8216%
$\widetilde{S_3}^{**}$	$\widetilde{n}_3^{**} = 200$	7 years	6 months	7.1124%
$\widetilde{S_4}^{**}$	$\widetilde{n}_4^{**} = 100$	10 years	6 months	7.2466%
$\widetilde{S_5}^{**}$	$\widetilde{n}_5^{**} = 100$	5 years	6 months	6.9475%
$\widetilde{S_1}^{*}$	$\widetilde{n}_1^{*} = 200$	4 years	1 year	6.9402%
$\widetilde{S_2}^{*}$	$\widetilde{n}_2^{*} = 300$	6 years	1 year	7.1668%
$\widetilde{S_3}^{*}$	$\widetilde{n}_3^{*} = 100$	7 years	1 year	7.2404%

Names of the types of swaps used are displayed in the first column of this Table 2. The numbers of swaps used for each type are presented in the second column. Maturities of the types of swaps considered are given in the third column. We have written in the fourth column the corresponding swap payment frequency, as semi-annually or annually frequency-based. Each swap is assumed to have the notional value of 1 000 000 Euros. The fair rate_swap of each swap, as mentioned in Subsection 2.1, is given in the fifth column. As we assume that the present time corresponds to the time-inceptions for all of these swaps, then the portfolio under consideration has zero value (or portf_value =0 Euros).

3.2.1 Bounds for the future change value of the portfolio to hedge

As explained in Subsection 2.4, in order to decide to hedge or not the considered swap portfolio, it is valuable to have a projection of the low and high bounds for the portfolio change value at the given horizon and under a (more and less severe) parallel shift of the interest rate.

By using our Proposition 2.2 then the result is summarized in Table 3.

Table 3: Characteristics of the portfolio swap to hedge

	ϵ^*	change_value_portf (ϵ^*)
change_value_portf_swap$_{min}$	-3%	-2.39×10^7
change_value_portf_swap$_{max}$	3%	1.93×10^7

In Table 3, by ϵ^* we denote the value of $\varepsilon \in [-3\%, 3\%]$ which allows to attain the minimum or maximum of the portfolio change value. This results is performed by making use of the MatLab **fmincom** function.

It is seen here that in the worst case, the potential loss in case of dealing with just a naked portfolio position can attain the size of 20 Millions of Euros, which corresponds roughly to 20 swaps. So it might be useful to hedge the position as we will consider below in subsubsection 3.2.3.

3.2.2 Estimate of the remainder term

Here we would like to grasp the size of the remainder term associated with the above swap portfolio, when one consider the three parts decomposition of the portfolio change value, at some order p. The result is summarized in Table 4.

The first column of this last table contains the expansion order p used in the approximation resulting from our Lemma 2.4. Remind that the Remainder term depends on ρ, with $0 < \rho < \varepsilon(\cdot) \leq \varepsilon^{\bullet\bullet}$ or $-\varepsilon^{\bullet} \leq \varepsilon(\cdot) < \rho < 0$. It may be observed in Table 4 that just limiting the expansion to the first or second

Table 4: Higher bound Remainder portfolio swap for different order case3

order p	ρ^*	Bound_Remain	higher_Bound_Remain
1	-3%	4.81×10^6	9.58×10^6
2	-3%	2.86×10^5	5.20×10^5
5	-3%	9.54	27.04
12	-3%	0.11×10^{-9}	0.15×10^{-9}

order leads to approximation errors of order 10^5 or more. This mean that in the framework of swap portfolio, just making use sensitivities of order one and two (similarly to the duration and convexity for the case of a single bond position) is not enough to expect to get an efficient and acceptable hedging operation. The particular situation considered here gives credence that at least an approximation order more than $p = 5$ is required.

3.2.3 Hedging illustration

To hedge the previous portfolio introduced in Table 2 above, we make use of another swap portfolio made by one type of payer swap S_1^{**}, and three types of receiver swaps S_1^*, S_2^* and S_3^*. The characteristics of all of these instruments are summarized in Table 5.

Table 5: Characteristics of the hedging instruments

type	number	maturity	frequency	rate_swap
S_1^{**}	n_1^{**}	2	6 months	6.41%
S_1^*	n_1^*	3	1 year	6.76%
S_2^*	n_2^*	10	1 year	7.37%
S_3^*	n_3^*	8	6 months	7.17%

The amount required for the hedging depends actually on the number of instruments used for that purpose, and consequently is not known in advance. However, it is common among investors/hedgers to keep back a priori some maximal allowed amount D. Here we determine this last from the relation

$$D = \gamma \Big\{ \sum_{i^{**}=1}^{\widetilde{I}^{**}} \widetilde{\mathcal{N}}^{**}_{i^{**}} \widetilde{n}^{**}_{i^{**}} + \sum_{i^{*}=1}^{\widetilde{I}^{*}} \widetilde{\mathcal{N}}^{*}_{i^{*}} \widetilde{n}^{*}_{i^{*}} \Big\}$$

where γ is taken to be equal to 5%. As each notional value $\widetilde{\mathcal{N}}^{**}_{i^{**}}$ or $\widetilde{\mathcal{N}}^{*}_{i^{*}}$ is equal to 1 000 000 Euros, then it is clear that $D = 65\,000\,000$.

The numbers $\mathbf{n_1^{**}}, \mathbf{n_1^{*}}, \mathbf{n_2^{*}}$ and $\mathbf{n_3^{*}}$ of swaps $S_1^{**}, S_1^{*}, S_2^{*}$ and S_3^{*} respectively required for the hedging, as described in the above in Lemma 2.1 in subsection 2.3, are determined here by using the IBM ILOG CPLEX' solver. After 0.3 second running time (in our computer processor: AMD Sempron(tm) M120 2.10 GHz)), we obtain the results summarized in Table 6.

Table 6: Result of hedging operation Case1

n_1^{**}	n_1^{*}	n_2^{*}	n_3^{*}	Max Profit or Loss	proportion
0	122	0	84	883 737.24	1.35%

The real Profits or Losses (PL) corresponding to some shifts $\varepsilon \in [-3\%, 3\%]$ are presented in the second column of Table 7. So by $PLport$ we mean the PL corresponding to the naked portfolio change value (that is the portfolio PL in absence of hedging).

Profits and losses for the hedging_instruments, denoted here $PLinst$ and defined in (19) are displayed in the third column. In the fourth column one can see the PL for the overall portfolio (portfolio to hedge and hedging portfolio). These last quantities include the hedging costs as defined in (20).

By *ret_port_cov*, in the fifth column, we mean the ratio

$$ret\_port\_cov = \frac{PLport\_cov}{D}.$$

It may be noted that it is not the return linked to the covered portfolio as we just take as a basis the maximal amount allowed for the hedging operation. Indeed for swaps whose the initial values may be equal to zero, the notion of return should be taken with care as it is analyzed by A. Meucci [10]. Observe that the portfolio to hedge is not assumed to be unwound at the considered horizon, and the amount D is freezed for the hedge though the cost really involved in the operation is strictly less than D.

Table 7: Wealth for any shift $\varepsilon \in [-3\%, 3\%]$ after the Hedging operation

ε	PLport	PLinst	PLport_cov	ret_port_cov	ret_port
−3%	−23 889 286.01	23 614 943.23	−857 471.72	−1.32%	−36.75%
−2.5%	−19 631 335.92	19 388 781.40	−825 683.46	−1.27%	−30.20%
−2%	−15 513 978.52	15 291 407.26	−805 700.19	−1.24%	−23.87%
−1.5%	−11 530 194.97	11 318 597.12	−794 726.80	−1.22%	−17.74%
−1%	−7673 324.12	7 466 276.54	−790 176.52	−1.22%	−11.81%
−0.5%	−3 937 044.78	3 730 514.93	−789 658.79	−1.21%	−6.06%
0%	−315 358.88	107 520.20	−790 967.62	−1.22%	−0.49%
0.5%	3 197 424.50	−3 406 366.24	−792 070.68	−1.22%	4.92%
1%	6 606 704.45	−6 814 674.43	−791 098.92	−1.22%	10.16%
1.5%	9 917 602.67	−10 120 810.52	−786 336.78	−1.21%	15.26%
2%	13 134 977.33	−13 328 061.30	−776 212.91	−1.19%	20.21%
2.5%	16 263 436.15	−16 439 598.58	−759 291.37	−1.17%	25.02%
3%	19 307 348.95	−19 458 483.39	−734 263.37	−1.13%	29.70%

For the last sixth column by *ret_port* we mean the ratio

$$ret\_port = \frac{PLport}{D}.$$

The compensation between the loss related to the portfolio to hedge and the gain associated with the hedging portfolio may be understood from the alternated signs for the quantities displayed in the second and third columns. For $\varepsilon = 0\%$ one has $ret\_port = -0.49\%$. This an indication that the time-passage matters in hedging, and consequently should be taken into account as is the case for the sensitivities we have introduced in this paper. For the interest rate shift $\varepsilon = -2\%$ it may be seen, from the last two columns, that $ret\_port\_cov = -1.24\%$ and $ret\_port = -23.87\%$. This means that a loss appears though the portfolio position is hedged or not. However the magnitude is clearly more important than the one involved in absence of hedge. Under the shift $\varepsilon = 2\%$ one has $ret\_port\_cov = -1.19\%$ and $ret\_port = 20.21\%$.

That is, in absence of the hedging operation, the considered portfolio has lead to an important gain. The hedge has an effect to get at worst a loss, but the corresponding magnitude (when taking into account D as a reference basis) is fortunately small.

The cost of the hedging instruments is about 583 128.94. Here the resulting loss can be viewed as the price of uncertainty and fear about the interest rate behavior at the considered horizon. At this point, it may be important to recall that the hedging operation has mainly as purpose to roughly maintain the portfolio at its initial level, but not to make any profit.

3.3 Case2: Hedging a bond portfolio by a swap portfolio

The portfolio to cover is made by five types of bonds $\widetilde{B_1}^{**}, \ldots, \widetilde{B_5}^{**}$ in long positions and three types of bonds $\widetilde{B_1}^{*}, \ldots, \widetilde{B_3}^{*}$ in short positions whose the characteristics are summarized in Table 8.

Table 8: Characteristics of the bond portfolio to hedge.

type	number	cpn	maturity	unit_value	number×unit_value
$\widetilde{B_1}^{**}$	$\widetilde{n_1}^{**} = 1\,000$	3%	3 years	889.82	889 822.50
$\widetilde{B_2}^{**}$	$\widetilde{n_2}^{**} = 1\,500$	5%	$4 years$	922.08	1 383 119.18
$\widetilde{B_3}^{**}$	$\widetilde{n_3}^{**} = 500$	7%	$5 years$	984.24	492 121.75
$\widetilde{B_4}^{**}$	$\widetilde{n_4}^{**} = 750$	4%	10 years	754.36	565 770.14
$\widetilde{B_5}^{**}$	$\widetilde{n_5}^{**} = 500$	4%	12 years	721.21	360 609.61
$\widetilde{B_1}^{*}$	$\widetilde{n_1}^{*} = 1\,000$	4%	2 years	944.62	944 621.69
$\widetilde{B_2}^{*}$	$\widetilde{n_2}^{*} = 900$	5%	3 years	942.20	847 988.22
$\widetilde{B_3}^{*}$	$\widetilde{n_3}^{*} = 1\,000$	6%	4 years	955.80	955 807.18

The type of bonds are presented in the first column of Table 8. The number of each type of bond is displayed in the second column. All of the considered bonds in the portfolio are assumed to have the same facial value 1 000 Euros. Their (annual) coupon rates and maturities are displayed respectively in the third and fourth columns of this Table. Unit market values of the bonds, computed by making use of the zero yield curve defined in subsection 3.1, are written in the fifth column. Therefore the portfolio initial value is seen here to be equal to 943 025 Euros. The bond portfolio sensitivities results and computations are not detailed in the present work, but they can be consulted in our working paper [5].

The hedging swap portfolio is assumed to be made by five types of payer swaps $S_1^{**}, \ldots, S_5^{**}$, and three types of receiver swaps $S_1^*, \ldots, S_3^*$. The characteristics of these hedging instruments are described in Table 9.

Table 9: Characteristics of the hedging instruments Case2

type	number	maturity	frequency	rate_swap
S_1^{**}	n_1^{**}	3 years	6 months	6.6490%
S_2^{**}	n_2^{**}	4 years	6 months	6.8216%
S_3^{**}	n_3^{**}	7 years	6 months	7.1124%
S_4^{**}	n_4^{**}	10 years	6 months	7.2466%
S_5^{**}	n_5^{**}	5 years	6 months	6.9475%
S_1^*	n_1^*	4 years	1 year	6.9402%
S_2^*	n_2^*	6 years	1 year	7.1668%
S_3^*	n_3^*	7 years	1 year	7.2404%

Each considered swap is assumed to have the notional value 1 000 000 Euros. For the payer swaps the payment dates is done in a semi-annual basis, while they are annually based for the receiver swaps. To simplify all of the swaps are at-par, that is they are supposed to be incepted at the current time . So they have an initial zero market value. The integer numbers $\mathbf{n_1^{**}}, \ldots, \mathbf{n_5^{**}}$, $\mathbf{n_1^{*}}, \ldots, \mathbf{n_3^{*}}$, mentioned in the second column two of Table10, are unknown. We seek to find them in order to perform the hedging in accordance with Proposition 2.1 in Subsection 2.3 with the maximal amount D calculated here from

$$D = \gamma\Big(\sum_{i^{**}=1}^{\widetilde{I}^{**}} \widetilde{B}_{i^{**}}^{**} \widetilde{n}_{i^{**}}^{**} + \sum_{i^{*}=1}^{\widetilde{I}^{*}} \widetilde{B}_{i^{*}}^{*} \widetilde{n}_{i^{*}}^{*} \Big)$$

with $\gamma = 5\%$. It may be seen here that $D = 325\,962.32$.

With the same solver and machine, used in the previous hedging operation (swap portfolio by swap portfolio) after 0.16 second machine running time, we obtain the allocation result summarized in Table 10.

Table 10: Result of hedging operation Case2

${n_1}^{**}$	${n_2}^{**}$	${n_3}^{**}$	${n_4}^{**}$	${n_5}^{**}$	${n_1}^{*}$	${n_2}^{*}$	${n_3}^{*}$	Loss	proportion
0	0	0	2	1	1	0	1	9619	1%

It seen in this Table10 that the possible maximum profit or loss is here 9 619, which corresponds to 1% of the bond portfolio initial value 95 883.15.

Next we provide in Table 11 the hedging operation effect after any parallel shift ε of the yield curve for $\varepsilon \in [-3\%; 3\%]$.

Table 11: Wealth associated with the hedging operation Case2

ε	PLport	PLinst	PLport_cov	ret_port_cov	ret_port
-3%	$357\,230,52$	$-366\,849,69$	$-9\,619,17$	$-2,95\%$	$109,59\%$
$-2,5\%$	$292\,360,69$	$-301\,025,82$	$-8\,665,12$	$-2,66\%$	$89,69\%$
-2%	$230\,355,99$	$-238\,254,93$	$-7\,898,94$	$-2,42\%$	$70,67\%$
$-1,5\%$	$171\,082,34$	$-178\,389,94$	$-7\,307,60$	$-2,24\%$	$52,49\%$
-1%	$114\,412,30$	$-121\,290,84$	$-6\,878,54$	$-2,11\%$	$35,10\%$
$-0,5\%$	$60\,224,74$	$-66\,824,40$	$-6\,599,66$	$-2,02\%$	$18,48\%$
0%	$8\,404,48$	$-14\,863,81$	$-6\,459,33$	$-1,98\%$	$2,58\%$
$0,5\%$	$-41\,157,98$	$34\,711,63$	$-6\,446,35$	$-1,98\%$	$-12,63\%$
1%	$-88\,566,77$	$82\,016,77$	$-6\,550,00$	$-2,01\%$	$-27,17\%$
$1,5\%$	$-133\,920,93$	$127\,160,95$	$-6\,759,98$	$-2,07\%$	$-41,08\%$
2%	$-177\,314,64$	$170\,248,21$	$-7\,066,43$	$-2,17\%$	$-54,40\%$
$2,5\%$	$-218\,837,50$	$211\,377,57$	$-7\,459,93$	$-2,29\%$	$-67,14\%$
3%	$-258\,574,72$	$250\,643,25$	$-7\,931,47$	$-2,43\%$	$-79,33\%$

Similar comments related to Table 11, as the ones done for Table7, may be also given, so for shortness we do not provide the details. However if trying to compare the hedging results for the Case1 (i.e. Swap by Swap) and the Case2 (i.e. Bond by Swap), we observe that at the shift level -2%, then $\left|\frac{ret\_port}{ret\_port\_cov}\right| \approx 19.25$ for the swap Case1, while $\left|\frac{ret\_port}{ret\_port\_cov}\right| \approx 29.20$ for the bond Case2.

In contrast at the shift level 2%, we obtain then $\left|\frac{ret\_port}{ret\_port\_cov}\right| \approx 16.98$ for the swap Case1, while $\left|\frac{ret\_port}{ret\_port\_cov}\right| \approx 25.07$ for the bond Case2.

It means that the hedging effect is more important in the case for a bond portfolio when compared with the one for a swap portfolio.

4 Conclusion

1. Matching the sensitivities of the portfolio to hedge and the hedging in-

struments is among the various common approaches used by practitioners. For the case of a position sensitive to interest rate and under the assumption of interest rate parallel shift (PS), we have seen in Section 2.2 that the hedging operation leads to a nonlinear and integer minimization problem. Actually, this last can be reduced to a Mixed Integer Linear Problem which may be solved by making use of standard solvers.

2. In contrast with the well-known classical bond duration and convexity, we have used high order sensitivities taking into account the passage of time. This is useful in order to reduce the portfolio change approximation whose the quality depends on notional sizes of the considered instruments. It may be noted that very often bond and swap are seen as linear instruments with respect to zero-coupons, but they are not with respect to the one-factor risk/opportunity which is here the interest rate PS.

3. Before considering the systematic analysis, it seems natural first to raise the question whether the hedging operation deserves to be performed. Our answer here is given by providing deterministic bounds of the nude portfolio change under the projected market situation.

4. In the hedging framework, we are able here to derive deterministic bound for the hedging error under a view of interest rate PS inside a given interval. This is especially interesting in comparison with standard variance hedging errors, since here the economical loss/gain resulting from the hedge operation becomes visible.

5. This work has been done in the framework of a PS of the interest rate curve at the hedging time horizon. Though the PS assumption deserves theoretical and practical consideration, it appears that such a situation is less realistic, since in most of concrete cases the interest rate curve moves in a non parallel fashion. Nevertheless, the ideas presented in the present paper can be explored to tackle the issue under the situation driven by a one factor uncertainty, as in the case of Vasicek or Cox-Ingersoll-Ross models. This full detail has been recently analyzed by the third author in [13].

6. The study performed in this work still belongs to the single-curve pre-crisis theoretical approach. However the ideas we have introduced here may be explored to analyze the hedging with swaps under the present interest rate framework, where multiple-curve valuation is now the market practice. Under the consideration of discounting and forwarding curves in separate manner, and always in the spirit of interest rate PS, then it would appear that we are lead to consider a two-factors problem, which should be more difficult to handle when comparing with the one factor setting considered in the present paper. Many more details are expected to be developed in our future next project.

References

[1] F. Mercurio, Interest rates and the credit crunch: new formulas and market models, (2009), http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1332205.

[2] H. Jaffal, Y. Rakotondratsimba and A. Yassine, Hedging with swaps under a parallel shift of the yield curve, (2012), http://papers.ssrn.com.

[3] S. Lajili and Y. Rakotondratsimba, Enhancement of the bond duration-convexity approximation, *International J. Economic Finance*, **4**(3), (2012).

[4] H. Jaffal, A. Yassine and Y. Rakotondratsimba, Enhancement of the bond portfolio Immunization under a parallel shift of the yield curve, *Journal of Finance and Investment Analysis*, **1**(2), (2012).

[5] H. Jaffal, A. Yassine and Y. Rakotondratsimba, Enhancement of the Fischer-Weil bond technique immunization, (2011), http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2191536

[6] M. Choudry, Using bond futures contracts for trading and hedging, (2004), http://yieldCurve.com.

[7] N. Carcano and H. Dallo, Alternative models for hedging yield curve risk: an empirical comparison, *Swiss Finance Institute, Research Paper Series*, **10-31**, (2010), http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1635291.

[8] M.E. Posner and C. Wu, Linear max-min programming, *Mathematical programming*, **20**(1), (1981), 166-172.

[9] F. Diebold and C. Li, Forecasting the term structure of government bond yields, *Journal of Econometrics*, **130**, (2006), 337-364.

[10] A. Meucci, Return calculation for leveraged securities and portfolios, http://papers.ssrn.com/sol3/papers. cfm?abstract_id=1675067, (2010).

[11] M. Junger and S. Thienel, The ABACUS system for branch-and-cut-and-price algorithms in integer programming and combinatorial optimization, *Softawre-Practice and Experience (SFE)*, **30**, (2000), 1325-1352, http://cs.ucsb.edu/ cappello/595n/abacus.pdf

[12] C. Blum and A. Roli, Metaheuristics in combinatorial optimization: Overview and conceptual comparison, *ACM Computing Surveys*, **35**(3), (2003), 268-308.

[13] Y. Rakotondratsimba, Interest rate sensitivies under the Vasicek and Cox-Ingersoll-Ross models, (2011), http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1977902.